

Hypersurfaces of paraquaternionic space forms ¹

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Abstract

In this paper we obtain some properties for real lightlike hypersurfaces of paraquaternionic space forms and give an example.

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1 Introduction

It is well-known that the lightlike hypersurfaces are of interest in mathematical physics, owing to their extensive use in general relativity and electromagnetism. The general theory of lightlike submanifolds has been developed by Kupelli [4] and Bejancu-Duggal [1]. In [3], the present authors and R. Mazzocco have begun the study of the real lightlike hypersurfaces of paraquaternionic manifolds. In this paper we obtain new properties for this kind of hypersurfaces and give some obstructions to the existence of lightlike hypersurfaces in paraquaternionic space forms.

2 Lightlike hypersurfaces

Let $(\overline{M}, \overline{g})$ be a $(m+2)$ -dimensional semi-Riemannian manifold with index $q \in \{1, 2, \dots, m+1\}$ and let (M, g) be a hypersurface of \overline{M} , with $g = \overline{g}|_M$. We say [1] that M is a lightlike hypersurface of \overline{M} if g is of constant rank m .

If we consider the vector bundle TM^\perp whose fibres are:

$$T_p M^\perp = \{Y_p \in T_p \overline{M} \mid \overline{g}_p(X_p, Y_p) = 0, \quad \forall X_p \in T_p M\}, \quad \forall p \in M$$

then we can easily see that M is lightlike if and only if TM^\perp is a distribution of rank 1 on M .

Let $S(TM)$ be the complementary distribution of TM^\perp in TM , which is called a screen distribution. This distribution is non-degenerate [1]. Thus we have direct orthogonal sum decomposition $TM = S(TM) \perp TM^\perp$. On the other hand, if $S(TM)^\perp$ is the orthogonal complementary vector bundle to $S(TM)$ in $T\overline{M}|_M$, we have direct orthogonal sum $T\overline{M}|_M = S(TM) \perp S(TM)^\perp$.

From [1] we know that, if $(M, g, S(TM))$ is a lightlike hypersurface of \overline{M} , then there exists a unique vector bundle $ltr(TM)$ of rank 1 over M so that for any non-zero section ξ of TM^\perp on a coordinate neighborhood $U \subset M$, there exists a unique section N of $ltr(TM)$ on U satisfying:

$$\overline{g}(N, \xi) = 1, \quad \overline{g}(N, N) = \overline{g}(W, W) = 0, \quad \forall W \in \Gamma(S(TM)|_U)$$

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The vector bundle $ltr(TM)$ is called the null transversal vector bundle of M with respect to $S(TM)$. Thus, we have

$$T\overline{M}|_M = S(TM) \perp (TM^\perp \oplus ltr(TM)) = TM \oplus ltr(TM)$$

Let $(M, g, S(TM))$ be a lightlike hypersurface of \overline{M} . If $\overline{\nabla}$ is the Levi-Civita connection on \overline{M} , then we have

$$\overline{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \forall X, Y \in \Gamma(TM) \quad (2.1)$$

$$\overline{\nabla}_X V = -A_V X + \nabla_X^\perp V, \quad \forall X \in \Gamma(TM), \quad \forall V \in \Gamma(ltr(TM)) \quad (2.2)$$

where $\nabla_X Y, A_V X \in \Gamma(TM)$ and $h(X, Y), \nabla_X^\perp V \in \Gamma(ltr(TM))$.

∇ is a symmetric linear connection on M called an induced linear connection, ∇^\perp is a linear connection on the null transversal bundle $ltr(TM)$, h is a $\Gamma(ltr(TM))$ -valued symmetric bilinear form and A_V is the shape operator of M with respect to V . If $\{\xi, N\}$ is a pair of sections on $U \subset M$ as above, then we define a symmetric $F(U)$ -bilinear form B and a 1-form τ on U by

$$B(X, Y) = \overline{g}(h(X, Y), \xi), \quad \tau(X) = \overline{g}(\nabla_X^\perp N, \xi), \quad \forall X, Y \in \Gamma(TM|_U)$$

Thus, from (2.1) and (2.2), locally we have

$$\overline{\nabla}_X Y = \nabla_X Y + B(X, Y)N \quad (2.3)$$

$$\overline{\nabla}_X N = -A_N X + \tau(X)N \quad (2.4)$$

If we denote by P the projection of TM on $S(TM)$, we have the decompositions:

$$\nabla_X PY = \nabla_X^* PY + C(X, PY)\xi \quad (2.5)$$

$$\nabla_X \xi = -A_\xi^* X + \epsilon(X)\xi \quad (2.6)$$

where $\nabla_X^* PY, A_\xi^* X \in S(TM)$ and ϵ is a 1-form on U . Moreover, it is easy to see that $\epsilon = -\tau$.

3 Real lightlike hypersurfaces of paraquaternionic manifolds

Let \overline{M} be a smooth manifold. We say that a rank-3 subbundle σ of $End(T\overline{M})$ is an almost paraquaternionic structure on \overline{M} if a local basis $\{J_1, J_2, J_3\}$ exists on sections of σ , such that for all $\alpha \in \{1, 2, 3\}$ we have

$$J_\alpha^2 = -\epsilon_\alpha, \quad J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = \epsilon_{\alpha+2} J_{\alpha+2} \quad (\text{indices mod } 3)$$

where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$.

Let $(\overline{M}, \overline{g})$ be a semi-Riemannian manifold and σ an almost paraquaternionic structure on \overline{M} . The metric \overline{g} is said to be adapted to the paraquaternionic structure σ if it satisfies $\overline{g}(J_\alpha X, J_\alpha Y) = \epsilon_\alpha \overline{g}(X, Y)$ for all $\alpha \in \{1, 2, 3\}$ and for all vector fields X, Y on M and any local basis $\{J_1, J_2, J_3\}$ of σ , where $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$. Moreover $(\overline{M}, \overline{g}, \sigma)$ is said to be an almost hermitian paraquaternionic manifold. Moreover, if the Levi-Civita connection of \overline{g} satisfies the following conditions for all $\alpha \in \{1, 2, 3\}$:

$$\overline{\nabla}_X J_\alpha = -\epsilon_\alpha [\omega_{\alpha+2}(X) J_{\alpha+1} - \omega_{\alpha+1}(X) J_{\alpha+2}] \quad (\text{indices mod } 3) \quad (3.1)$$

for any vector field X on \overline{M} , $\omega_1, \omega_2, \omega_3$ being local 1-forms over the open for which $\{J_1, J_2, J_3\}$ is a local basis of σ and $\epsilon_1 = 1, \epsilon_2 = \epsilon_3 = -1$, then $(\overline{M}, \overline{g}, \sigma)$ is said to be a paraquaternionic Kähler manifold (see [2]).

Let $(\overline{M}^{4m}, \overline{g}, \sigma)$ be an almost hermitian paraquaternionic manifold and let (M, g) be a real lightlike hypersurface of \overline{M} . We remark that we can choose the screen distribution $S(TM)$ such that it contains $J_\alpha(TM)^\perp$ as a vector subbundle, because we have: $g(J_\alpha\xi, \xi) = 0, \forall \xi \in \Gamma(TM|_U)$, $J_\alpha\xi$ are tangent to M and thus $J_\alpha(TM)^\perp$ is a 3-rank distribution on M such that $J_\alpha(TM)^\perp \cap TM^\perp = \{0\}$.

If $\{\xi, N\}$ is a pair of sections on $U \subset M$ as in above section, then $\overline{g}(J_\alpha N, \xi) = 0 = \overline{g}(J_\alpha N, N)$, and we deduce that $J_\alpha N \in \Gamma(S(TM))$. It is easy to see that $J_\alpha(TM)^\perp \oplus J_\alpha N$ is a vector subbundle of $S(TM)$ of rank 6. Then there exists non-degenerate distribution D_0 on M such that $S(TM) = \{D_1 \oplus D_2\} \perp D_0$, where $D_1 = J_1\xi \perp J_2\xi \perp J_3\xi$ and $D_2 = J_1N \perp J_2N \perp J_3N$.

We remark that the distribution D_0 is invariant with respect to J_α for all $\alpha \in \{1, 2, 3\}$ (see [3]). We consider now the local lightlike vector fields: $V_\alpha = -J_\alpha N, U_\alpha = -J_\alpha\xi$ for all $\alpha \in \{1, 2, 3\}$. Then any local vector field on M is expressed as follows:

$$X = SX + \sum_{\alpha=1}^3 \epsilon_\alpha f_\alpha(X) V_\alpha \tag{3.2}$$

where S is the projection on the almost paraquaternionic distribution $D = \{TM^\perp \perp D_1\} \perp D_0$ and f_α are 1-forms locally defined on M by: $f_\alpha(X) = g(X, U_\alpha)$, for all $\alpha \in \{1, 2, 3\}$.

By using (3.2) we derive:

$$J_\alpha X = \Phi_\alpha X + f_\alpha(X) N$$

where $\Phi_\alpha X$ are the tangential components of $J_\alpha X$ for all $\alpha \in \{1, 2, 3\}$.

Theorem 3.1. *Let $(\overline{M}, \overline{g}, \sigma)$ be a paraquaternionic Kähler manifold and let M be a lightlike hypersurface of \overline{M} . Then M is totally geodesic if and only if: $A_N X, A_\xi^* X, \nabla_X^* Z_0 \notin \Gamma(D_2)$, for all $X \in \Gamma(TM), \forall N \in \Gamma(\text{ltr}(TM)), \forall \xi \in \Gamma(TM^\perp), \forall Z_0 \in \Gamma(D_0)$.*

Proof. By using (2.3), (2.4) and (3.1) we obtain for any $X \in \Gamma(TM)$ and $Z_0 \in \Gamma(D_0)$:

$$\begin{aligned} h(X, J_\alpha Z_0) &= -\epsilon_\alpha [\omega_{\alpha+2}(X) J_{\alpha+1} Z_0 - \omega_{\alpha+1}(X) J_{\alpha+2} Z_0] \\ &\quad + J_\alpha (\nabla_X^* Z_0) - C(X, Z_0) U_\alpha - B(X, Z_0) V_\alpha - \nabla_X J_\alpha Z_0 \end{aligned}$$

The proof is now complete, since by the definition of a lightlike hypersurface, M is totally geodesic if and only if: $h(X, U_\alpha) = h(X, V_\alpha) = h(X, Z_0) = h(X, \xi) = 0, \alpha \in \{1, 2, 3\}$, for all $Z_0 \in \Gamma(D_0)$ and $\xi \in \Gamma(TM^\perp)$. □

Theorem 3.2. *Let $(\overline{M}, \overline{g}, \sigma)$ be a paraquaternionic Kähler manifold and let M be a lightlike hypersurface of \overline{M} . Then D_0 is integrable if and only if: $C(X, Y) = C(Y, X), C(X, J_\alpha Y) = C(Y, J_\alpha X), B(X, J_\alpha Y) = B(Y, J_\alpha X)$ for all $X, Y \in \Gamma(D_0)$ and $\alpha \in \{1, 2, 3\}$.*

Proof. D_0 is integrable if and only if: $\forall X, Y \in \Gamma(D_0) \Rightarrow [X, Y] \in \Gamma(D_0)$ and from the general decomposition $TM = \{(D_1 \oplus D_2)\} \perp D_0 \perp TM^\perp$ we deduce that D_0 is integrable if and only if: $\overline{g}([X, Y], N) = \overline{g}([X, Y], V_\alpha) = \overline{g}([X, Y], U_\alpha) = 0$ for all $X, Y \in \Gamma(D_0), N \in \Gamma(\text{ltr}(TM))$ and $\alpha \in \{1, 2, 3\}$. For $X, Y \in \Gamma(D_0)$ and $N \in \Gamma(\text{ltr}(TM))$, by using (2.5) we deduce:

$$\overline{g}([X, Y], N) = C(X, Y) - C(Y, X) \tag{3.3}$$

From (2.5) and (3.1), we obtain:

$$\overline{g}([X, Y], V_\alpha) = C(X, J_\alpha Y) - C(Y, J_\alpha X), \quad \forall \alpha \in \{1, 2, 3\} \tag{3.4}$$

and similarly we deduce

$$\overline{g}([X, Y], U_\alpha) = B(X, J_\alpha Y) - B(Y, J_\alpha X), \quad \forall \alpha \in \{1, 2, 3\} \tag{3.5}$$

The proof is now complete from (3.3), (3.4) and (3.5). □

Corollary 3.1. *Let $(\overline{M}, \overline{g}, \sigma)$ be a paraquaternionic Kähler manifold and let M be a totally geodesic lightlike hypersurface of \overline{M} . Then the distribution D is parallel.*

4 Non-existence of real lightlike hypersurfaces in paraquaternionic space form

Let $(\overline{M}, \overline{g}, \sigma)$ be a paraquaternionic Kähler manifold and let X be a non-lightlike vector on \overline{M} . Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$, denoted by $PQ(X)$, is called a paraquaternionic 4-plane. Any 2-plane in $PQ(X)$ is called a paraquaternionic plane. A paraquaternionic Kähler manifold is said to be a paraquaternionic space form if its paraquaternionic sectional curvatures are equal to a constant.

We recall now the following technical results [3].

Lemma 4.1. *Let $(\overline{M}, \overline{g}, \sigma)$ be a paraquaternionic Kähler manifold and let M be a lightlike hypersurface of \overline{M} . Then we have for all $\alpha \in \{1, 2, 3\}$:*

$$(\nabla_X \Phi_\alpha)Y = -\epsilon_\alpha[\omega_{\alpha+2}(X)\Phi_{\alpha+1}(Y) - \omega_{\alpha+1}(X)\Phi_{\alpha+2}(Y)] + f_\alpha(Y)A_NX - B(X, Y)V_\alpha \tag{4.1}$$

$$(\nabla_X f_\alpha)Y = -\epsilon_\alpha[f_{\alpha+1}(Y)\omega_{\alpha+2}(X) - f_{\alpha+2}(Y)\omega_{\alpha+1}(X)] - B(X, \Phi_\alpha Y) - f_\alpha(Y)\tau(X) \tag{4.2}$$

(indices mod 3) for all $X, Y \in \Gamma(TM)$.

Let (M, g) be a lightlike hypersurface of a semi-Riemannian manifold $(\overline{M}, \overline{g})$. The null sectional curvature of M at $p \in M$ with respect to ξ_p , is the real number $K_{\xi_p}(M)$ defined by:

$$K_{\xi_p}(M) = \frac{R(X_p, \xi_p, X_p, \xi_p)}{g(X_p, X_p)} \tag{4.3}$$

where X_p is non-null vector in T_pM .

Theorem 4.1. *There are no real lightlike hypersurfaces with positively or negatively null sectional curvature of paraquaternionic space forms.*

Proof. Let $\overline{M}(c)$ be a paraquaternionic space form and let M be a real lightlike hypersurface of $\overline{M}(c)$. By using (4.2) and (4.3) we obtain:

$$K_\xi(M) = \frac{3c}{4\|X\|^2} \sum_{\alpha=1}^3 \epsilon_\alpha f_\alpha^2(X)$$

If we take now $X \in \Gamma(D_0)$ we deduce that $K_\xi(M) = 0$, which proves our assertion. □

Theorem 4.2. *There are no real lightlike hypersurfaces with positively or negatively Ricci curvature of paraquaternionic space forms.*

Proof. Let M be a real lightlike hypersurface of a paraquaternionic space form $\overline{M}(c)$. We have:

$$\begin{aligned} \text{Ric}(X, Y) = & \sum_{i=1}^{4m-8} g(Z_i, Z_i)g(R(X, Z_i)Y, Z_i) + g(R(X, \xi)Y, N) + \\ & + \sum_{\alpha=1}^3 [g(R(X, U_\alpha)Y, V_\alpha) + g(R(X, V_\alpha)Y, U_\alpha)] \end{aligned} \tag{4.4}$$

where $\{\{Z_i\}_{i=1, 4m-8}, \{U_\alpha\}_{\alpha=1, 3}, \{V_\alpha\}_{\alpha=1, 3}\}$ is a basis of $S(TM)|_U$. By straightforward computations using (2.3), (2.4), (2.5), (2.6), (4.1) and (4.2) in (4.4) we obtain $\text{Ric}(X, \xi) = 0$. □

5 Example

We consider the paraquaternionic manifold $(\mathbb{R}_4^8, \bar{g}, \{J_1, J_2, J_3\})$ where the metric \bar{g} and the structures J_1, J_2, J_3 are given by

$$\begin{aligned}\bar{g}((x_i)_{i=\overline{1,8}}, (y_i)_{i=\overline{1,8}}) &= -\sum_{i=1}^4 x_i y_i + \sum_{i=5}^8 x_i y_i \\ J_1((x_i)_{i=\overline{1,8}}) &= (-x_2, x_1, -x_4, x_3, -x_6, x_5, -x_8, x_7) \\ J_2((x_i)_{i=\overline{1,8}}) &= (-x_7, x_8, -x_5, x_6, -x_3, x_4, -x_1, x_2) \\ J_3((x_i)_{i=\overline{1,8}}) &= (x_8, x_7, x_6, x_5, x_4, x_3, x_2, x_1)\end{aligned}$$

We define now a hypersurface M of $(\mathbb{R}_4^8, \bar{g}, \{J_1, J_2, J_3\})$ by

$$f((t_i)_{i=\overline{1,7}}) = (t_1, t_2, t_1 + t_3, t_4 + t_5, t_3 + t_5 + t_6, t_6 + t_7, t_7, t_2)$$

Thus the tangent space TM is spanned by $\{W_i\}_{i=\overline{1,7}}$, where

$$\begin{aligned}W_1 &= (1, 0, 1, 0, 0, 0, 0, 0), & W_2 &= (0, 1, 0, 0, 0, 0, 0, 1), & W_3 &= (0, 0, 1, 0, 1, 0, 0, 0) \\ W_4 &= (0, 0, 0, 1, 0, 0, 0, 0), & W_5 &= (0, 0, 0, 1, 1, 0, 0, 0), & W_6 &= (0, 0, 0, 0, 1, 1, 0, 0) \\ W_7 &= (0, 0, 0, 0, 0, 1, 1, 0)\end{aligned}$$

If $X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8) \in TM^\perp$, from conditions $\bar{g}(X, W_i) = 0$ ($i \in \{1, \dots, 7\}$) we obtain

$$x_1 = x_3 = x_4 = x_5 = x_6 = x_7 = 0, \quad x_2 = x_8$$

Hence,

$$X = (0, x_2, 0, 0, 0, 0, 0, x_2) = x_2 W_2$$

Thus $TM^\perp = Sp(W_2)$ and M is lightlike hypersurface of \mathbb{R}_4^8 .

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