δ-Ideals in MS-Algebras

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Abstract

The concepts of δ-Ideals and principal δ-Ideals are introduced in an MS-algebra and many properties of these ideals are studied. It is observed that the class of all δ-Ideals forms a complete distributive lattice and the class of all principal δ-Ideals forms a de Morgan algebra. A characterization of δ-Ideals in terms of principal δ-Ideals is given. Finally, many properties of δ-Ideals are studied with respect to homomorphisms.

Keywords: De Morgan algebras; MS-algebras; Ideals; Filters; Homomorphisms

Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [1]. Blyth and Varlet [2] defined a subclass of Ockham algebras so called MS-algebras which generalizes both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman [3]. The class of all MS-algebras forms an equational class. Blyth and Varlet characterized the subvarieties of MS-algebras in Ref. [4]. Recently, Luo and Zeng [5] characterized the MS-algebras on which all congruences are in a one-to-one correspondence with the kernel ideals. In Ref. [6], Rao introduced the concepts of boosters and β-filters of MS-algebras. In Ref. [7], Rao introduced and characterized the concept of δ-Ideals in pseudo-complemented distributive lattices. Many various properties of Ockham algebras and MS-algebras are considered in Refs. [9-14].

In this paper, we defined δ-Ideals and principal δ-Ideals in MS-algebras and some basic properties of δ-Ideals and principal δ-Ideals are studied. It is proved that the class $P(L)$ of all δ-Ideals of an MS-algebra $L$ is a complete distributive lattice. It is proved that the set of all principal δ-Ideals of an MS-algebra can be made into a de Morgan algebra. A set of equivalent conditions is obtained to characterize δ-Ideals of MS-algebras by means of principal δ-Ideals. Finally, some properties of δ-Ideals are studied with respect to homomorphisms. The concept of δ-Ideals preserving homomorphism from an MS-algebra $L$ into another MS-algebra $L'$ is introduced as a homomorphism $h$ satisfying the condition $h(δ(F))=δ(h(F))$, for any δ-Ideals $F$ of $L$, where $F$ is a filter of $L$. It is proved that the images and the inverse images, under this homomorphism, of a δ-Ideals are again δ-Ideals. If an MS-algebras $L$ is homomorphic to an MS-algebra $L'$, then the lattice $M(L)$ of all principal δ-Ideals of $L$ is homomorphic to $M(L')$ the lattice of all principal δ-Ideals of $L'$, and the lattice $P(L)$ of all δ-Ideals of $L$ is homomorphic to the lattice $P(L')$ of all δ-Ideals of $L'$.

Preliminaries

In this section, we present certain definitions and results. We refer the reader to Ref. [1,2,4,9] as a guide references.

Definition 2.1

A de Morgan algebra is an algebra $(L, ∨, ∧, °, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L, ∨, ∧, 0, 1)$ is a bounded distributive lattice and the unary operation of involution satisfies :

$$x°x = x, x°°x = 1$$

Definition 2.2

An MS-algebra is an algebra $(L, ∨, ∧, °, 0, 1)$ of type $(2, 2, 1, 0, 0)$ where $(L, ∨, ∧, 0, 1)$ is a bounded distributive lattice and the unary operation ° satisfies :

$$x°x = x, x°°x = 1$$

We recall some of the basic properties of MS-algebras which were proved in Ref. [2].

Theorem 2.3

For any two elements $a, b$ of an MS-algebra $L$, we have:

1. $0° = 1$
2. $a ≤ b ⇒ b° ≤ a°$
3. $a°° = a°$
4. $(a ∧ b)° = a° ∧ b°$
5. $(a ∨ b)°° = a°° ∨ b°°$
6. $(a ∧ b)°° = a°° ∧ b°°$

For any MS-algebra $L$, let $I(L)$ denote to the set of all ideals of $L$. It is known that $(I(L); ∨, ∧)$ is a distributive lattice, where $I ∧ f = I ∧ f$ and $I ∨ f = f ∨ f$ for all $f ∈ I$. Also, $[a)=\{x ∈ L | x ≤ a\}[(a]=\{x ∈ L | x ≥ a\}$ is a principal ideal (filter) of $L$ generated by $a$.

Any MS-algebra $L$ we can define the set of closed elements $L^c(a)e=\{x ∈ L | x = a°\}$. It is known that $(L^c(a)e, ∨, ∧, 0, 1)$ is a de Morgan subalgebra of $L$. An element $a ∈ L$ is called a dense element if $a°=0$. Then the set $D(L)$ of all dense elements of $L$ forms a filter in $L$. An element $x ∈ L$ is called a fixed point of $L$ if $x°=x$.

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Received December 30, 2015; Accepted January 25, 2016; Published January 28, 2016


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Properties of $\delta$-Ideals

In this Section, the concept of $\delta$-Ideals and principal $\delta$-Ideals are introduced in MS-algebras. Many properties of $\delta$-Ideals and principal $\delta$-Ideals are investigated in the class of all MS-algebras. We observed that the class of all principal $\delta$-Ideals of an MS-algebra $L$ is a de Morgan algebra. It is proved that the class of all $\delta$-Ideals of any MS-algebra forms a complete distributive lattice. A characterization of $\delta$-Ideals in terms of principal $\delta$-Ideals is obtained.

**Definition 3.1**

Let $L$ be an MS-algebra. Then for any filter $F$ of $L$, de ne the set $\delta(F)$ as follows:

$$\delta(F) = \{ x \in L : x^\omega \in F \}$$

Clearly, $\delta([1]) = \{0\}$ and $\delta([0]) = L$. The following two Lemmas are direct consequence of the above definition.

**Lemma 3.2**

Let $L$ be an MS-algebra. Then $\delta(F)$ is an ideal of $L$.

**Proof:** Clearly $0 \in \delta(F)$. Let $x, y \in F$. Then $x^\omega, y^\omega \in F$. Hence $(x \land y)^\omega = x^\omega \land y^\omega \in F$. Thus $x \land y \in F$. Again, let $x \in \delta(F)$ and $r^\omega \leq x$. Then $r^\omega \leq x^\omega \in F$ implies $r \in F$. Therefore $\delta(F)$ is an ideal of $L$.

**Lemma 3.3**

Let $L$ be an MS-algebra. Then for any two filters $G, F$ of $L$, we have the following:

1. $F \land \delta(F) = \phi$, whenever $L \in S$.
2. $x \in \delta(F)$ implies $x^\omega \in \delta(F)$.
3. $x \in F$ implies $x^\omega \in \delta(F)$.
4. $F = L$ if and only if $\delta(F) = L$.
5. $F \subseteq G$ implies $\delta(F) \subseteq \delta(G)$.
6. $\delta(D(L)) = \{0\}$.
7. $\delta(F)$ is a prime, whenever $F$ is a prime filter of $L$.

**Proof:** (1) Suppose $x \in F \land \delta(F)$. Then $x \in F$ and $x^\omega \in F$. Since $F$ is a filter and $L$ is a Stone algebra, we get $0 = x^\omega \land x \in F$, which is a contradiction. Therefore $F \land \delta(F) = \phi$.

2. Let $x \in \delta(F)$. Then $x^\omega = x^\omega \land x^\omega \in \delta(F)$.

3. Let $x \in F$. Then $x^\omega \geq x \in F$ implies $x^\omega \in \delta(F)$.

4. Let $F = L$. Then $0 = 0^\omega \in F$ implies $1 = 0^\omega \in \delta(F)$. Therefore $\delta(F) = L$. Conversely, let $\delta(F) = L$. Then $1 = 0^\omega \in F$. Hence $0 = 1^\omega \in \delta(F)$. Then $\delta(F) = L$.

5. Let $F \subseteq G$. Suppose $x \in \delta(F)$. Then $x^\omega \in F \subseteq G$. Therefore $x \in \delta(G)$ and $\delta(F) \subseteq \delta(G)$.

6. Let $x \in \delta(D(L))$. Then $x^\omega \in \delta(D(L))$. Hence $x \leq x^\omega = 0$. Therefore $\delta(D(L)) = \{0\}$.

7. Let $F$ be a prime filter of $L$. Assume $x \land y \in \delta(F)$ and $x \neq F$. Then $x^\omega \lor y^\omega = (x \lor y)^\omega \neq 2$ and $y^\omega \in F$. Since $F$ is prime filter, then $x^\omega \in F$. Hence $x \in \delta(F)$. Therefore $\delta(F)$ is prime ideal of $L$.

The concept of $\delta$-Ideals is introduced in the following.

**Definition 3.4**

Let $L$ be an MS-algebra. An ideal $I$ of $L$ is called a $\delta$-Ideal if $I = \delta(I)$ for some filter $F$ of $L$.

**Example 3.5**

Let $L = (0, \{x, y, z\}, 1:\{0 < x < y < z < 1\})$ be a five element chain and $x^\omega = y^\omega = z^\omega = 0$. Clearly, $(L, \land)$ is an MS-algebra. We observe that the ideals $\{0\}$, $\{0, x\}$ and $L$ are $\delta$-Ideals of $L$ but the ideals $\{0, x, y\}$ and $\{0, x, y, z\}$ are not.

**Lemma 3.6**

A proper $\delta$-ideal of an MS-algebra $L$ contains no dense element.

**Proof:** Let $I$ be a proper $\delta$-Ideal. Then $I = \delta(F)$ for some filter $F$ of $L$. Suppose $x \in \delta(F) \land \delta(L)$. Then we get $0 = x^\omega \in F$, which is a contradiction. Therefore $\delta(F) \land \delta(L) = \phi$.

The following lemma produces some more examples for $\delta$-Ideals of an MS-Algebra from the subvariety $K_\gamma$.

**Lemma 3.7**

Let $L$ be an MS-algebra from $K_\gamma$. Then we have

1. $L \land$ is a $\delta$-Ideal of $L$.
2. Every prime ideal $P$ with $P \land L \subseteq \phi$ and $L \subseteq P$ is a $\delta$-Ideal of $L$, whenever $L$ has no fixed point.

**Proof:** (1) It is known that, if $L \in K_\gamma$, then $L = (x \lor x^\omega : x \in L)$ is a filter of $L$. Then $x \lor x^\omega \in L$ and $x \lor x^\omega \in L^\omega$ for all $x \in L$. It is enough to deduce that $\delta(L^\omega) = L$. Let $x \in L$. Then $x \lor x^\omega \in L^\omega$. Therefore $\delta(L) \subseteq L$. Consequently $L^\omega$ is a $\delta$-Ideal of $L$.

(2) Suppose that $P$ is a prime ideal of $L$ such that $P \land L' = \phi$ and $L' \subseteq P$. Let $x \in P$. Then $x \land x^\omega \in L'$ and $x \land x^\omega \in L'$. Hence $x \land x^\omega \neq P$. Thus we get $x \neq P$, which yields that $x \neq L$. Thus $x \in \delta(P)$. Therefore $P \subseteq \delta(L)$. Conversely, let $x \in \delta(L)$. Then $x \neq P$. Now $x \neq P$ and $P$ is prime imply $x \in P$. Hence $\delta(L) \subseteq \delta(P)$. Therefore $P$ is a $\delta$-Ideal of $L$.

Now, let us denote the set of all $\delta$-Ideals of $L$ by $\delta(L)$. Then, in the following Theorem, we prove that $\delta(L)$ forms a complete distributive lattice.

**Theorem 3.8**

Let $L$ be an MS-algebra. Then $\delta(L)$ forms a complete distributive lattice.

**Proof:** It is obviously that $\{0\}$ and $L$ are the smallest and the greatest $\delta$-Ideals of $L$. Now, for every two $\delta$-Ideals $I$ and $J$ we prove that $I \land J$ and $I \lor J$ are again $\delta$-Ideals. Since $I$ and $J$ are $\delta$-Ideals, then there exist filters $F$ and $G$ of $L$ such that $I = \delta(F)$ and $J = \delta(G)$. So we have to show the following:

$$\delta(F \land G) = \delta(F) \land \delta(G)$$

Since $F \subseteq G \subseteq F$ and $F \land G \subseteq G$, then by Lemma 3.2(5), we get $\delta(F \land G) \subseteq \delta(G) \land \delta(G)$. Conversely, let $x \in \delta(G) \land \delta(G)$. Then $x \in F \land G$. Hence $x \in \delta(F \land G)$. Therefore $\delta(F \land G) \subseteq \delta(G) \land \delta(G)$. Now, $\delta(F \land G)$ is a $\delta$-Ideal of $L$. Since $\delta(F)$, $\delta(G)$ $\subseteq \delta(F \land G)$, then $\delta(F \land G)$ is an upper bound of $\delta(F)$ and $\delta(G)$ in $P(L)$. Let $\delta(H)$ be a $\delta$-Ideal of $L$ such that $\delta(F \land G) \subseteq \delta(H)$ and $\delta(G) \subseteq \delta(H)$ where $H$ is a filter of $L$. We claim that $\delta(F \land G) \subseteq \delta(H)$. Let $x \in \delta(F \lor G)$, then $x \lor x \in F \lor G$. Hence $x = x \lor g$ for some $f \in F$ and $g \in G$. Since $f^\omega \in \delta(F)$ and $g^\omega \in \delta(G)$ (see Lemma 3.3(3)), then $f^\omega \in \delta(H)$ and $g^\omega \in \delta(H)$. Now we have...
\[ f \in \delta(H) \text{ and } g \in \delta(H) \Rightarrow f \vee g \in \delta(H) \]
\[ \Rightarrow x^{\text{MS-}}(f \vee g)^{\text{MS-}} \in \delta(H) \]
\[ \Rightarrow x \in \delta(H) \text{ by Lemma 3.3(2).} \]

Hence \( \delta(F \vee G) \) is the supremum of both \( \delta(F) \) and \( \delta(G) \) in \( \mathcal{P}(L) \). Therefore \( \mathcal{P}(L), \cap, \vee, \{0, L\} \) is a bounded sublattice of the lattice \( \mathcal{L}(L) \) of all ideals of \( L \). Hence \( \mathcal{P}(L) \) is a bounded distributive lattice. It is clear that \( \mathcal{P}(L) \) is a partially ordered set with respect to set-inclusion. Then by the extension of the properties \( \delta(F \cap G) = \delta(F) \cap \delta(G) \) and \( \delta(F \vee G) = \delta(F) \vee \delta(G) \), we can obtain that \( \mathcal{P}(L) \) is a complete lattice. Therefore \( \mathcal{P}(L) \) is a complete distributive lattice.

**Definition 3.9**

A \( \delta \)-Ideal \( I \) of an MS-algebra \( L \) is called principal \( \delta \)-Ideal if there exists \( x \in L \) such that \( I = \delta(I(x)) \).

It is observed in the following Theorem that any principal ideal generated by a closed element of an MS-algebra is a \( \delta \)-Ideal.

**Theorem 3.10**

Let \( L \) be an MS-algebra. Then for any \( x \in L \), \( \delta(I(x)) \) is a principal \( \delta \)-Ideal of \( L \).

**Proof:** It is easy to show that \((x^\text{a}) \subseteq \delta(I(x))\). Let \( a \in (x^\text{a}) \). Then \( a \leq x \). Hence \( a^\text{a} \geq x^\text{a} \). \( a^\text{a} \leq a \). Thus \( a \in \delta(I(x)) \). Conversely, suppose that \( a \in \delta(I(x)) \). Then \( a \in \delta(I(x)) \) implies \( a^\text{a} \geq x \). Hence \( a \leq a^\text{a} \geq x \). This yields that \( a \in (x^\text{a}) \). Therefore \( (x^\text{a}) \) is a principal \( \delta \)-Ideal of \( L \).

Some properties of principal \( \delta \)-Ideal are given in the following:

**Lemma 3.11**

Let \( L \) be an MS-algebra. Then we have the following statements:

1. For all \( a \in L \), \( \delta(I(a)) = (a^a) \),
2. For all \( a \in L \), \( \delta(I(a)) = \delta(a^a) \),
3. For all \( d \in D(L), \delta(I(d)) = \{0\} \),
4. For all \( x \in F \), \( \delta(I(x)) = \delta(F) \) for any filter \( F \) of \( L \).

**Proof:** (1) It is clear from the above Theorem 3.10.

(2) Using (1) and the fact, \( a^a = a \), we get,

(3) \( \delta(I(a)) = (a^a) = \delta(a) \).

(5) For every \( d \in D(L), \delta(I(d)) = \delta(d) = \{0\} \).

(6) Let \( x \in F \). Suppose \( y \in \delta(I(x)) \). Then we get,

\[ y^a \geq x \in F \]
\[ \Rightarrow y^a \in F \]
\[ \Rightarrow y \in \delta(F) \]

Therefore \( \delta(I(x)) \subseteq \delta(F) \).

Let us denote that \( M^*(L) = \{\delta(I(x)) : x \in L\} \). Then, in the following Theorem, it is observed that \( M^*(L) \) is a de Morgan algebra.

**Theorem 3.12:** For any MS-algebra \( L \), \( M^*(L) \) is a sublattice of the lattice \( \mathcal{P}(L) \) of all \( \delta \)-Ideals of \( L \) and \( M^*(L) \) can be made into a de Morgan algebra. Moreover, the mapping \( x \mapsto (x^a) \) is a dual homomorphism of \( L \) into \( M^*(L) \).

**Proof:** Let \( \delta(I(x)), \delta(I(y)) \in M^*(L) \) for some \( x, y \in L \). Then we get \( \delta((x) \cap \delta((y)) = \delta((x \cap y)) \subseteq \delta(I(x \cap y)) \subseteq \delta((x^a) \cap \delta((y^a)) \subseteq \delta(I(x^a) \cap \delta(I(y^a)) \subseteq M^*(L) \).

Also, \( \{0\} = \delta(I(0)) \subseteq M^*(L) \) and \( L = \delta(I(0)) \subseteq M^*(L) \). Hence \( M^*(L) \) is a bounded sublattice of \( \mathcal{P}(L) \) and hence a distributive lattice. Now, define a unary operation on \( M^*(L) \) by \( \delta(I(x)) \). Then we have \( \delta(I(x)) \cap \delta(I(y)) \subseteq \delta(I(x \cap y)) \subseteq \delta(I(x \cap y)) \subseteq M^*(L) \).

Therefore \( M^*(L) \) is a de Morgan algebra. The remaining part can be easily observed. A characterization of \( \delta \)-Ideals in terms of principal \( \delta \)-Ideals is investigated in the following.

**Theorem 3.13:** For any ideal \( I \) in an MS-algebra \( L \), then the following conditions are equivalent:

1. \( I \) is a \( \delta \)-Ievals
2. \( I = \cup_{a \in I} \delta(I(a^a)) \)
3. For any \( x, y \in L \), \( \delta(I(x)) = \delta(I(y)) \) and \( x \in I \).

**Proof:** (1) \( \Rightarrow \) (2): Let \( I \) be a \( \delta \)-Ideal. Then \( I = \delta(F) \) for some filter \( F \) of \( L \). Let \( x \in I \). So we get \( x \in I \Rightarrow \delta(F) \Rightarrow x^a \in F \)

\[ \Rightarrow x^a \in \delta(I(x^a)) \subseteq \delta(F) \]
\[ \Rightarrow x \in \delta(I(x)) \subseteq \bigcup_{a \in I} \delta(I(a)) \].

Then \( I \subseteq \bigcup_{a \in I} \delta(I(a^a)) \). Conversely, let \( x \in \bigcup_{a \in I} \delta(I(a^a)) \). Then we have,

\[ x \in \bigcup_{a \in I} \delta(I(a^a)) \Rightarrow x \in \delta(I(y^a)) \text{ for some } y \in I \]
\[ \Rightarrow x \in (y^a) \subseteq I \text{ as } y^a \in I \]
\[ \Rightarrow x \in \bigcup_{a \in I} (I(a^a)) \subseteq I \].

Then \( I = \bigcup_{a \in I} \delta(I(a^a)) \).

(2) \( \Rightarrow \) (3): Let \( I = \bigcup_{a \in I} \delta(I(a^a)) \). Suppose \( \delta(I(x^a)) = \delta(I(y^a)) \) and \( x \in I \).

Then we get

\[ \delta(I(x^a)) = \delta(I(y^a)) \text{ and } x \in I \Rightarrow \delta(I(y^a)) = \delta(I(x^a)) \subseteq \bigcup_{a \in I} \delta(I(a^a)) = I \]
\[ \Rightarrow y^a \subseteq I \]
\[ \Rightarrow y^a \in I \Rightarrow y \in I \].

(3) \( \Rightarrow \) (1): Assume the condition (3). Consider \( F = \{x \in L : x^a \in I\} \).

Let \( x, y \in F \). Then \( x^a, y^a \in I \). Hence \( x \wedge y^a = x^a \vee y^a \in F \). Thus \( x \wedge y \in F \).

Now let \( x \in F \) and \( x \in L \) such that \( z \geq x \). Then \( x^a \leq x^a \in I \) implies \( z^a \in I \).

Thus \( x \in F \) and \( F \) is a filter of \( L \). We claim that \( I = \delta(F) \).

Let \( x \in \delta(F) \).

Then we get,

\[ x \in \delta(F) \Rightarrow x^a \in F \]
\[
\Rightarrow x^e \in I \\
\Rightarrow x \in I \Rightarrow \delta(F) \subseteq I.
\]

For the converse, let \(y \in I\). We have,
\[
y \in I \text{ and } \delta((y^e)=\delta((y^e)) \Rightarrow y^e \in I \text{ by (3)}
\]
\[
\Rightarrow y^e \in F
\]
\[
\Rightarrow y \in \delta(F)
\]
\[
\Rightarrow I \subseteq \delta(F).
\]

Therefore \(I\) is a \(\delta\)-ideal.

**\(\delta\)-Ideals and Homomorphisms of \(MS\)-algebras**

In this section, some properties of the homomorphic images and the inverse images of \(\delta\)-ideals are studied. By a homomorphism on an \(MS\)-algebra \(L\), we mean a lattice homomorphism \(h\) satisfying \((h(x))^e=h(x^e)\) for all \(x \in L\).

**Theorem 4.1**

Let \(h: L \rightarrow M\) be a homomorphism of an \(MS\)-algebra \(L\) onto an \(MS\)-algebra \(M\). Then we have,

1. for any \(a \in L\), \(h(\delta([a]))=\delta(h([a]))\),
2. for any \(\delta\)-ideal \(I\) of \(L\), \(h(I)\) is a \(\delta\)-ideal of \(M\),
3. for any \(\delta\)-ideal \(I\) of \(L\), \(I=\bigcup_{\alpha} \delta((h([\alpha]^{\ast}))\).

Let \(F\) be any \(MS\)-algebra. Then we have,
\[
x \in \delta(h(F)) \Rightarrow x=h(y), y \in \delta(F)
\]
\[
x^e=h(y^e), y^e \in F
\]
\[
x \in h(\delta(F)).
\]

**Theorem 4.2**

Let \(f: L \rightarrow M\) be a homomorphism of an \(MS\)-algebra \(L\) into an \(MS\)-algebra \(M\). Then we have,

1. for any \(\delta\)-ideal \(H\) of \(M\), \(f^{-1}(H)\) is a \(\delta\)-ideal of \(L\),
2. \(\text{Coker } f\) is a \(\delta\)-ideal of \(L\).

**Proof:** (1) Since \(H\) is a \(\delta\)-ideal of \(M\), then \(H=\delta(F)\) for some filter \(F\) of \(M\). We claim \(f^{-1}(H)=\delta(f^{-1}(F))\), where \(f^{-1}(H)\) is an ideal of \(L\).

\[
x \in f^{-1}(H) \Rightarrow f(x)=y, y \in H=\delta(F)
\]
\[
\Rightarrow (f(x))^e=f(x^e)=y^e \in F
\]
\[
\Rightarrow x^e \in f^{-1}(F)
\]
\[
\Rightarrow x \in \delta(f^{-1}(F)).
\]

Conversely, \(x \in \delta(f^{-1}(F))\). Then we have,
\[
x \in \delta(f^{-1}(F)) \Rightarrow x^e \in f^{-1}(F)
\]
\[
\Rightarrow (f(x))^e=f(x^e)=1
\]
\[
\Rightarrow f(x) \in \delta(F)=H
\]
\[
\Rightarrow x^e \in f^{-1}(H)
\]
\[
\Rightarrow \delta(f^{-1}(H)) \subseteq f^{-1}(H)
\]

Therefore \(f^{-1}(H)\) is a \(\delta\)-ideal of \(L\).

(2) Since \(f\) is a homomorphism, then \(\text{Coker } f = [x \in L: f(x)=0]\) and \(\text{Coker } f = [x \in L: f(x)=1]\) are ideal and filter of \(L\) respectively. We claim \(\text{Coker } f = (\text{Coker } f)\). Now
\[
x \in \text{Coker } f \Rightarrow f(x)=0
\]
\[
\Rightarrow f(x^e)=f(x^e)=1
\]
\[
\Rightarrow x^e \in \text{Coker } f
\]
\[
\Rightarrow x \in \delta(\text{Coker } f).
\]

Then \(\delta(\text{Coker } f) \subseteq \text{Coker } f\). Conversely,
\[
x \in \delta(\text{Coker } f) \Rightarrow x^e \in \text{Coker } f
\]
\[
\Rightarrow f(x^e)=f(x^e)=1
\]
\[
\Rightarrow f(x)^e=f(x^e)=0
\]
\[
\Rightarrow x \in \text{ker } f
\]
\[
\Rightarrow \delta(\text{Coker } f) \subseteq \text{ker } f\). Therefore \(\text{ker } f\) is a \(\delta\)-ideal of \(L\).

**Theorem 4.3**

Let \(h: L \rightarrow L_1\) be an onto homomorphism between \(MS\)-algebras \(L\) and \(L_1\). Then we have,

1. \(M^e(L)\) is homomorphic of \(M^e(L_1)\),
2. \(\text{Im } h\) is homomorphic of \(\text{Im } h\).

**Proof:** (1) Define \(g:M^e(L) \rightarrow M^e(L_1)\) by \(g(h([a])))=\delta([a]))\). Clearly, \(g([0])=0\) and \(g(1)=1\). For every \(\delta([a]))\), \(\delta([b])) \in M^e(L)\) we get,
\[ g(\delta(a) \cap \delta((b))) = \delta(h(a \wedge b)) \]
\[ = \delta((h(a) \wedge h(b))) \]
\[ = \delta(h(a)) \cap \delta(h(b)) \]
\[ = g(\delta(a)) \cap g(\delta(b)), \]
and
\[ g(\delta(a) \vee \delta((b))) = g(\delta(a \vee b)) \]
\[ = \delta((h(a) \vee h(b))) \]
\[ = \delta(h(a)) \vee \delta(h(b)) \]
\[ = g(\delta(a)) \vee g(\delta(b)), \]
also,
\[ g(\delta(a)) = g(\delta(a^*)) \]
\[ = \delta(h(a^*))) \]
\[ = \delta(h(a)) \]
\[ = g(\delta(a)) \]
Therefore \( g \) is a homomorphism of de Morgan algebras \( M^*(L) \) and \( M^*(L'_{-}). \)

(2) Define the map \( \pi: I^*(L) \to I^*(L_{-}) \) by \( \pi(I) = \delta(h(F)) \) where \( I = \delta(F). \)
It is clear that \( \pi(0_I) = 0_{I_{-}} \) and \( \pi(L) = L_{-}. \) Let \( I, J \in I^*(L). \) Then \( I = \delta(F) \) and \( J = \delta(G), \) where \( F \) and \( G \) are filters of \( L. \) Then we get,
\[ \pi(I \vee J) = \delta(h(F \vee G)) \]
\[ = \delta(h(F)) \vee \delta(h(G)) \]
\[ = \pi(I) \vee \pi(J), \]
and
\[ \pi(I \cap J) = \delta(h(F \cap G)) \]
\[ = \delta(h(F) \cap h(G)) \]
\[ = \delta(h(F)) \cap \delta(h(G)) \]
\[ = \pi(I) \cap \pi(J) \]
Therefore \( \pi \) is a \((0, 1)\)-lattice homomorphism and the proof is completed.

References