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Research Article

Histories Distorted by Partial Isometries

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Abstract In quantum dynamical systems, a history is defined by a pair (M, γ), consisting of a type I factor M, acting on a Hilbert space H, and an E_0-group $\gamma = (\gamma_t)_{t\in\mathbb{R}}$, satisfying certain additional conditions. In this paper, we distort a given history (M, γ), by a finite family $\mathcal{G}$ of partial isometries on H. In particular, such a distortion is dictated by the combinatorial relation on the family $\mathcal{G}$. Two main purposes of this paper are (i) to show the existence of distortions on histories, and (ii) to consider how distortions work. We can understand Sections 3, 4 and 5 as the proof of the existence of distortions (i), and the properties of distortions (ii) are shown in Section 6.

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1 Introduction

Directed graphs, both finite and infinite, play a role in a myriad of areas of applications: electrical networks of resistors, the internet and random walks in probability theory, to mention only a few. Some of the results in the subject stress combinatorial aspects of the graphs (e.g. [4,21,24,26,27,28,29,30,32,33]), while others have a more analytic slant. Applications to quantum theory fall in the latter group, and that is where Hilbert space and noncommutative operators play a role (e.g. [1,5,6,18,22,23,34,35] through [25,36]).

In [14,6,7], algebraic structures induced by directed graphs, called *graph groupoids*, have been studied. They are indeed categorical groupoids (e.g. see [31]). Under the suitable representations of graph groupoids, one understands that the elements of graph groupoids are operators on Hilbert spaces. In particular, we have the *canonical representation* of graph groupoids (e.g. [6,7,8,10]). Such representations are the groupoidal version of the well-known left regular representations of groups (e.g. [37]). It lets us construct von Neumann algebras generated by graph groupoids. Via canonical representation, the edges and vertices of graphs assign Hilbert-space operators.

Independently, in [15,18,19], we consider the *embedding representation* of graph groupoids, and study the corresponding C^*-subalgebras in fixed operator algebras $B(H)$. In particular, the author and Jorgensen showed that if partial isometries on a Hilbert space H are given, then they generate a corresponding graph (and hence an embedded subgroupoid of $B(H)$). This provides a connection from Hilbert-space operators to graphs.

The study of graph groupoids is not only interested in operator theory and quantum physics, but also interested in statistical algebra. For instance, graph groupoids have been used to enlarge the understandings for fractal structures. Graph groupoids with *fractal property* are said to be *graph fractaloids* (e.g. [11,13,16,17,20,9]). There are sufficiently many graph fractaloids which are not fractal groups.

1.1 Motivation

This paper is highly motivated by the recent works of Arveson (e.g. see [2,3] and the cited papers therein) and those of the author and Jorgensen (see [15,18]).

In [2,3], the *histories* are defined and investigated mathematically, and the groupoid actions induced by partial isometries on a fixed Hilbert space H are considered in [15,18].

This paper starts with the following questions:

(1.1) Are histories distorted?
(1.2) How can we distort histories?

1.2 Technical overview

Let H be a fixed separable infinite dimensional Hilbert space and $B(H)$ an operator algebra consisting of all (bounded linear) operators on H. Assume that $M \subseteq B(H)$ is a *type I (sub) factor*, and let $\gamma = (\gamma_t)_{t\in\mathbb{R}}$ be an

E_0-*group*, satisfying (i) the *fixed past* property, (ii) *irreducibility* and (iii) *trivial infinitely remote past* property. The pair (M, γ) is called a *history* (of M in $B(H)$) (see Section 2.4 below).

In [11, 12], *framings on graphs* have been considered; *measure-space-framings* and *group-framings*. We are interested in group-framings under the assumption that groups Γ are topological groups, and the graphs G are regarded as discrete topological spaces consisting of all vertices and edges. The *group-framed graph* G_Γ *of* G *with its group-frame* Γ is a topological space which is neither a combinatorial graph nor a group, however it acts like a usual graph, and generates the (categorical) groupoid $\mathbb{G}_\Gamma$. Interestingly, $\mathbb{G}_\Gamma$ is characterized by the *product groupoid* $\Gamma \times \mathbb{G}$. The group-framing $\mathbb{G}_\gamma$ makes us "distort" the E_0-group γ of a given history (M, γ) by the graph groupoid $\mathbb{G}$ of G. In other words, $\mathbb{G}$ distorts the fixed history (M, γ) if $\mathbb{G}$ acts on H.

Like in our real life, two kinds of distortions may happen; *inner distortions* (the distortions happened inside the paradigm) and the *outer distortions* (the distortions happened outside the paradigm).

In [18], it is shown that if we have a finite family $\mathcal{G}$ of partial isometries on a Hilbert space H, then the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is characterized by the groupoid C^*-algebra $C^*_\pi(\mathbb{G})$ in $B(H)$, where $\mathbb{G}$ is the graph groupoid of a certain graph $G_\mathcal{G}$ induced by $\mathcal{G}$. In this paper, the "special" case introduced in [15] is considered. But one may/can apply the same approach and techniques for the general case of [18].

We provide the positive answer for our first question (1.1) in Sections 3, 4 and 5. The properties of distorted histories is shown in Section 6, as an answer for the second question (1.2).

2 Definitions and notations

We first introduce the main objects we use throughout the paper.

2.1 Partial isometries on a Hilbert space

We say that an operator $a \in B(H)$ is a *partial isometry* if the operators $a^*a \in B(H)$ is a projection. The characterizations of partial isometries are well known: the operator a is a partial isometry if and only if $a = aa^*a$, if and only if aa^* is a projection, if and only if the adjoint a^* of a is a partial isometry in $B(H)$, if and only if $a^* = a^*aa^*$. Recall that an operator p in $B(H)$ is a *projection*, if p is self-adjoint and idempotent. That is, $p^* = p = p^2$ in $B(H)$. The projections a^*a and aa^*, induced by a partial isometry a, are called the *initial projection* and the *final projection* of a, respectively.

Every partial isometry a has its *initial space*

$$H^a_{\text{init}} = \big(a^*a\big)H,$$

and its *final space*

$$H^a_{\text{fin}} = \big(aa^*\big)H,$$

which are (closed) subspaces of H. Notice that every partial isometry a is a unitary from H^a_{init} onto H^a_{fin}, in the sense that

$$a^*a = 1_{H^a_{\text{init}}}, \quad aa^* = 1_{H^a_{\text{fin}}},$$

where $1_\mathcal{K}$ means the identity operator on an arbitrary Hilbert space $\mathcal{K}$.

Suppose a_1 and a_2 are partial isometries on H, and assume that the initial space $H^{a_1}_{\text{init}}$ of a_1 is (not only Hilbert-space isomorphic but also) identically the same as the final space $H^{a_2}_{\text{fin}}$ of a_2 in H. Or, equivalently, the projections $a_1^*a_1$ and $a_2a_2^*$ are (not only unitarily equivalent but also) exactly the same projection on H. Then the product a_1a_2 of the operators a_1 and a_2 is again a partial isometry on H. Indeed, if we denote the identical projections $a_1^*a_1$ and $a_2a_2^*$ by p, then

$$\big(a_1a_2\big)\big(a_1a_2\big)^*\big(a_1a_2\big) = a_1a_2a_2^*a_1^*a_1a_2 = a_1\big(a_2a_2^*\big)\big(a_1^*a_1\big)a_2 = a_1p^2a_2 = a_1pa_2 = a_1a_2.$$

Therefore, the operator a_1a_2 is a partial isometry with its initial space $H^{a_1a_2}_{\text{init}} = H^{a_2}_{\text{init}}$ and its final space $H^{a_1a_2}_{\text{fin}} = H^{a_1}_{\text{fin}}$ on H.

In general, even though a_1 and a_2 are partial isometries, the product a_1a_2 is not a partial isometry. Denote the final projection $a_2a_2^*$ of a_2 and the initial projection $a_1^*a_1$ of a_1 by p_2 and p_1, respectively. Then we have obtained

$$\big(a_1a_2\big)\big(a_1a_2\big)^*\big(a_1a_2\big) = a_1a_2a_2^*a_1^*a_1a_2 = a_1p_2p_1a_2 \neq a_1a_2,$$

in general, because p_2p_1 is not a projection, in general. Recall that p_2p_1 is a projection if $p_1p_2 = p_2p_1$.

2.2 Directed graphs and graph groupoids

Recently, countable directed graphs have been studied extensively in pure and applied mathematics. Not only are they connected with certain noncommutative structures but also they let us visualize such structures. Moreover, the visualization has a nice matricial expressions, (sometimes, operator-valued matricial expressions depending on) adjacency matrices or incidence matrices of the given graph (e.g. [14, 30, 34]). In particular, the partial isometries in an operator algebra can be expressed by directed graphs (see [15, 18]).

A *graph* is a set of objects called vertices (or points or nodes) connected by links called edges (or lines). In a *directed graph*, the two directions are counted as being distinct directed edges (or arcs). A graph is depicted in a diagrammatic form as a set of dots (for vertices), jointed by curves (for edges). Similarly, a directed graph is depicted in a diagrammatic form as a set of dots jointed by arrowed curves, where the arrows point the direction of the directed edges.

More combinatorially, for us, a directed graph G means the pair $(V(G), E(G))$, equipped with the fixed direction on $E(G)$, where $V(G)$ is the *vertex set* of G, and $E(G)$ is the *edge set* of G. Since all edges are directed, if $e \in E(G)$, then it has its initial vertex v and its terminal vertex v'. Sometimes, we denote this edge e by $e = vev'$, to emphasize the initial and the terminal vertices v and v'. Remark here that the vertices v and v' are not necessarily distinct in $V(G)$. For example, if e is a loop-edge, then $v = v'$ in $V(G)$.

Now, let $e_k = v_k e_k v'_k$ be edges in $E(G)$, with $v_k, v'_k \in V(G)$, for $k = 1, 2$. Assume that $v'_1 = v_2$ in $V(G)$. Then we have a finite path $e_1 e_2$, connecting the vertex v_1 to the vertex v'_2, on G. Inductively, we can have finite paths generated by edges.

Denote the set of all *finite paths* of G by $\mathrm{FP}(G)$. We call $\mathrm{FP}(G)$ the *finite path set of* G. Clearly, the edge set $E(G)$ is contained in $\mathrm{FP}(G)$. Moreover, all elements in $\mathrm{FP}(G)$ are the words in $E(G)$. That is, if w is a finite path in $\mathrm{FP}(G)$, then it is represented as a word in $E(G)$: if $e_1, \ldots, e_n$ are connected directed edges in the order $e_1 \to e_2 \to \cdots \to e_n$ on G, for $n \in \mathbb{N}$, then we can express w by $e_1 \cdots e_n$ in $\mathrm{FP}(G)$. If there exists a finite path $w = e_1 \cdots e_n$ in $\mathrm{FP}(G)$, where $n \in \mathbb{N} \setminus \{1\}$, we say that the directed edges $e_1, \ldots, e_n$ are *admissible*.

The *length* $|w|$ *of* w is defined to be n, which is the cardinality of the admissible edges generating w. Also, we say that finite paths $w_1 = e_{11} \cdots e_{1k_1}$ and $w_2 = e_{21} \cdots e_{2k_2}$ are *admissible* if $w_1 w_2 = e_{11} \cdots e_{1k_1} e_{21} \cdots e_{2k_2}$ is again an element of $\mathrm{FP}(G)$, where $e_{11}, \ldots, e_{1k_1}, e_{21}, \ldots, e_{2k_2} \in E(G)$. Otherwise, we say that w_1 and w_2 are *not admissible*.

Suppose that w is a finite path in $\mathrm{FP}(G)$, connecting the vertex v_1 to the vertex v_2. Then we write $w = v_1 w$ or $w = w v_2$ or $w = v_1 w v_2$, for emphasizing the initial vertex of w, respectively, the terminal vertex of w, respectively, both the initial vertex and the terminal vertices of w. Suppose $w = v_1 w v_2$ in $\mathrm{FP}(G)$ with $v_1, v_2 \in V(G)$. Then we also say that "v_1 *and* w *are admissible*" and "w *and* v_2 *are admissible*". Notice that even though the elements w_1 and w_2 in $V(G) \cup \mathrm{FP}(G)$ are admissible, w_2 and w_1 are not admissible, in general. For instance, if $e_1 = v_1 e_1 v_2$ is an edge with $v_1, v_2 \in V(G)$ and $e_2 = v_2 e_2 v_3$ is an edge with $v_3 \in V(G)$, such that $v_3 \neq v_1$, then there is a finite path $e_1 e_2$ in $\mathrm{FP}(G)$, but there is no finite path $e_2 e_1$, equivalently, the finite path $e_2 e_1$ is undefined.

The *free semigroupoid* $\mathbb{F}^+(G)$ of G is defined by a set

$$\mathbb{F}^+(G) = \{\emptyset\} \cup V(G) \cup \mathrm{FP}(G),$$

with its binary operation $(\cdot)$ on $\mathbb{F}^+(G)$, defined by

$$(w_1, w_2) \longmapsto w_1 \cdot w_2 = \begin{cases} w_1 & \text{if } w_1 = w_2 \text{ in } V(G), \\ w_1 & \text{if } w_1 \in \mathrm{FP}(G),\ w_2 \in V(G) \text{ and } w_1 = w_1 w_2, \\ w_2 & \text{if } w_1 \in V(G),\ w_2 \in \mathrm{FP}(G) \text{ and } w_2 = w_1 w_2, \\ w_1 w_2 & \text{if } w_1, w_2 \text{ in } \mathrm{FP}(G) \text{ and } w_1 w_2 \in \mathrm{FP}(G), \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\emptyset$ is the *empty word* in $V(G) \cup E(G)$. Sometimes, the free semigroupoid $\mathbb{F}^+(G)$ of a certain graph G does not contain the empty word $\emptyset$. For instance, the free semigroupoid of the one-vertex-multi-loop-edge graph does not have the empty word. But, in general, the empty word $\emptyset$ is contained in the free semigroupoid, whenever $|V(G)| > 1$. So, if there is no confusion, then we usually assume that the empty word is contained in free semigroupoids. This binary operation $(\cdot)$ on $\mathbb{F}^+(G)$ is called the *admissibility*. That is, the algebraic structure $(\mathbb{F}^+(G), \cdot)$ is the free semigroupoid of G. For convenience, we denote $(\mathbb{F}^+(G), \cdot)$ simply by $\mathbb{F}^+(G)$.

For the given countable directed graph G, we can define a new countable directed graph G^{-1} which is the opposite directed graph of G, with

$$V\big(G^{-1}\big) = V(G), \quad E\big(G^{-1}\big) = \big\{e^{-1} : e \in E(G)\big\},$$

where $e^{-1} \in E(G^{-1})$ is the opposite directed edge of $e \in E(G)$, called the *shadow of* $e \in E(G)$. That is, if $e = v_1 e v_2$ in $E(G)$ with $v_1, v_2 \in V(G)$, then $e^{-1} = v_2 e^{-1} v_1$ in $E(G^{-1})$ with $v_2, v_1 \in V(G^{-1}) = V(G)$. This new directed graph G^{-1} is said to be *the shadow of* G. It is trivial that

$$\left(G^{-1}\right)^{-1} = G.$$

This relation shows that the admissibility on the shadow G^{-1} is oppositely preserved by that on G.

A new countable directed graph $\widehat{G}$ is called the *shadowed graph of* G if it is a directed graph with

$$V\big(\widehat{G}\big) = V(G) = V\big(G^{-1}\big), \quad E\big(\widehat{G}\big) = E(G) \cup E\big(G^{-1}\big).$$

Definition 1. Let G be a countable directed graph and $\widehat{G}$ the shadowed graph of G, and let $\mathbb{F}^+(\widehat{G})$ be the free semigroupoid of $\widehat{G}$. Define the reduction (RR) on $\mathbb{F}^+(\widehat{G})$ by

$$ww^{-1} = v, \quad w^{-1}w = v', \tag{RR}$$

whenever $w = vwv'$ in $\mathrm{FP}(\widehat{G})$, with $v, v' \in V(\widehat{G})$. The subset of $\mathbb{F}^+(\widehat{G})$, satisfying this reduction (RR), is denoted by $\mathbb{F}_r^+(\widehat{G})$. And this set $\mathbb{F}_r^+(\widehat{G})$ with the inherited admissibility $(\cdot)$ from $\mathbb{F}^+(\widehat{G})$ is called the graph groupoid of G. Denote $(\mathbb{F}_r^+(\widehat{G}), \cdot)$ by $\mathbb{G}$. Define the reduced finite path set $\mathrm{FP}_r(\widehat{G})$ of $\mathbb{G}$ by

$$\mathrm{FP}_r\left(\widehat{G}\right) \overset{\text{def}}{=} \mathbb{G} \setminus \left(V\big(\widehat{G}\big) \cup \{\emptyset\}\right).$$

All elements of $\mathrm{FP}_r(\widehat{G})$ are said to be reduced finite paths of $\widehat{G}$.

Remark that all elements of a graph groupoid $\mathbb{G}$ are reduced words in $E(\widehat{G})$.

2.3 Groupoids and groupoid actions

Every graph groupoid is indeed a (categorical) groupoid. This means that a graph groupoid has a (rough but rich) algebraic structures.

Definition 2. One says an algebraic structure $(\mathcal{X}, \mathcal{Y}, s, r)$ is a (categorical) groupoid if it satisfies that (i) $\mathcal{Y} \subset \mathcal{X}$, (ii) there exists a partially-defined binary operation $(x_1, x_2) \mapsto x_1 x_2$, for all $x_1, x_2 \in \mathcal{X}$, depending on the source map s and the range map r satisfying the followings:

(ii-1) $x_1 x_2$ is well determined, whenever $r(x_1) = s(x_2)$, for $x_1, x_2 \in \mathcal{X}$,
(ii-2) $(x_1 x_2) x_3 = x_1 (x_2 x_3)$, if they are well defined, for $x_1, x_2, x_3 \in \mathcal{X}$,
(ii-3) if $x \in \mathcal{X}$, then there exist $y, y' \in \mathcal{Y}$, such that $s(x) = y$ and $r(x) = y'$, satisfying $x = yxy'$,
(ii-4) if $x \in \mathcal{X}$, then there exists a unique groupoid-inverse x^{-1} satisfying $xx^{-1} = s(x)$ and $x^{-1}x = r(x)$.

For example, every group Γ is a groupoid $(\Gamma, \{e_\Gamma\}, s, r)$, where e_Γ is the group-identity and $s = r$. The subset $\mathcal{Y}$ of a groupoid $\mathcal{X}$ is said to be the *base of* $\mathcal{X}$.

Remark that we can naturally assume that there exists the *empty element* $\emptyset$ in a groupoid $\mathcal{X}$. The empty element $\emptyset$ represents the undefinedness of the operation. By adding $\emptyset$, we can make the partially-defined binary operation on a groupoid be well defined. Notice that if $|\mathcal{Y}| = 1$ (equivalently, if $\mathcal{X}$ is a group), then the empty word $\emptyset$ is not contained in the groupoid $\mathcal{X}$. However, in general, whenever $|\mathcal{Y}| \geq 2$, a groupoid $\mathcal{X}$ always contain the empty word. So, if there is no confusion, we automatically assume that the empty element $\emptyset$ is contained in $\mathcal{X}$.

It is easy to check that our graph groupoid $\mathbb{G}$ of a graph G is indeed a groupoid with its base $V(\widehat{G})$.

Let $\mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k)$ be groupoids, for $k = 1, 2$. We say that a map $f : \mathcal{X}_1 \to \mathcal{X}_2$ is a *groupoid morphism* if (i) $f(\mathcal{Y}_1) \subseteq \mathcal{Y}_2$, (ii) $s_2(f(x)) = f(s_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$ and (iii) $r_2(f(x)) = f(r_1(x))$ in $\mathcal{X}_2$, for all $x \in \mathcal{X}_1$. If a groupoid morphism f is bijective, then we say that f is a *groupoid-isomorphism*. If there is a groupoid-isomorphism, then the groupoids $\mathcal{X}_1$ and $\mathcal{X}_2$ are *groupoid-isomorphic*. Algebraically, if f is a groupoid-morphism, then

$$f\big(w_1 w_2\big) = f\big(w_1\big) f\big(w_2\big) \quad \text{in } \mathcal{X}_2,$$

for all $w_1, w_2 \in \mathcal{X}_1$.

Recall that if two graphs G_1 and G_2 have graph-isomorphic shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$, then the corresponding graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic.

Let $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ be a groupoid. We say that $\mathcal{X}$ *acts on a set* X if there exists a *groupoid action* π *of* $\mathcal{X}$, such that $\pi(x) : X \to X$ is a well-defined function, and

$$\pi\big(x_1 x_2\big) = \pi\big(x_1\big) \circ \pi\big(x_2\big), \quad \text{on } X,$$

for all $x_1, x_2 \in \mathcal{X}$, where $(\circ)$ means the usual composition. Sometimes, we call the set X a $\mathcal{X}$-*set*.

2.4 E_0-groups and histories

Let M be a von Neumann algebra acting on a Hilbert space H. That is, $M \subseteq B(H)$. Define the *commutant* M' *of* M by

$$M' \overset{\text{def}}{=} \{x \in B(H) : xm = mx, \ \forall m \in M\} \subseteq B(H).$$

Recall that M is a *factor* if

$$M \cap M' = \mathbb{C} \cdot 1_M \overset{*\text{-iso}}{=} \mathbb{C},$$

where "$\overset{*\text{-iso}}{=}$" means "being $*$-isomorphic."

A von Neumann algebra M is said to be of *type* I if there exists the minimal subspace H_0 of H, such that M is $*$-isomorphic to $B(H_0)$. If H_0 is finite dimensional, with $\dim H_0 = n$, then we say M is of *type* I_n, for $n \in \mathbb{N}$, and if H_0 is infinite dimensional, then we say M is of *type* I_∞.

Definition 3. An E_0*-semigroup* is a one-parameter semigroup $\alpha = (\alpha_t)_{t \in \mathbb{R}_0^+}$, acting on M, where $\mathbb{R}_0^+ \overset{\text{def}}{=} \{t \in \mathbb{R} : t \geq 0\}$, satisfying the followings:

(i) every α_t is a $*$-endomorphism on M, for all $t \in \mathbb{R}_0^+$,
(ii) $\alpha_t(1_M) = 1_M$, for all $t \in \mathbb{R}$,
(iii) $\alpha_0 = \text{id}_M$, where id_M means the identity map on M,
(iv) $\alpha_{t_1} \circ \alpha_{t_2} = \alpha_{t_1 + t_2}$, on M, for all $t_1, t_2 \in \mathbb{R}_0^+$.

Recall that $(\mathbb{R}_0^+, +)$ is a semigroup (or a monoid). Thus, by (iii) and (iv), $\alpha = (\alpha_t)_{t \in \mathbb{R}_0^+}$ forms a semigroup under the composition. If the collection $\alpha = (\alpha_t)_{t \in \mathbb{R}_0^+}$ satisfies the conditions (i), (iii) and (iv) (without (ii)), then we call α, an E*-semigroup* (e.g. [2,3]). If an E-semigroup α satisfies an additional condition (ii), then this α is said to be an E_0-semigroup. Similarly, E_0-groups are defined as follows.

Definition 4. An E_0-group is a one-parameter group $\gamma = (\gamma_t)_{t \in \mathbb{R}}$, such that (i) the sub-classes

$$\gamma^+ = (\gamma_t)_{t \in \mathbb{R}_0^+}, \quad \gamma^- = (\gamma_{-t})_{t \in \mathbb{R}_0^+}$$

are E_0-semigroups, (ii) $\gamma_0 = \text{id}_M$ and (iii) the inverses γ_t^{-1} of γ_t are identified with γ_{-t}, for all $t \in \mathbb{R}$.

Notice now that every E_0-group $\gamma = (\gamma, \circ)$ and the "flow" group $\mathbb{R} = (\mathbb{R}, +)$ are group-isomorphic.

Consider certain pairs (M, γ), consisting of type I factors M, and E_0-groups $\gamma = (\gamma_t)_{t \in \mathbb{R}}$ acting on M.

Definition 5. Let M be a type I-(sub)factor in a fixed operator algebra $B(H)$, and let $U = (U_t)_{t \in \mathbb{R}}$ be a one-parameter unitary group on H, where $U_t \in B(H)$ are unitaries, for all $t \in \mathbb{R}$. Define the actions γ_t, acting on M, by

$$\gamma_t(m) \overset{\text{def}}{=} U_t m U_t^*, \quad \forall t \in \mathbb{R}.$$

Then one has the corresponding E_0-group $\gamma = (\gamma_t)_{t \in \mathbb{R}}$, acting on M. Assume that M and γ satisfy the followings:

(2.4.1) fixed past: $\gamma_t(M) \subseteq M$, for all $t < 0$,
(2.4.2) irreducibility: $(\cup_{t \in \mathbb{R}} \gamma_t(M))'' = B(H)$,
(2.4.3) trivial infinitely remote past: $\cap_{t \in \mathbb{R}} \gamma_t(M) = \mathbb{C} \cdot 1_H$.

Then the pair (M, γ) is said to be a history in $B(H)$.

Remark that if a group γ is defined as above, then it is indeed an E_0-group, acting on M: let

$$\gamma_t(m) \overset{\text{def}}{=} U_t m U_t^*, \quad \forall m \in M, \ t \in \mathbb{R}.$$

Then $\gamma = (\gamma_t)_{t \in \mathbb{R}}$ is a group satisfying the followings:

(i) $\gamma_t(M) = U_t M U_t^* \subseteq M$, in $B(H)$, for all $t \in \mathbb{R}$,
(ii) $\gamma_t(1_M) = U_t 1_M U_t^* = U_t U_t^* = 1_{B(H)}$, for all $t \in \mathbb{R}$,

(iii) for any $t_1, t_2 \in \mathbb{R}$, and $m \in M$, we obtain that

$$\gamma_{t_1} \circ \gamma_{t_2}(m) = U_{t_1}\big(U_{t_2} m U_{t_2}^*\big)U_{t_1}^* = U_{t_1}U_{t_2} m \big(U_{t_1}U_{t_2}\big)^* = U_{t_1+t_2} m U_{t_1+t_2}^*$$

since $U = (U_t)_{t\in\mathbb{R}}$ is a one-parameter unitary group on H

$$= \gamma_{t_1+t_2}(m).$$

(iv) for all $m \in M$,

$$\gamma_t \circ \gamma_{t^{-1}}(m) = U_0 m U_0^{-1} = m = U_0^{-1} m U_0 = \gamma_{t^{-1}} \circ \gamma_t(m),$$

by (iii), since $U_0 \overset{\text{def}}{=} \text{id}_M$.

Therefore, the group γ is an E_0-group, acting on M.

Recall that if X is an arbitrary subset in $B(H)$, then the double commutant X'' of X is the von Neumann algebra $vN(X)$, generated by the set X, by the famous *double-commutant theorem*. So, in the condition (2.4.2), the left-hand side means the von Neumann algebra $vN(\gamma(M))$, generated by $\gamma_t(M)$'s, for all $t \in \mathbb{R}$.

It is useful to think of the group γ as actions of the time-flow, and the von Neumann algebra M as events or facts or something happened in the past. Then, the condition (2.4.1) means that the past M is fixed; the condition (2.4.2) means that the history M is understood fully or wholly inside a fixed paradigm $B(H)$, and the condition (2.4.3) means the history of M started from the triviality, represented by $\mathbb{C}$.

3 Graph families of partial isometries

In [15], we observed the C^*-subalgebras of a fixed operator algebra $B(H)$, generated by "certain" finite families of partial isometries. And in [18], we extend the results of [15] to the "general" case, where we have arbitrary finite families of partial isometries on H. In this paper, we only consider the structures introduced in [15]. We remark that the results of this paper are extendable to the general cases of [18]. However, the settings of [15] are more natural and reasonable for our purpose.

In the rest of this paper, we fix a Hilbert space H, and the corresponding operator algebra $B(H)$, regarded as a paradigm where histories are embedded.

Definition 6. Let $\mathcal{G} = \{a_1, \ldots, a_N\}$ be a finite family of partial isometries in $B(H)$, for $N \in \mathbb{N}$. One says that the family $\mathcal{G}$ constructs a finite directed graph G if there exists G, such that

(i) $|E(G)| = |\mathcal{G}|$ and $|V(G)| = |\mathcal{G}_{pro}|$, where

$$\mathcal{G}_{pro} \overset{\text{def}}{=} \{a^*a : a \in \mathcal{G}\} \cup \{aa^* : a \in \mathcal{G}\}.$$

Equivalently, there exist bijections $g_E : E(G) \to \mathcal{G}$ and $g_V : V(G) \to \mathcal{G}_{pro}$, such that

$$g_E(e) = g_E(vev') = g_V(v)g_E(e)g_V(v'),$$

where $g_V(v)$ and $g_V(v')$ are the initial and the final projections of $g_E(e)$, respectively,

(ii) $e_1 \cdots e_n$ is a nonempty finite path on a graph G if and only if the corresponding operator $g_E(e_1)\cdots g_E(e_n)$ is a well-defined "partial isometry" in $B(H)$, for all $n \in \mathbb{N}$,

(iii) two edges e_1 and e_2 are not admissible ($e_1e_2 = \emptyset$) if and only if $g_E(e_1)g_E(e_2) = 0_H$.

The family $\mathcal{G}$ is called a G-family of partial isometries in $B(H)$. Conversely, the graph G is called the $\mathcal{G}$-graph.

The above conditions (ii) and (iii) can be rewritten as follows: two edges e_1 and e_2 are admissible if and only if the initial spaces $H_{\text{init}}^{g_E(e_1)}$ and the final space $H_{\text{fin}}^{g_E(e_2)}$ are (not only Hilbert-space isomorphic, but also) identically the same in H, as subspaces of H. Equivalently, the edges e_1 and e_2 are not admissible if and only if $H_{\text{init}}^{g_E(e_1)} \cap H_{\text{init}}^{g_E(e_2)} = \{0_H\}$. That is,

$$e_1e_2 \neq \emptyset \Longleftrightarrow H_{\text{init}}^{g_E(e_1)} \cap H_{\text{fin}}^{g_E(e_2)} \equiv H_{\text{init}}^{g_E(e_1)} \equiv H_{\text{fin}}^{g_E(e_2)},$$

in H, and

$$e_1e_2 = \emptyset \Longleftrightarrow H_{\text{init}}^{g_E(e_1)} \cap H_{\text{fin}}^{g_E(e_2)} \equiv \{0_H\},$$

where "$\equiv$" means "being identically the same in H."

Let $\mathcal{G} = \{a_1, \ldots, a_N\}$ be a finite family of partial isometries on H, and assume that the initial spaces and the final spaces of all elements are infinite dimensional. Then the family

$$\widehat{\mathcal{G}} \overset{\text{def}}{=} \mathcal{G} \cup \mathcal{G}^*$$

generates a C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$, where

$$\mathcal{G}^* \overset{\text{def}}{=} \left\{a^* : a \in \mathcal{G}\right\}.$$

That is,

$$C^*(\mathcal{G}) = \overline{\mathbb{C}\left[\widehat{\mathcal{G}}\right]} \quad \text{in B(H)},$$

where $\overline{X}$ means the topological closure of X in $B(H)$.

A graph G is *connected* if, for any pair (v_1, v_2) of "distinct" vertices, there always exists at least one reduced finite path $w \in \text{FP}_r(\widehat{G})$, such that $w = v_1 w v_2$ and $w^{-1} = v_2 w^{-1} v_1$. If not, the graph G is said to be *disconnected*.

Assume that a graph G is disconnected. Then there exists $t \in \mathbb{N} \setminus \{1\}$, and full-subgraphs $G_1, \ldots, G_t$ of G, such that (i) G_j are connected full-subgraphs of G, (ii) $V(G) = \sqcup_{j=1}^{t} V(G_j)$ and $E(G) = \sqcup_{j=1}^{t} E(G_j)$, (iii) the family $\{G_1, \ldots, G_t\}$ is the "minimal" family satisfying (i) and (ii). The full-subgraphs $G_1, \ldots, G_t$ are called the *connected components of* G.

If G is disconnected, with its connected components $G_1, \ldots, G_t$, for $t \in \mathbb{N} \setminus \{1\}$, then the shadowed graph $\widehat{G}$ of G is a disconnected graph with its connected components $\widehat{G_1}, \ldots, \widehat{G_t}$, where $\widehat{G_j}$ are the shadowed graphs of G_j, for $j = 1, \ldots, t$. Also, the graph groupoid $\mathbb{G}$ of G is partitioned by the graph groupoids $\mathbb{G}_j$ of G_j, for $j = 1, \ldots, t$,

$$\mathbb{G} \overset{\text{Groupoid}}{=} \underset{j=1}{\overset{t}{\sqcup}} \mathbb{G}_j,$$

set-theoretically, and it is groupoid-isomorphic to the direct product

$$\mathbb{G} = \underset{j=1}{\overset{t}{\oplus}} \mathbb{G}_j,$$

algebraically.

Assumption. From now on, all given graphs are "connected" and "finite."

Recall that a graph G is *finite* if $|V(G)| < \infty$, and $|E(G)| < \infty$.

Let G be a connected finite graph, with

$$\left|V\left(\widehat{G}\right)\right| = n, \quad \left|E\left(\widehat{G}\right)\right| = 2N \quad (\text{equivalently, } \left|E(G)\right| = N).$$

We will give an *indexing* on $V(\widehat{G})$ by $\{1, \ldots, n\}$. That is, we will let

$$V\left(\widehat{G}\right) = \left\{v_1, \ldots, v_n\right\} = V(G).$$

By indexing the vertices, we can index the elements of $E(G)$ and $E(\widehat{G})$ (for the fixed indices on $V(\widehat{G})$), as follows:

$$E(G) = \left\{ e_{m:ij} \,\middle|\, \begin{array}{l} m = 1, \ldots, k_{ij}, k_{ij} \neq 0 \\ e_{m:ij} = v_i e_{m:ij} v_j \end{array} \right\}, \quad E(\widehat{G}) = \left\{ x_{m:ij} \,\middle|\, \begin{array}{l} x_{m:ij} = e_{m:ij}, \text{ if } x_{m:ij} \in E(G) \\ x_{m:ij} = e_{m:ji}^{-1}, \text{ if } x_{m:ij} \in E\left(G^{-1}\right) \\ m = 1, \ldots, k_{ij},\ k_{ij} \neq 0 \end{array} \right\},$$

where k_{ij} means the cardinality of edges connecting v_i to v_j, in G (not in $\widehat{G}$). By the finiteness of G,

$$k_{ij} < \infty, \quad \text{whenever } k_{ij} \neq 0.$$

Clearly, "$k_{ij} = 0$" means that "there is no edge connecting v_i to v_j" in G.

For instance, if we have a graph G

$$G = \begin{array}{ccc} v_1 \bullet & \rightrightarrows & \bullet v_2 \\ & \searrow & \\ & & \bullet v_3 \end{array},$$

then

$$k_{12} = 2, \quad k_{13} = 1, \quad k_{23} = k_{32} = k_{21} = k_{31} = 0.$$

The admissibility on $\mathbb{G}$ is in fact independent from the choice of indexings (see [15,18]). It means that if we fix one indexing on $V(\widehat{G})$ (and hence, that on $E(\widehat{G})$), and if we fix other indexing on $V(\widehat{G})$, then the indexed graphs are graph-isomorphic from each other.

Under the above settings, the graph groupoid $\mathbb{G}$ has its *matricial graph representation* $(\mathcal{H}_G, \pi)$, where $\mathcal{H}_G$ is a Hilbert space defined by

$$\mathcal{H}_G \overset{\text{def}}{=} \oplus_{j=1}^{n} \left(\mathbb{C}\xi_{v_j}\right),$$

which is Hilbert-space isomorphic to $\mathbb{C}^{\oplus\, n}$ $(n = |V(G)|)$, and where

$$\pi : \mathbb{G} \longrightarrow B\left(\mathcal{H}_G\right)$$

is a groupoid action satisfying that

$$\pi\left(v_j\right) = P_j, \quad \pi\left(x_{m:ij}\right) = \begin{cases} E_{m:ij} & \text{if } x_{m:ij} = e_{m:ij}, \\ E_{m:ji}^{*} & \text{if } x_{m:ij} = e_{m:ji}^{-1}, \end{cases}$$

where P_j is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero (j, j)-entry 1, and $E_{m:ij}$ is the matrix in $M_n(\mathbb{C})$, having its only nonzero (i, j)-entry ω^m, where ω is the root of unity of the polynomial $z^{k_{ij}}$, whenever $i \neq j$, or it is the diagonal matrix in $M_n(\mathbb{C})$, having its only nonzero (j, j)-entry $e^{i\,\theta_{m:jj}}$, where $\theta_{m:jj} \in \mathbb{R} \setminus \{0\}$, satisfying that $\theta_{m_1:jj} \neq \theta_{m_2:jj}$, whenever $m_1 \neq m_2$ in $\{1, \ldots, k_{jj}\}$, whenever $i = j$.

That is, if v_j is a vertex, then

$$\pi\left(v_j\right) = \overset{j\text{th}}{\begin{pmatrix} 0 & & & & & & 0 \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & & & & 0 \end{pmatrix}} j\text{th},$$

and if $e_{m:ij} \in E(G), k_{ij} \neq 0$, then

$$\pi\left(e_{m:ij}\right) = \overset{i\text{th}}{\begin{pmatrix} 0 & & 0 \\ & & \\ & \omega^m & \\ & & \\ 0 & & 0 \end{pmatrix}} j\text{th},$$

and if $e_{m:jj} \in E(G), k_{jj} \neq 0$, then

$$\pi\left(e_{m:jj}\right) = \overset{j\text{th}}{\begin{pmatrix} 0 & & & & & & \\ & \ddots & & & & & \\ & & 0 & & & & \\ & & & e^{i\theta_{m:jj}} & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}} j\text{th},$$

for all $m \in \{1, \ldots, k_{ij}\}$ and $i, j \in \{1, \ldots, n = |V(\widehat{G})|\}$.

Under this matricial graph representation $(\mathcal{H}_G, \pi)$ of $\mathbb{G}$, the graph groupoid $\mathbb{G}$ generates the *matricial graph C^*-algebra*

$$\mathcal{M}_G \overset{\text{def}}{=} C^*_\pi(\mathbb{G}) \subseteq B\big(\mathcal{H}_G\big) = M_n(\mathbb{C}).$$

Thus the C^*-subalgebra $C^*(\mathcal{G})$ of $B(H)$ is $*$-isomorphic to *the affiliated matricial graph C^*-algebra* $\mathcal{M}_G(H_0) \subset M_n(B(H_0))$, as a C^*-subalgebra of $B(H)$, where

$$\mathcal{M}_G\big(H_0\big) = \big(\mathbb{C} \cdot 1_{H_0}\big) \otimes_{\mathbb{C}} \mathcal{M}_G, \quad M_n\big(B\big(H_0\big)\big) = \left\{ \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \vdots \\ T_{n1} & \cdots & T_{nn} \end{pmatrix} \,\middle|\, T_{ij} \in B\big(H_0\big) \right\},$$

where H_0 is a subspace of H, which is Hilbert-space isomorphic to the initial and the final spaces of all elements of $\mathcal{G}$.

Notice that the subspace H_0 of H always exists, whenever $\mathcal{G}$ is a G-family and G is "connected". Let $a_j \in \mathcal{G}$, with its initial space $H^{a_j}_{\text{init}}$, for $j = 1, \ldots, N = |E(G)|$. By definition, $H^{a_j}_{\text{init}}$ and the final space $H^{a_j}_{\text{fin}}$ are Hilbert-space isomorphic, for all $j = 1, \ldots, N$. By the connectedness of G, for any $k \in \{1, \ldots, N\}$, the subspaces $H^{a_k}_{\text{init}}$ and $H^{a_j}_{\text{init}}$ are Hilbert-space isomorphic, too. Therefore, we can have a subspace H_0 of H.

And the graph groupoid $\mathbb{G}$ has its affiliated matricial graph representation (H_G, π^G), where

$$H_G \overset{\text{def}}{=} H_0 \otimes \mathcal{H}_G \overset{\text{Hilbert}}{=} H_0^{\otimes n} \overset{\text{Subspace}}{\subseteq} H, \quad \pi^G = 1_{H_0} \otimes \pi,$$

where $(\mathcal{H}_G, \pi)$ is the matricial graph representation of $\mathbb{G}$. Under this representation, we construct the groupoid C^*-subalgebra $C^*_{\pi^G}(\mathbb{G})$, generated by $\mathbb{G}$, in $B(H_G) \subseteq B(H)$. Notice that $C^*_{\pi^G}(\mathbb{G})$ is $*$-isomorphic to the affiliated matricial graph C^*-algebra $\mathcal{M}_G(H_0)$ (see [15, 18]).

Notice also that $\mathcal{M}_G(H_0)$ and $\mathcal{M}_G$ are $*$-isomorphic, however, we want to emphasize the affiliation H_0 in $B(H)$. So, we distinguish the notations $\mathcal{M}_G(H_0)$ and $\mathcal{M}_G$. That is, we understand $\mathcal{M}_G(H_0)$ as a C^*-subalgebra of $B(H)$, relatively, we understand $\mathcal{M}_G$ as a C^*-subalgebra of $M_n(\mathbb{C}) = B(\mathcal{H}_G)$.

Theorem 7 (see [15, 18]). *Let $\mathcal{G}$ be a G-family of finite partial isometries on H, and let $\mathbb{G}$ be the graph groupoid of G. Then the C^*-subalgebra $C^*(\mathcal{G})$, generated by $\mathcal{G}$, is $*$-isomorphic to the groupoid C^*-algebra $C^*_{\pi^G}(\mathbb{G})$ in $B(H_G)$, as C^*-subalgebras of $B(H)$. Moreover, the groupoid C^*-algebra $C^*_{\pi^G}(\mathbb{G})$ is $*$-isomorphic to the affiliated matricial graph C^*-algebra $\mathcal{M}_G(H_0)$. That is,*

$$C^*(\mathcal{G}) \overset{*\text{-iso}}{=} C^*_{\pi^G}(\mathbb{G}) \overset{*\text{-iso}}{=} \mathcal{M}_G\big(H_0\big),$$

in $B(H_G) \subseteq B(H)$, where (H_G, π^G) is the H_0-affiliated matricial graph representation of $\mathbb{G}$.

4 Group-framed groupoids

In this section, we consider a new algebraic structure, called *group-framed groupoids*, containing both group property and groupoid property.

4.1 Group-framing on graphs and corresponding groupoids

Throughout this section, let Γ be a topological group, and let G be a countable directed graph with its graph groupoid $\mathbb{G}$. Recall that Γ is a *topological group* if the binary operation on Γ is continuous under a topology for the set Γ.

Understand G as a topological space $V(G) \cup E(G)$, equipped with the discrete topology. Construct the Cartesian product topological space

$$\Gamma \times G = \big\{(g, w) : g \in G,\ w \in G\big\},$$

equipped with the product topology of Γ and G. Here, the notation "$w \in G$" means that

$$w \in V(G) \cup E(G).$$

Definition 8. The topological space $\Gamma \times G$ is called the group-framed graph of G with its group-frame Γ. By G_Γ one denotes this group-framed graph. And the construction of G_Γ is called the group-framing of G with Γ.

Notice that G_Γ is simply a topological space, which is neither a (pure algebraic) group nor a (pure combinatorial) graph. And remark that this topological space is "oriented," by the direction on G. Roughly speaking, the group-framing is understood as the process attaching a line Γ on (every vertex and every edge of) a given graph G.

As in graph theory, we define the partition $\{V(G_\Gamma), E(G_\Gamma)\}$ of G_Γ as follows:

$$V\big(G_\Gamma\big) = \big\{(g, v) : g \in \Gamma,\ v \in V(G)\big\}, \quad E\big(G_\Gamma\big) = \big\{(g, e) : g \in \Gamma,\ e \in E(G)\big\}.$$

The sets $V(G_\Gamma)$ and $E(G_\Gamma)$ are called the *framed-vertex set* and the *framed-edge set*, respectively. Indeed, the oriented topological space G_Γ is partitioned by

$$G_\Gamma = V\big(G_\Gamma\big) \sqcup E\big(G_\Gamma\big).$$

Define now the shadow G_Γ^{-1} of G_Γ by a new oriented topological space,

$$G_\Gamma^{-1} \overset{\text{def}}{=} \Gamma^{-1} \times G^{-1},$$

where G^{-1} is the shadow of G, and $\Gamma^{-1} = \{g^{-1} : g \in \Gamma\}$ (Γ and Γ^{-1} are identically the same). That is,

$$(g, w) \in G_\Gamma \Longleftrightarrow \big(g^{-1}, w^{-1}\big) \in G_\Gamma^{-1},$$

for all $(g, w) \in G_\Gamma$. We call $(g^{-1}, w^{-1}) \in G_\Gamma^{-1}$ the *shadow of* $(g, w) \in G_\Gamma$, and, by $(g, w)^{-1}$, we denote (g^{-1}, w^{-1}). We call this new group-framed graph G_Γ^{-1} the *framed-shadow of* G_Γ. By the very definition, the topological space G_Γ^{-1} is oppositely oriented for G_Γ.

Similarly, we can define the *framed shadowed graph* $\widehat{G_\Gamma}$ *of* G_Γ by a new group-framed graph,

$$\widehat{G_\Gamma} \overset{\text{def}}{=} \Gamma \times \widehat{G},$$

where $\widehat{G}$ is the shadowed graph of G.

Let $\widehat{G_\Gamma}$ be the framed shadowed graph of the group-framed graph G_Γ. Then we can define the "admissibility" on $\widehat{G_\Gamma}$:

$$\big(g_1, e_1\big)\big(g_2, w_2\big) \overset{\text{def}}{=} \big(g_1 g_2, e_1 e_2\big),$$

for all $(g_k, e_k) \in \widehat{G_\Gamma}$, where $g_1 g_2$ is the product in Γ, and $e_1 e_2$ is the admissible product on $\widehat{G}$. This binary operation, acting on $\widehat{G_\Gamma}$, is said to be the *framed-admissibility*.

Definition 9. Let G be a directed graph with its graph groupoid $\mathbb{G}$, and let Γ be a group. Let G_Γ be the Γ-framed graph. Then the groupoid $\mathbb{G}_\Gamma$ generated by $\widehat{G_\Gamma}$ is called the framed(-graph) groupoid.

The empty element $\emptyset_\Gamma$ of $\mathbb{G}_\Gamma$ is defined to be the elements having their forms

$$(g, \emptyset), \quad \forall g \in \Gamma,$$

where $\emptyset$ is the empty word of the graph groupoid $\mathbb{G}$.

This new algebraic structure $\mathbb{G}_\Gamma$ is indeed a well-defined groupoid in the sense of Section 2.3, which is neither a group nor a graph groupoid.

4.2 Product groupoids

In this section, we observe the general case. Let $\mathcal{X}_k = (\mathcal{X}_k, \mathcal{Y}_k, s_k, r_k)$ be groupoids, in the sense of Section 2.3, for $k = 1, 2$. Define a new groupoid $\mathcal{X}$ from $\mathcal{X}_1$ and $\mathcal{X}_2$. Define the set $\mathcal{X}$ by

$$\mathcal{X} \overset{\text{def}}{=} \big\{\big(x_1, x_2\big) : x_1 \in \mathcal{X}_1,\ x_2 \in \mathcal{X}_2\big\},$$

denoted also by $\mathcal{X}_1 \times \mathcal{X}_2$. Define the binary operation on $\mathcal{X}$ by

$$\big(x_1, x_2\big)\big(x_1', x_2'\big) \overset{\text{def}}{=} \big(x_1 x_2, y_1 y_2\big),$$

for all $(x_1, x_2), (x_1', x_2') \in \mathcal{X}$, where the first entry $x_1 x_2$ is the product in $\mathcal{X}_1$, and the second entry $x_1' x_2'$ is the product in $\mathcal{X}_2$. And define the source map s and the range map r on $\mathcal{X}$ by

$$s\big(\big(x_1, x_2\big)\big) \overset{\text{def}}{=} \big(s_1\big(x_1\big), s_2\big(x_2\big)\big), \quad r\big(\big(x_1, x_2\big)\big) \overset{\text{def}}{=} \big(r_1\big(x_1\big), r_2\big(x_2\big)\big),$$

for all $(x_1, x_2) \in \mathcal{X}$. Then it is easy to check that

$$s(\mathcal{X}) \cup r(\mathcal{X}) = \mathcal{Y}_1 \times \mathcal{Y}_2.$$

Denote the set $\mathcal{Y}_1 \times \mathcal{Y}_2$ by $\mathcal{Y}$.

Definition 10. The quadruple $\mathcal{X} = (\mathcal{X}, \mathcal{Y}, s, r)$ obtained in the above paragraph is called the product groupoid of $\mathcal{X}_1$ and $\mathcal{X}_2$.

By our definition of framed groupoids, we can realize that our group-framed groupoids are product groupoids.

Proposition 11. *Let $\varGamma$ be a group and G a directed graph, and let $G_\varGamma$ be the group-framed graph of G with the group-frame $\varGamma$. Then the framed groupoid $\mathbb{G}_\varGamma$ of $G_\varGamma$ is groupoid-isomorphic to the product groupoid $\varGamma \times \mathbb{G}$, where $\mathbb{G}$ is the graph groupoid of G.*

Remark 12. In Section 4.1, the symbol "×" in $\varGamma \times G$ means the topological Cartesian product. In the above proposition, the symbol "×" in $\varGamma \times \mathbb{G}$ means the (topological) algebraic groupoid product in the sense of Definition 10.

Proof. By definition of $\mathbb{G}_\varGamma$, there exists a morphism

$$\varPhi : \mathbb{G}_\varGamma \longrightarrow \varGamma \times \mathbb{G}$$

such that

$$\varPhi\big([g, w]\big) \overset{\text{def}}{=} (g, w),$$

for all $[g, w] \in \mathbb{G}_\varGamma$. (Here, we denote the element (g, w) of $\mathbb{G}_\varGamma$ by $[g, w]$ to distinguish the element (g, w) of $\varGamma \times \mathbb{G}$.) It is a well-defined bijection, and it satisfies that

$$\varPhi\big(\big[g_1, w_1\big]\big[g_2, w_2\big]\big) = \varPhi\big(\big[g_1g_2, w_1w_2\big]\big) = \big(g_1g_2, w_1w_2\big) = \big(g_1, w_1\big)\big(g_2, w_2\big) = \varPhi\big(\big(g_1, w_1\big)\big)\varPhi\big(\big(g_2, w_2\big)\big),$$

for all $[g_1, w_1], [g_2, w_2] \in \mathbb{G}_\varGamma$. Therefore, the bijection $\varPhi$ is a groupoid-homomorphism, and hence it is a groupoid-isomorphism. In particular, the above formula implies that

$$\varPhi\big(s\big([g, w]\big)\big) = \varPhi\big(\big[e_\varGamma, v\big]\big) = \big(e_\varGamma, v\big) = s\big(\varPhi\big([g, w]\big)\big), \quad \varPhi\big(r\big([g, w]\big)\big) = \varPhi\big(\big[e_\varGamma, v'\big]\big) = \big(e_\varGamma, v'\big) = r\big(\varPhi\big([g, w]\big)\big),$$

for all $[g, w] \in \mathbb{G}_\varGamma$. □

Now, consider the isomorphism theorem.

Theorem 13. *Let $\varGamma_k$ be groups and G_k directed graphs with their graph groupoids $\mathbb{G}_k$, and let $G^k_{\varGamma_k}$ be the framed graphs of G_k with group-frames $\varGamma_k$, for $k = 1, 2$. Then the framed groupoids $\mathbb{G}^k_{\varGamma_k}$ of $G^k_{\varGamma_k}$ are groupoid-isomorphic if and only if (i) $\varGamma_1$ and $\varGamma_2$ are group-isomorphic, and (ii) the shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic.*

Proof. (⇐) Assume that $\varGamma_k$ are group-isomorphic, and the shadowed graphs $\widehat{G_k}$ are graph-isomorphic, for $k = 1, 2$. It is well known that if $\widehat{G_k}$ are graph-isomorphic, then the graph groupoids $\mathbb{G}_k$ are groupoid-isomorphic, for $k = 1, 2$. Thus the product groupoids

$$\varGamma_k \times \mathbb{G}_k$$

are groupoid-isomorphic, for $k = 1, 2$. Since the framed groupoids $\mathbb{G}^k_{\varGamma_k}$ are groupoid-isomorphic to the product groupoids $\varGamma_k \times \mathbb{G}_k$, the framed groupoids $\mathbb{G}^k_{\varGamma_k}$ are groupoid-isomorphic, for $k = 1, 2$.

(⇒) Assume that either (i) $'\varGamma_k$ are not group-isomorphic or (ii) the shadowed graphs $\widehat{G_k}$ are not graph-isomorphic. Then, clearly, the product groupoids $\varGamma_k \times \mathbb{G}_k$ are not groupoid-isomorphic, and hence the framed groupoids $\mathbb{G}^k_{\varGamma_k}$ are not groupoid-isomorphic, for $k = 1, 2$. □

Assumption. In the rest of this paper, we use the group-framed groupoids $\mathbb{G}_\varGamma$ and the product groupoids $\varGamma \times \mathbb{G}$, alternatively.

5 E_0-groupoids

Let M be a type I factor contained in $B(H)$, and let $\gamma = (\gamma_t)_{t\in\mathbb{R}}$ be an E_0-group (in the sense of Section 2.4), acting on M. That is, γ on M satisfies that

(1) $\gamma_0 = \mathrm{id}_M$, where id_M is the identity map on M,
(2) γ_t are $*$-endomorphisms on M, for all $t \in \mathbb{R}$,

(3) $\gamma_t(1_M) = 1_M$, for all $t \in \mathbb{R}$,
(4) $\gamma_{t_1} \circ \gamma_{t_2} = \gamma_{t_1+t_2}$, for all $t_1, t_2 \in \mathbb{R}$.

We use this E_0-group as a group-frame on a certain (finite, connected) directed graph G.

Let G be the corresponding graph of a G-family of partial isometries in $B(H)$ with its graph groupoid $\mathbb{G}$ as in Section 3. Then, like in Section 4, we have the corresponding group-framed graph G_γ, and its framed groupoid $\mathbb{G}_\gamma \overset{\text{Groupoid}}{=} \gamma \times \mathbb{G}$. We call $\mathbb{G}_\gamma$ the *E_0-groupoid*.

However, in Section 5.1, we introduce E_0-groupoids in a different way (in a dynamical-system point of view), which is more useful (algebraically, and operator-theoretically). The key idea of the construction of our E_0-groupoids (in the sense of Section 5.1) is that γ and $\mathbb{R}$ are group-isomorphic. That is, there exists a group-isomorphism,

$$\gamma_t \in \gamma \longmapsto t \in \mathbb{R}.$$

In Section 5.2, we establish C^*-dynamical systems induced by the E_0-groupoids.

5.1 E_0-groupoids

Let $\mathcal{G} = \{a_1, \ldots, a_N\}$ be a finite family of partial isometries on H, and assume that $\mathcal{G}$ construct a connected finite directed graph G, equivalently, the family $\mathcal{G}$ is a G-family in $B(H)$. Then, the groupoid $\mathbb{G}_{\mathcal{G}}$, generated by $\mathcal{G}$, is groupoid-isomorphic to the graph groupoid $\mathbb{G}$ of G.

Define now the topological space $\mathbb{G}_{\mathbb{R}}$ by the group-framed groupoid of the framed graph $G_{\mathbb{R}}$ of the $\mathcal{G}$-graph G, with the group-frame $\mathbb{R} = (\mathbb{R}, +)$. Then, by Section 4.2, this $\mathbb{R}$-framed groupoid $\mathbb{G}_{\mathbb{R}}$ is groupoid-isomorphic to the product groupoid $\mathbb{R} \times \mathbb{G}$, under the binary operation defined by

$$\big(t_1, w_1\big) \cdot \big(t_2, w_2\big) \overset{\text{def}}{=} \big(t_1 + t_2, w_1 w_2\big),$$

for $(t_k, w_k) \in \mathbb{R} \times \mathbb{G}$, for $k = 1, 2$. Notice that since $\mathbb{G}$ is generated by $\mathcal{G}$, we can understand $w \in \mathbb{G}$ as an operator on H, too.

Definition 14. Let $\mathbb{G}_{\mathbb{R}}$ be the group-framed groupoid induced by the $\mathbb{R}$-framed graph $G_{\mathbb{R}}$, where G is the $\mathcal{G}$-graph, where $\mathcal{G}$ is the finite family of partial isometries in $B(H)$. One calls $\mathbb{G}_{\mathbb{R}}$ the flowed groupoid of $\mathbb{G}$ with the flow $\mathbb{R}$. The binary operation $(\cdot)$ on $\mathbb{G}_{\mathbb{R}}$ is called the flowed admissibility.

Now, fix the groupoid action π^G of $\mathbb{G}$, acting on $H_G = H_0 \otimes \mathcal{H}_G$, in the sense of Section 3. Recall that H_0 is the subspace of H, which is Hilbert-space isomorphic to the initial and final spaces of all elements of $\mathcal{G}$, and $\mathcal{H}_G$ is a subspace of H, which is Hilbert-space isomorphic to $\mathbb{C}^{\oplus |V(G)|}$, and hence the tensor product Hilbert space H_G is a subspace of H.

Depending on π^G and the given E_0-group γ, we define an action γ_G of $\mathbb{G}_{\mathbb{R}}$, acting on M, by

$$\gamma_G : (t, w) \in \mathbb{G}_{\mathbb{R}} \longmapsto \gamma_G\big((t, w)\big) \overset{\text{def}}{=} \gamma_{t,w},$$

where

$$\gamma_{t,w}(m) \overset{\text{def}}{=} \pi_w^G \gamma_t(m) \pi_w^{G\,*}, \quad \forall m \in M.$$

Recall that $\pi_w^{G\,*} = \pi_{w^{-1}}^G$, for all $w \in \mathbb{G}$. So, without loss of generality, we can regard the action γ_G of $\mathbb{G}_{\mathbb{R}}$ as a family $(\gamma_{t,w})_{w \in \mathbb{G}}^{t \in \mathbb{R}}$ of actions. Notice that each element $\gamma_{t,w}$ of γ_G is a $*$-endomorphism on M, in $B(H)$, for all $(t, w) \in \mathbb{G}_{\mathbb{R}}$.

The following proposition shows, for any $(t, w) \in \mathbb{G}_{\mathbb{R}}$,

$$\gamma_{t,w}(m) = \gamma_t(m) \pi_{ww^{-1}}^G, \quad \forall m \in M.$$

This means that the action γ_G of $\mathbb{G}_{\mathbb{R}}$ sends an element $m \in M$ to the element $\gamma_t(m)$ located at the corner of $B(H)$ determined by a projection $\pi_{ww^{-1}}^G$.

Proposition 15. *Let $\gamma_G = (\gamma_{t,w})_{w \in \mathbb{G}}^{t \in \mathbb{R}}$ be an action of $\mathbb{G}_{\mathbb{R}}$, acting on M, given as above. Then, for any $(t, w) \in \mathbb{G}_{\mathbb{R}}$, one has*

$$\gamma_{t,w}(m) = \gamma_t(m) \pi_{ww^{-1}}^G, \quad \forall m \in M,$$

where $\gamma = (\gamma_t)_{t \in \mathbb{R}}$ is the fixed E_0-group, acting on M.

Proof. Let $(t, w) \in \mathbb{G}_{\mathbb{R}}$. Then, for any $m \in M$,

$$\gamma_{t,w}(m) = \pi_w^G \gamma_t(m) \pi_{w^{-1}}^G = \gamma_t(m) \pi_w^G \pi_{w^{-1}}^G$$

since π_w^G (or $\pi_{w^{-1}}^G$) is a matrix, having only nonzero (i, j)-entry (resp. (j, i)-entry), having its form $\alpha \cdot 1_{H_0}$, for some $\alpha \in \mathbb{C}$, whenever $w = v_i w v_j$, in the affiliated matricial graph C^*-algebra $\mathcal{M}_G(H_0)$, which is $*$-isomorphic to the matricial graph C^*-algebra $\mathcal{M}_G$

$$= \gamma_t(m) \pi_{ww^{-1}}^G. \qquad \square$$

The above action γ_G satisfies the followings:

(5.1.1) $\gamma_{t,w}(1_M) = \pi_{ww^{-1}}^G$, a projection on H_G, for all $(t, w) \in \mathbb{G}_{\mathbb{R}}$. In particular, $\gamma_{t,\emptyset} = 0_H$, for all $t \in \mathbb{R}$. Indeed, we can compute

$$\gamma_{t,w}\big(1_M\big) = \gamma_t\big(1_M\big) \pi_{ww^{-1}}^G = 1_M \cdot \pi_{ww^{-1}}^G = \pi_{ww^{-1}}^G,$$

by the previous proposition. Also, if $w = \emptyset$ in $\mathbb{G}$, then

$$\gamma_{t,\emptyset}(m) = \gamma_t(m) \pi_\emptyset^G = 0_H, \quad \forall m \in M,$$

since $\pi_\emptyset^G = 0_H$.

(5.1.2) $\gamma_{0,w}(m) = m \pi_{ww^{-1}}^G$, for all $m \in M$, by the previous proposition;

$$\gamma_{0,w}(m) = \gamma_0(m) \pi_{ww^{-1}}^G = m \pi_{ww^{-1}}^G,$$

since $\gamma_0 = \mathrm{id}_M$.

(5.1.3) $\gamma_{t,w}(m^*) = (\gamma_{t,w^{-1}}(m))^*$, for all $m \in M$, $(t, w) \in \mathbb{G}_{\mathbb{R}}$.

Fix $(t, w) \in \mathbb{G}_{\mathbb{R}}$. Then, for any $m \in M$,

$$\gamma_{t,w}\big(m^*\big) = \pi_w^G \gamma_t\big(m^*\big) \pi_{w^{-1}}^G = \pi_w^G \gamma_t(m)^* \pi_{w^{-1}}^G = \big(\pi_{w^{-1}}^G \gamma_t(m) \pi_w^G\big)^* = \big(\gamma_{t,w^{-1}}(m)\big)^*.$$

The formula (5.1.3) guarantees that

$$\big(\gamma_{t,w}(m)\big)^* = \gamma_{t,w^{-1}}\big(m^*\big), \quad \text{for } m \in M.$$

(5.1.4) For $(t_1, w_1), (t_2, w_2) \in \mathbb{G}_{\mathbb{R}}$,

$$\gamma_{t_1,w_1} \circ \gamma_{t_2,\, w_2} = \gamma_{t_1+t_2, w_1 w_2}, \quad \text{on } M.$$

Indeed, we obtain

$$\begin{aligned} \gamma_{t_1,w_1}\big(\gamma_{t_2,w_2}(m)\big) &= \gamma_{t_1,w_1}\Big(\pi_{w_2}^G \gamma_t(m) \pi_{w_2^{-1}}^G\Big) = \pi_{w_1}^G \Big(\gamma_{t_1}\Big(\pi_{w_2}^G \gamma_{t_2}(m) \pi_{w_2^{-1}}^G\Big)\Big) \pi_{w_1^{-1}}^G \\ &= \pi_{w_1}^G \Big(\gamma_{t_1}\Big(\gamma_{t_2}(m) \pi_{w_2 w_2^{-1}}^G\Big)\Big) \pi_{w_1^{-1}}^G \end{aligned}$$

by the previous proposition

$$= \pi_{w_1}^G \Big(\gamma_{t_1}\big(\gamma_{t_2}(m)\big) \pi_{w_2 w_2^{-1}}^G\Big) \pi_{w_1^{-1}}^G$$

since $\pi_{w_2 w_2^{-1}}^G$ is a projection, having its form of a matrix with only one nonzero diagonal entry 1_{H_0}

$$= \pi_{w_1}^G \Big(\gamma_{t_1+t_2}(m) \pi_{w_2 w_2^{-1}}^G\Big) \pi_{w_1^{-1}}^G$$

since

$$\begin{aligned} \gamma_{t_1} \circ \gamma_{t_2} = \gamma_{t_1+t_2} &= \pi_{w_1}^G \Big(\pi_{w_2}^G \big(\gamma_{t_1+t_2}(m)\big) \pi_{w_2^{-1}}^G\Big) \pi_{w_1^{-1}}^G = \pi_{w_1 w_2}^G \big(\gamma_{t_1+t_2}(m)\big) \pi_{w_2^{-1} w_1^{-1}}^G \\ &= \pi_{w_1 w_2}^G \big(\gamma_{t_1+t_2}(m)\big) \pi_{(w_1 w_2)^{-1}}^G = \gamma_{t_1+t_2,\, w_1 w_2}(m), \end{aligned}$$

for all $m \in M$. By the previous observations (5.1.1) through (5.1.4), we can conclude that the collection $\gamma_G = (\gamma_{t,w})_{w \in \mathbb{G}}^{t \in \mathbb{R}}$ is indeed an action of the flowed groupoid $\mathbb{G}_{\mathbb{R}}$, acting on M, in $B(H)$.

Lemma 16. *The collection $\gamma_G = (\gamma_{t,w})_{w\in\mathbb{G}}^{t\in\mathbb{R}}$ is a groupoid action of the flowed groupoid $\mathbb{G}_{\mathbb{R}}$ of G, acting on M, in $B(H)$.*

Moreover, by the previous lemma (in particular, by (5.1.4)), we obtain the following theorem, characterizing the algebraic structure of γ_G.

Theorem 17. *Let $\mathbb{G}_{\mathbb{R}}$ be the flowed groupoid, and let $\gamma_G = (\gamma_{t,w})_{w\in\mathbb{G}}^{t\in\mathbb{R}}$ be the groupoid action of $\mathbb{G}_{\mathbb{R}}$, acting on M, in $B(H)$. Then the action γ_G is a groupoid (under the usual composition), which is groupoid-isomorphic to $\mathbb{G}_{\mathbb{R}}$.*

Proof. Define the morphism

$$\Phi : \big(\gamma_G, \circ\big) \longrightarrow \mathbb{G}_{\mathbb{R}}$$

by

$$\gamma_{t,w} \in \gamma_G \overset{\Phi}{\longmapsto} (t, w) \in \mathbb{G}_{\mathbb{R}}.$$

Then, it is a bijection. And it satisfies that

$$\Phi\big(\gamma_{t_1,w_1} \circ \gamma_{t_2,w_2}\big) = \Phi\big(\gamma_{t_1+t_2,\, w_1w_2}\big) = \big(t_1 + t_2, w_1w_2\big) = \big(t_1, w_1\big)\big(t_2, w_2\big) = \Phi\big(\gamma_{t_1,w_1}\big)\Phi\big(\gamma_{t_2,w_2}\big),$$

for all $\gamma_{t_k,w_k} \in \gamma_G$, for $k = 1, 2$. Therefore, the bijective morphism Φ is a groupoid-isomorphism, and hence $(\gamma_G, \circ)$ is actually groupoid-isomorphic to the flowed groupoid $\mathbb{G}_{\mathbb{R}}$. □

By the previous theorem, we can alternatively use γ_G and $\mathbb{G}_{\mathbb{R}}$.

Definition 18. Let $\gamma_G = (\gamma_{t,w})_{w\in\mathbb{G}}^{t\in\mathbb{R}}$ be given as above. Then one calls γ_G the E_0-groupoid induced by $\mathcal{G}$ and γ, where $\mathcal{G}$ is the given G-family of finite partial isometries on H, and γ is the fixed E_0-group.

Let $\gamma_{G_1} = (\gamma_{t,w})_{w\in\mathbb{G}_1}^{t\in\mathbb{R}}$ and $\alpha_{G_2} = (\alpha_{s,y})_{y\in\mathbb{G}_2}^{s\in\mathbb{R}}$ be E_0-groupoids, acting on the type I factor M, in $B(H)$, where α and γ are E_0-groups. We say that they are *equivalent*, if there exists a $*$-automorphism $\varphi : M \to M$, and a groupoid-isomorphism $\Phi : \mathbb{G}_1 \to \mathbb{G}_2$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\gamma_{t,w}} & M \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ M & \xrightarrow[\alpha_{t,\Phi(w)}]{} & M, \end{array}$$

for all $t \in \mathbb{R}$. The following theorem provides the characterization of equivalence of E_0-groupoids.

Theorem 19. *Let $\mathcal{G}_k$ be G_k-families of partial isometries, constructing finite connected directed graphs G_k, for $k = 1, 2$. Assume that $\gamma = (\gamma_t)_{t\in\mathbb{R}}$ and $\alpha = (\alpha_t)_{t\in\mathbb{R}}$ are E_0-groups, and suppose $\gamma_{G_1} = (\gamma_{t,w})_{w\in\mathbb{G}_1}^{t\in\mathbb{R}}$ and $\alpha_{G_2} = (\alpha_{t,w})_{w\in\mathbb{G}_2}^{t\in\mathbb{R}}$ are the E_0-groupoid induced by $\mathcal{G}_k$ and γ_k, for $k = 1, 2$, where $\mathbb{G}_k$ are the groupoid generated by $\mathcal{G}_k$, which are the graph groupoids of G_k, for $k = 1, 2$. Then the conditions (i) and (ii) hold if and only if the E_0-groupoids γ_{G_1} and α_{G_2} are equivalent, where*

(i) *the E_0-groups γ and α are equivalent,*
(ii) *the shadowed graphs $\widehat{G_k}$ of G_k are graph-isomorphic, for $k = 1, 2$.*

Proof. ($\Rightarrow$) Suppose α and γ are equivalent. Then there exists a $*$-automorphism $\varphi : M \to M$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\gamma_{t}} & M \\ \varphi\big\downarrow & & \big\downarrow\varphi \\ M & \xrightarrow[\alpha_{t}]{} & M, \end{array}$$

for all $t \in \mathbb{R}$. Also, assume that two shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic. Then, by Section 2.3, the graph groupoids $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic. This shows that the flowed graph groupoids $\mathbb{G}_{1:\mathbb{R}}$ and $\mathbb{G}_{2:\mathbb{R}}$ are groupoid-isomorphic, too. So, by definition, the E_0-groupoids γ_{G_1} and α_{G_2} are equivalent, via the $*$-automorphism φ.

($\Leftarrow$) Assume now that the E_0-groupoids γ_{G_1} and α_{G_2} are equivalent. This shows that there exists a $*$-automorphism $\varphi : M \to M$, such that the following diagram commutes:

$$\begin{array}{ccc} M & \xrightarrow{\gamma_{t,w}} & M \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ M & \xrightarrow[\alpha_{t,\Phi(w)}]{} & M, \end{array}$$

where $\Phi : \mathbb{G}_1 \to \mathbb{G}_2$ is a groupoid-isomorphism. Then, by the existence of the groupoid-isomorphism Φ, the shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$ are graph-isomorphic: define a graph-isomorphism $g : \widehat{G_1} \to \widehat{G_2}$, by $g = \Phi|_{V(\widehat{G_1}) \cup E(\widehat{G_1})}$. Also, by the previous commuting diagram, we have

$$\varphi\big(\gamma_{t,w}(m)\big) = \varphi\Big(\gamma_t(m)\pi^{G_1}_{ww^{-1}}\Big) = \varphi\big(\gamma_t(m)\big)\pi^{G_1}_{ww^{-1}}$$

since $\pi^{G_1}_{ww^{-1}}$ is a projection, having its form of a matrix, having only nonzero (i, j)-entry $s \cdot 1_{H_0}$, for $s \in \mathbb{C}$.

$$= \alpha_t\big(\varphi(m)\big)\pi^{G_2}_{\Phi(w)\Phi(w)^{-1}} = \alpha_{t,\Phi(w)}\big(\varphi(m)\big).$$

Since $\mathbb{G}_1$ and $\mathbb{G}_2$ are groupoid-isomorphic from each other, via Φ, the projections $\pi^{G_1}_{w_1 w_1^{-1}}$ and $\pi^{G_2}_{\Phi(w)\Phi(w)^{-1}}$ are identical in the matricial graph C^*-algebras $\mathcal{M}_{G_1}$ and $\mathcal{M}_{G_2}$. Therefore, the above formula shows that

$$\varphi\big(\gamma_t(m)\big) = \alpha_t\big(\varphi(m)\big), \quad \forall m \in M.$$

Thus the E_0-groups γ and α are equivalent. □

The above theorem shows that the equivalence on γ_G is determined by both the graph-isomorphisms on $\widehat{G}$ and the equivalence on γ.

5.2 C^*-dynamical systems

Now, let M be a type I factor in $B(H)$, and let $\gamma_G = (\gamma_{t,w})^{t\in\mathbb{R}}_{w\in\mathbb{G}}$ be the E_0-groupoid, which is groupoid-isomorphic to the flowed groupoid $\mathbb{G}_{\mathbb{R}}$ of $G_{\mathbb{R}}$, where G is the $\mathcal{G}$-graph of a finite graph-family $\mathcal{G}$ of partial isometries on H. Then we naturally define the groupoid C^*-dynamical system

$$\big(M, \mathbb{G}_{\mathbb{R}}, \gamma_G\big).$$

Also, such a dynamical system generates a C^*-algebra

$$M \times_{\gamma_G} \mathbb{G}_{\mathbb{R}},$$

which is the groupoid crossed product C^*-algebra. That is, $M \times_{\gamma_G} \mathbb{G}_{\mathbb{R}}$ is the C^*-algebra generated by M and $\mathbb{G}_{\mathbb{R}}$ satisfying γ_G-action.

If $\mathbb{G}_{\mathbb{R}}$ acts on M inside $B(H)$, we may say that the C^*-dynamical system $(M, \mathbb{G}_{\mathbb{R}}, \gamma_G)$ is *inner* (in $B(H)$). And if $\mathbb{G}_{\mathbb{R}}$ acts on M in $B(\mathcal{K})$, for some Hilbert space $\mathcal{K} \supset H$, then we may say that $(M, \mathbb{G}_{\mathbb{R}}, \gamma_G)$ is *outer* (in $B(H)$). In fact, there is no big difference between "innerness" and "outerness," under the C^*-setting. But we want to emphasize the originally given structure (or paradigm) $B(H)$, where a fixed history (M, γ) is embedded.

6 Distorted histories

Let $B(H)$ be given as before, and let (M, γ) be a history in $B(H)$, where M is a type I (sub)factor in $B(H)$, and $\gamma = (\gamma_t)_{t\in\mathbb{R}}$ is an E_0-group, determined by a one-parameter unitary group $U = (U_t)_{t\in\mathbb{R}}$, satisfying the followings:

(i) fixed past: $\gamma_t(M) \subseteq M$, for all $t < 0$,
(ii) irreducibility: $(\cup_{t\in\mathbb{R}}\gamma_t(M))'' = B(H)$,
(iii) trivial infinitely remote past: $\cap_{t\in\mathbb{R}}\gamma_t(M) = \mathbb{C} \cdot 1_H$.

We may/can understand that the history (M, γ) is a group C^*-dynamical system $(M, \mathbb{R}, \gamma)$, induced by the group $\mathbb{R}$ acting on the "results or events" M, via the group-action γ, which means "what happened, is happening, and will happen."

6.1 Inner distorted histories

Let (M, γ) be a given history in $B(H)$. In this section, we are interested in the distortions occurred by finite families $\mathcal{G}$ of partial isometries "in" $B(H)$.

Let $\mathcal{G} = \{a_1, \ldots, a_N\}$ be a finite family of partial isometries on H, and assume that $\mathcal{G}$ constructs a connected finite directed graph G. Then the family $\mathcal{G}$ generates a corresponding subgroupoid of $B(H)$, which is groupoid-isomorphic to the graph groupoid $\mathbb{G}$ of G. Then, for this graph groupoid $\mathbb{G}$, we obtain the flowed groupoid $\mathbb{G}_{\mathbb{R}}$, which is groupoid-isomorphic to the product groupoid $\mathbb{R} \times \mathbb{G}$. And, as an action of $\mathbb{G}_{\mathbb{R}}$, acting on M, we can construct the E_0-groupoid $\gamma_G = (\gamma_{t,w})_{w \in \mathbb{G}}^{t \in \mathbb{R}}$, induced by $\mathcal{G}$ and γ. This E_0-groupoid γ_G can explain how to distort a fixed history (M, γ) by $\mathcal{G}$.

Definition 20. Let $\mathcal{G} \subset B(H)$ be a G-family of partial isometries. The pair (M, γ_G) is called the inner distorted history of (M, γ) distorted by $\mathcal{G}$.

Similar to histories, we can understand the inner distorted history (M, γ_G) as a C^*-dynamical system $(M, \mathbb{G}_{\mathbb{R}}, \gamma_G)$. We can get the following fundamental properties of an inner distorted history (M, γ_G) of the fixed history (M, γ).

Theorem 21. *Let (M, γ_G) be an inner distorted history of a history (M, γ), distorted by a finite family $\mathcal{G}$ of partial isometries on H Then.*

(1) $\gamma_{t,w}(M) = M\pi_{ww^{-1}}^G = \pi_{ww^{-1}}^G M \pi_{ww^{-1}}^G$, *for all $t < 0$, and $w \in \mathbb{G}$,*
(2) $(\cup_{t \in \mathbb{R},\, w \in \mathbb{G}} \gamma_{t,w}(M))'' = B(H_G)$, *where H_G is given in Section 3,*
(3) $\cap_{t \in \mathbb{R},\, w \in \mathbb{G}} \gamma_{t,w}(M) = \{0_H\}$.

Proof. (1) Recall that $\gamma_{t,w}(m) = \gamma_t(m)\pi_{ww^{-1}}^G$, for all $m \in M$, for $(t, w) \in \mathbb{G}_{\mathbb{R}}$. And, by definition, $\gamma_t(M) \subseteq M$, for all $t < 0$. More precisely, since γ_t are $*$-automorphisms, for all $t \in \mathbb{R}$, we have $\gamma_t(M) = M$, for all $t < 0$. Therefore,

$$\gamma_{t,w}(M) = \gamma_t(M)\pi_{ww^{-1}}^G \overset{*\text{-iso}}{=} M\pi_{ww^{-1}}^G,$$

for all $w \in \mathbb{G}$, whenever $t < 0$. So, the image $\gamma_{t,w}(M)$ of $\gamma_{t,w}$ is the compressed W^*-subalgebra of $B(H)$, which is $*$-isomorphic to M, cornered by the projection $\pi_{ww^{-1}}^G$, for $t < 0$, and $w \in \mathbb{G}$. That is,

$$\gamma_{t,w}(M) \overset{*\text{-iso}}{=} M\pi_{ww^{-1}}^G = \pi_{ww^{-1}}^G M \pi_{ww^{-1}}^G.$$

(2) By (1), we can have that

$$\left(\underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cup} \gamma_{t,w}(M)\right)'' = \left(\underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cup} \gamma_t(M)\pi_{ww^{-1}}^G\right)'' = \left(\underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cup} \pi_w^G \gamma_t(M)\pi_{w^{-1}}^G\right)'' \subseteq B\big(H_G\big),$$

since $\gamma_{t,w}(M)$ is the compressed W^*-subalgebra in $B(H_G) \subseteq B(H)$. Moreover,

$$\left(\underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cup} \gamma_{t,w}(M)\right)'' = \left(\underset{w \in \mathbb{G}}{\cup} \left(\pi_w^G \left(\underset{t \in \mathbb{R}}{\cup} \gamma_t(M)\right) \pi_{w^{-1}}^G\right)\right)'' = \left(\underset{w \in \mathbb{G}}{\cup} \pi_w^G B(H) \pi_{w^{-1}}^G\right)'' = \left(\underset{w \in \mathbb{G}}{\cup} B(H)\pi_{ww^{-1}}^G\right)''$$

by the irreducibility of the history $(M, \gamma) = B(H_G)$.

(3) It is clear, by the trivial infinitely remote past property of (M, γ), that

$$\underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cap} (\gamma_{t,w}(M)) = \underset{(t,w) \in \mathbb{G}_{\mathbb{R}}}{\cap} \left(\gamma_t(M)\pi_{ww^{-1}}^G\right) = \underset{w \in \mathbb{G}}{\cap} \left(\left(\underset{t \in \mathbb{R}}{\cap} \gamma_t(M)\right) \pi_{ww^{-1}}^G\right) = \underset{w \in \mathbb{G}}{\cap} \left(\mathbb{C} \cdot \pi_{ww^{-1}}^G\right) = \{0_H\},$$

since $(\mathbb{C} \cdot \pi_{v_1}^G) \cap (\mathbb{C} \cdot \pi_{v_2}^G) = \{0_H\}$ whenever $v_1 \neq v_2$ in $V(\widehat{G})$. □

The first property (1) of the above theorem means that the past M is understood only partially by $\mathcal{G}$. The second property (2) means that the history (M, γ) is compressed, and the compressed part of the history lies in the restricted paradigm $B(H_G)$, where $\mathcal{G}$ works. The third property (3) means that the distortion of (M, γ) has no common infinitely remote past.

Let (M, γ) be a fixed history in $B(H)$, and let $\mathcal{G}_k$ be finite families of partial isometries on H, constructing the corresponding connected finite directed graphs G_k, for $k = 1, 2$. We show that if G_1 and G_2 have the graph-isomorphic shadowed graphs $\widehat{G_1}$ and $\widehat{G_2}$, then the families $\mathcal{G}_1$ and $\mathcal{G}_2$ give the same distortion on (M, γ).

Theorem 22. *Let $\mathcal{G}_k$ be G_k-families of partial isometries on H, where G_k are connected finite directed graphs, for $k = 1, 2$, and let (M, γ) be a given history in $B(H)$. The graphs G_k have graph-isomorphic shadowed graphs $\widehat{G_k}$ if and only if the inner distorted histories (M, γ_{G_k}) of (M, γ) (in $B(H)$) are equivalent, for $k = 1, 2$.*

The proof of the above theorem is the direct consequence of the equivalence of E_0-groupoids obtained in Section 5.

So, we have the equivalence of inner distortions.

Definition 23. One says that two inner distorted histories (M, γ_{G_1}) and (M, γ_{G_2}) of a fixed history (M, γ) are equivalent if the E_0-groupoids γ_{G_1} and γ_{G_2} are equivalent in the sense of Section 5.

6.2 Outer distorted histories

As before, we let (M, γ) be a history in $B(H)$. Suppose $\mathcal{G}$ is a finite family of partial isometries on a Hilbert space K, constructing a connected finite directed graph G. (Remark that the Hilbert spaces H and K are not necessarily distinct. However, for convenience, we may assume that they are distinct.) Then the groupoid generated by $\mathcal{G}$ is a subgroupoid in $B(K)$, and it is groupoid-isomorphic to the graph groupoid $\mathbb{G}$ of the $\mathcal{G}$-graph G.

Construct a bigger Hilbert space $\mathcal{K}$, containing both H and K as its Hilbert subspaces. For instance, we may determine $\mathcal{K}$ by the Cartesian-product Hilbert space $H \times K$ or the tensor product Hilbert space $H \otimes K$, or the direct-product Hilbert space $H \oplus K$, and so on.

For our purpose, we choose $\mathcal{K}$ as $H \otimes K$,

$$\mathcal{K} \overset{\text{def}}{=} H \otimes K.$$

The operator algebra $B(\mathcal{K})$ is $*$-isomorphic to $B(H) \otimes_{\mathbb{C}} B(K)$. Then the $\mathcal{G}$-groupoid $\mathbb{G}$ is embedded in $B(K) \subset B(\mathcal{K})$.

Now, let $\gamma_G = (\gamma_{t,w})_{w \in \mathbb{G}}^{t \in \mathbb{R}}$ be the E_0-groupoid induced by $\mathcal{G}$ and γ. For any $(t, w) \in \mathbb{G}_{\mathbb{R}}$, γ_G satisfies that

$$\gamma_{t,w}(m) = \gamma_t(m)\pi_{ww^{-1}}^{G}, \quad \forall m \in M\big(\subset B(\mathcal{K})\big)$$

with

$$\pi_{ww^{-1}}^{G} \in B\big(K_G\big) \subseteq B(K)\big(\subset B(\mathcal{K})\big),$$

where

$$K_G = K_0 \otimes \left(\underset{v \in V(G)}{\oplus} \big(\mathbb{C} \cdot \xi_v\big) \right) \overset{\text{Hilbert}}{=} K_0^{\oplus |V(G)|} \hookrightarrow K.$$

Here, K_0 is the subspace of K, which is Hilbert-space isomorphic to the initial and the final spaces of all elements of $\mathcal{G}$ in $B(K) \subset B(\mathcal{K})$. So, under this setting, the pair (M, γ_G) is an inner distorted history of (M, γ) in the new (extended) paradigm $B(\mathcal{K})$. However, it is not an inner distorted history of (M, γ) in the originally given paradigm $B(H)$.

Definition 24. Let (M, γ_G) be given as above in $B(\mathcal{K})$. Then it is called an outer distorted history of (M, γ) by $\mathcal{G}$.

We may understand an outer distorted history (M, γ_G) as the history distorted by the activities γ_G of $\mathcal{G}$ $(\subset B(K))$, outside the paradigm $B(H)$.

An outer distorted history (M, γ_G) of the given history (M, γ) can be understood as a C^*-dynamical system $(M, \mathbb{G}_{\mathbb{R}}, \gamma_G)$, induced by the $\mathbb{R}$-framed groupoid $\mathbb{G}_{\mathbb{R}}$.

Basically, outer distorted histories have the same properties with inner distorted histories like in Section 5.1. The only difference between inner distorted histories and outer distorted histories is where they are working. That is, an inner distortion happens in $B(H)$, and an outer distortion happens in $B(\mathcal{K})$, containing $B(H)$.

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