Computer Science Systems Biology

Abd El-Mohsen, J Comput Sci Syst Biol 2016, 9:2
http://dx.doi.org/10.4172/jcsb.1000217

Review Article Open Access

δ-Ideals in *MS*-Algebras

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Abstract

The concepts of δ-Ideals and principal δ-Ideals are introduced in an *MS*-algebra and many properties of these ideals are studied. It is observed that the class of all δ-Ideals forms a complete distributive lattice and the class of all principal δ-Ideals forms a de Morgan algebra. A characterization of δ-Ideals in terms of principal δ-Ideals is given. Finally, many properties of δ-Ideals are studied with respect to homomorphisms.

Keywords: De Morgan algebras; *MS*-algebras; Ideals; Filters; Homomorphisms

Introduction

An Ockham algebra is a bounded distributive lattice with a dual endomorphism. The class of all Ockham algebras contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras [1]. Blyth and Varlet [2] defined a subclass of Ockham algebras so called *MS*-algebras which generalizes both de Morgan algebras and Stone algebras. These algebras belong to the class of Ockham algebras introduced by Berman [3]. The class of all *MS*-algebras forms an equational class. Blyth and Varlet characterized the subvarieties of *MS*-algebras in Ref. [4]. Recently, Luo and Zeng [5] characterized the *MS*-algebras on which all congruences are in a one-to-one correspondence with the kernel ideals. In Ref. [6], Rao, introduced the concepts of boosters and β-filters of *MS*-algebras. In Ref. [7], Rao introduced and characterized the concepts of *D*-filters and *e*- filters of *MS*-algebras. Also, in Ref. [8] Rao introduced and characterized the concept of δ-Ideals in pseudo-complemented distributive lattices. Many various properties of Ockham algebras and *MS*-algebras are considered in Ref. [9-14].

In this paper, we defined δ-Ideals and principal δ-Ideals in *MS*-algebras and some basic properties of δ-Ideals and principal δ-Ideals are studied. It is proved that the class $I^{\delta}(L)$ of all δ-Ideals of an *MS*-algebra L is a complete distributive lattice. It is proved that the set of all principal δ-Ideals of an *MS*-algebra can be made into a de Morgan algebra. A set of equivalent conditions is obtained to characterize δ-Ideals of *MS*-algebras by means of principal δ-Ideals. Finally, some properties of δ-Ideals are studied with respect to homomorphisms. The concept of δ-Ideals preserving homomorphism from an *MS*-algebra L into another *MS*-algebra L_1 is introduced as a homomorphism h satisfying the condition $h(\delta(F)=(h(F))$, for any δ-Ideals $I=\delta(F)$ of L, where F is a filter of L. It is proved that the images and the inverse images, under this homomorphism, of a δ-Ideals are again δ-Ideals. If an *MS*-algebras L is homomorphic to an *MS*-algebra L_1, then the lattice $M^{o}(L)$ of all principal δ-Ideals of L is homomorphic to $M^{o}(L_1)$ the lattice of all principal δ-Ideals of L_1 and the lattice $I^{\delta}(L)$ of all δ-Ideals of L is homomorphic to the lattice $I^{\delta}(L_1)$ of all δ-Ideals of L_1.

Preliminaries

In this section, we present certain definitions and results. We refer the reader to Ref. [1,2,4,9] as a guide references.

Definition 2.1

A de Morgan algebra is an algebra $(L, \vee, \wedge, \bar{\ }, 0, 1)$ of type (2, 2, 1, 0, 0) where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation of involution satisfies :

$$\bar{\bar{x}} = x, \overline{(x \vee y)} = \bar{x} \wedge \bar{y}, \bar{1} = 0$$

Definition 2.2

An *MS*-algebra is an algebra $(L, \vee, \wedge, {}^{\circ}, 0, 1)$ of type (2, 2, 1, 0, 0) where $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies :

$$x \le x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, 1^{\circ} = 0.$$

We recall some of the basic properties of *MS*-algebras which were proved in Ref. [2].

Theorem 2.3

For any two elements a, b of an *MS*-algebra L, we have

(1) $0^{\circ}=1$

(2) $a \le b \Rightarrow b^{\circ} \le a^{\circ}$

(3) $a^{\circ\circ\circ}=a^{\circ}$

(4) $(a \vee b)^{\circ}=a^{\circ} \wedge b^{\circ}$

(5) $(a \vee b)^{\circ\circ}=a^{\circ\circ} \vee b^{\circ\circ}$

(6) $(a \wedge b)^{\circ\circ}=a^{\circ\circ} \wedge b^{\circ\circ}$.

For any *MS*-algebra L, let $I(L)$ denote to the set of all ideals of L. It is known that $(I(L); \wedge, \vee)$ is a distributive lattice, where $I \wedge J= I \cap J$ and $I \vee J=\{i \vee j : i \in I, j \in J\}$. Also, $(a]=\{x \in L:x \le a\}$($[a)=\{x \in L:x \ge a\}$) is a principal ideal (filter) of L generated by a.

For any *MS*-algebra L we can define the set of closed elements $L^{\circ\circ}=\{a \in L:a=a^{\circ\circ}\}$. It is known that $(L^{\circ\circ}, \vee, \wedge, {}^{\circ}, 0, 1)$ is a de Morgan subalgebra of L. An element $a \in L$ is called a dense element if $a^{\circ}=0$. Then the set $D(L)$ of all dense elements of L forms a filter in L. An element $x \in L$ is called a fixed point of L if $x^{o}=x$.

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Received December 30, 2015; **Accepted** January 25, 2016; **Published** January 28, 2016

Citation: Abd El-Mohsen B (2016) δ-Ideals in *MS*-Algebras. J Comput Sci Syst Biol 9: 028-032. doi:10.4172/jcsb.1000217

Properties of δ-Ideals

In this Section, the concept of δ-Ideals and principal δ-Ideals are introduced in *MS*-algebras. Many properties of δ-Ideals and principal δ-Ideals are investigated in the class of all *MS*-algebras. We observed that the class of all principal δ-Ideals of an *MS*-algebra L is a de Morgan algebra. It is proved that the class of all δ-Ideals of any *MS*-algebra forms a complete distributive lattice. A characterization of δ-Ideals in terms of principal δ-Ideals is obtained.

Definition 3.1

Let L be an *MS*-algebra. Then for any filter F of L, de ne the set $\delta(F)$ as follows:

$\delta(F)= \{x \in L{:}x^{o} \in F\}$

Clearly, $\delta([1))=\{0\}$ and $\delta([0))=L$. The following two Lemmas are direct consequence of the above definition.

Lemma 3.2

Let L be an *MS*-algebra. Then $\delta(F)$ is an ideal of L.

Proof: Clearly $0 \in \delta(F)$. Let $x, y \in (F)$. Then $x^{o}, y^{o} \in F$. Hence $(x \vee y)^{o}=x^{o} \wedge y^{o} \in F$. Thus $x \vee y \in F$. Again, let $x \in \delta(F)$ and $r^{o} \le x$. Then $r^{o} \ge x^{o} \in F$ implies $r^{o} \in F$. Therefore $\delta(F)$ is an ideal of L.

Lemma 3.3

Let L be an *MS*-algebra. Then for any two filters F, G of L, we have the following:

(1) $F \cap \delta(F)=\phi$, whenever $L \in \mathbf{S}$,

(2) $x \in \delta(F)$ implies $x^{oo} \in \delta(F)$,

(3) $x \in F$ implies $x^{o} \in \delta(F)$,

(4) $F=L$ if and only if $\delta(F)= L$,

(5) $F \subseteq G$ implies $\delta(F) \subseteq \delta(G)$,

(6) $\delta(D(L))=\{0\}$,

(7) $\delta(F)$ is a prime, whenever F is a prime filter of L.

Proof: (1) Suppose $x \in F \cap \delta(F)$. Then $x \in F$ and $x^{o} \in F$. Since F is a filter and L is a Stone algebra, we get $0=x^{o} \wedge x \in F$, which is a contradiction. Therefore $F \cap \delta(F)=\phi$.

(2) Let $x \in \delta(F)$. Then $x^{ooo}=x^{o} \in F$ implies $x^{oo} \in \delta(F)$.

(3) Let $x \in F$. Then $x^{oo} \ge x \in F$ implies $x^{o} \in \delta(F)$.

(4) Let $F=L$. Then $0=0^{oo} \in F$ implies $1=0^{o} \in \delta(F)$. Therefore $\delta(F)= L$. Conversely, let $\delta(F)=L$. Then $1^{oo}=1 \in F$. Hence $0=1^{o}\in \delta(F)$. Then $\delta(F)= L$.

(5) Let $F \subseteq G$. Suppose $x \in \delta(F)$. Then $x^{o} \in F \subseteq G$. Therefore $x \in \delta(G)$ and $(\delta(F) \subseteq \delta(G)$.

(6) Let $x \in \delta(D(L))$. Then $x^{o} \in \delta(D(L)$. Hence $x\le x^{oo}=0$. Therefore $\delta(D(L))=\{0\}$.

(7) Let F be a prime filter of L. Assume $x \wedge y \in \delta(F)$ and $y \notin F$. Then $x^{o} \vee y^{o} =(x \wedge y)^{o}$ 2 F and $y^{o} \in F$. Since F is prime filter, then $x^{o} \in F$. Hence $x^{o} \in \delta(F)$. Therefore $\delta(F)$ is prime ideal of L.

The concept of δ-Ideals is introduced in the following.

Definition 3.4

Let L be an *MS*-algebra. An ideal I of L is called a δ-Ideal if $I=\delta(F)$ for some filter F of L.

Example 3.5

Let $L=(0, x, y, z, 1 : 0 < x < y < z < 1\}$ be a five element chain and $x^{o}=x$, $y^{o}= z^{o}=0$. Clearly $(L, {}^{o})$ is an *MS*-algebra. We observe that the ideals $\{0\}$, $\{0, x\}$ and L are δ-Ideals of L but the ideals $\{0, x, y\}$ and $\{0, x, y, z\}$ are not.

Lemma 3.6

A proper δ-ideal of an *MS*-algebra L contains no dense element.

Proof: Let I be a proper δ-Ideal. Then $I=\delta(F)$ for some filter F of L. Suppose $x \in \delta(F) \cap D(L)$. Then we get $0=x^{o} \in F$, which is a contradiction. Therefore $\delta(F) \cap D(L)= \phi$.

The following lemma produces some more examples for δ-Ideals of an *MS*-algebra from the subvariety $\mathbf{K}_2$.

Lemma 3.7

Let L be an *MS*-algebra from $\mathbf{K}_2$. Then we have

(1) $L\wedge$ is a δ-Ideal of L,

(2) Every prime ideal P with $P \cap L\wedge= \phi$ and $L\wedge \subseteq P$ is a δ-Ideal of L, whenever L has no fixed point.

Proof

(1) It is known that, if $L \in \mathbf{K}_2$, then $L^{\vee}=\{x \vee x^{o} : x \in L\}$ is a filter of L, $L^{\wedge}=\{ x \wedge x^{o}: x \in L\}$ is an ideal of L and $x \in L^{\wedge} \Leftrightarrow x^{o} \in L^{\vee}$ for all $x \in L$. It is enough to deduce that $\delta(L^{\vee})= L^{\wedge}$. Let $x \in \delta(L^{\vee})$. Then $x^{o} \in L^{\vee}$, which yields $x \le x^{oo} \in L^{\wedge}$. Then $x \in L^{\wedge}$. Conversely, let $x \in L^{\wedge}$. Then $x^{o} \in L^{\vee}$. Therefore $x \in \delta(L^{\vee})$. Consequently $L^{\wedge}$ is a δ-ideal of L.

(2) Suppose that P is a prime ideal of L such that $P \cap L^{\wedge}= \phi$ and $L^{\wedge} \subseteq P$. Let $x \in P$. Then $x \wedge x^{o} \in L^{\wedge}$ and $x \vee x^{o} \in L^{\vee}$. Hence $x \vee x^{o} \notin P$. Thus we get $x^{o} \notin P$, which yields that $x^{o} \in (L\text{-}P)$. Thus $x \in \delta(L\text{-}P)$. Therefore $P \subseteq \delta(L\text{-}P)$. Conversely, let $x \in \delta(L\text{-}P)$. Then $x^{o} \in (L\text{-}P)$. Thus $x^{o} \notin P$. Now $x \wedge x^{o} \in P$ and P is prime imply $x \in P$. Hence $\delta(L\text{-}P) \subseteq P$. Therefore P is a δ-ideal of L.

Now, let us denote the set of all δ-Ideals of L by $I^{\delta}(L)$. Then, in the following Theorem, we prove that $I^{\delta}(L)$ forms a complete distributive lattice.

Theorem 3.8

Let L be an *MS*-algebra. Then $I^{\delta}(L)$ forms a complete distributive lattice.

Proof: It is obviously that $\{0\}$ and L are the smallest and the greatest δ-Ideals of L. Now, for every two δ-Ideals I and J we prove that $I \cap J$ and $I \vee J$ are again δ-Ideals. Since I and J are δ-Ideals, then there exist filters F and G of L such that $I=\delta(F)$ and $J=\delta(G)$. So we have to show the following:

$\delta(F \cap G)=\delta(F) \cap \delta(G)$ and $\delta(F \vee G)=\delta(F) \vee \delta(G)$.

Since $F \cap G \subseteq F$ and $F \cap G \subseteq G$, then by Lemma 3.2(5), we get $\delta(F \cap G) \subseteq \delta(F) \cap \delta(G)$. Conversely, let $x \in \delta(F) \cap \delta(G)$. Then $x^{o} \in F \cap G$. Hence $x \in \delta(F \cap G)$. Therefore $\delta(F) \cap \delta(G) \subseteq \delta(F \cap G)$. Now, $\delta(F \vee G)$ is a δ-Ideal of L. Since $\delta(F), \delta(G) \subseteq \delta(F \vee G)$, then $\delta(F \vee G)$ is an upper bound of $\delta(F)$ and $\delta(G)$ in $I^{\delta}(L)$. Let $\delta(H)$ be a δ-Ideal of L such that $\delta(F) \subseteq \delta(H)$ and $\delta(G) \subseteq \delta(H)$ where H is a filter of L. We claim that $\delta(F \vee G) \subseteq \delta(H)$. Let $x \in \delta(F \vee G)$, then $x^{o} \in F \vee G$. Hence $x^{o} =f \wedge g$ for some $f \in F$ and $g \in G$. Since $f^{o} \in \delta(F)$ and $g^{o} \in \delta(G)$ (see Lemma 3.3(3)), then $f^{o} \in \delta(H)$ and $g^{o} \in \delta(H)$. Now we have

$$f^o \in \delta(H) \text{ and } g^o \in \delta(H) \Rightarrow f^o \vee g^o \in \delta(H)$$
$$\Rightarrow x^{oo}=(f \wedge g)^o \in \delta(H)$$
$$\Rightarrow x \in \delta(H) \text{ by Lemma 3.3(2).}$$

Hence $\delta(F \vee G)$ is the supreumum of both $\delta(F)$ and $\delta(G)$ in $I^\delta(L)$. Therefore $(I^\delta(L), \cap, \vee, \{0\}, L)$ is a bounded sublattice of the lattice $I(L)$ of all ideals of L. Hence $I^\delta(L)$ is a bounded distributive lattice. It is clear that $I^\delta(L)$ is a partially ordered set with respect to set-inclusion. Then by the extension of the properties $\delta(F \cap G)=\delta(F) \cap \delta(G)$ and $\delta(F \vee G)=\delta(F) \vee \delta(G)$, we can obtain that $I^\delta(L)$ is a complete lattice. Therefore $I^\delta(L)$ is a complete distributive lattice.

Definition 3.9

A δ-Ideal I of an *MS*-algebra L is called principal δ-Ideal if there exists $x \in L$ such that $I=\delta([x))$.

It is observed in the following Theorem that any principal ideal generated by a closed element of an *MS*-algebra is a δ- Ideal.

Theorem 3.10

Let L be an *MS*-algebra. Then for any $x \in L$, $\delta([x))$ is a principal δ-Ideal of L.

Proof: It is enough to show that $(x^o]=\delta([x))$. Let $a \in (x^o]$. Then $a \le x^o$. Hence $a^o \ge x^{oo} \ge x$ implies $a^o \in (x^o]$. Thus $a \in \delta([x))$. Conversely, suppose that $a \in \delta([x))$. Then $a \in \delta([x)$ implies $a^o \ge x$. Hence $a \le a^{oo} \le x$. This yields that $a \in (x^o]$. Therefore $(x^o]$ is a δ-Ideal of L.

Some properties of principal δ-Ideal are given in the following:

Lemma 3.11

Let L be an *MS*-algebra. Then we have the following statements:

(1) for all $a \in L$, $\delta([a))=(a^o]$,

(2) for all $a \in L$, $\delta([a))=\delta([a^{oo}))$,

(3) for all $d \in D(L)$, $\delta([d))=\{0\}$,

(4) for all $x \in F$, $\delta([x))=\delta(F)$ for any filter F of L.

Proof: (1) It is clear from the above Theorem 3.10.

(2) Using (1) and the fact, $a^{oo}=a^o$, we get,

(3) $\delta([a^{oo}))=(a^{ooo}]=(a^o]=\delta([a))$.

(5) For every $d \in D(L)$, we have $\delta([d))=d^o=(0]$.

(6) Let $x \in F$. Suppose $y \in \delta([x))$. Then we get,

$$y \in \delta([x)) \Rightarrow y^o \in [x)$$
$$\Rightarrow y^o \ge x \in F$$
$$\Rightarrow y^o \in F$$
$$\Rightarrow y^o \in \delta(F)$$

Therefore $\delta([x)) \subseteq \delta(F)$.

Let us denote that $M^o(L)=\{\delta([x)):x \in L\}=\{(x^o] : x \in L\}$. Then, in the following Theorem, it is observed that $M^o(L)$ is a de Morgan algebra.

Theorem 3.12: For any *MS*-algebra L, $M^o(L)$ is a sublattice of the lattice $I^\delta(L)$ of all δ-Ideals of L and $M^o(L)$ can be made into a de Morgan algebra. Moreover, the mapping $x \mapsto (x^o]$ is a dual homomorphism of L into $M^\delta(L)$.

Proof: Let $\delta([x)), \delta([y)) \in M^o(L)$ for some $x, y \in L$. Then we get $\delta([x)) \cap \delta([y))=\delta([x \vee y)) \in M^o(L)$ and $\delta([x)) \vee \delta([y))=\delta([x \vee y)) \in M^o(L)$. Also, $\{0\}=\delta([1)) \in M^o(L)$ and $L=\delta([0)) \in M^o(L)$. Hence $M^o(L)$ is a bounded sublattice of $I^\delta(L)$ and hence a distributive lattice. Now, define a unary operation on $^-$ $M^o(L)$ by $\overline{\delta([x))}=\delta([x^o))$. Then we have

$$\overline{\overline{\delta([x))}} = \delta([x^{oo}))$$
$$= (x^{ooo}]$$
$$= (x^o]$$
$$= \delta([x)),$$

and

$$\overline{\delta([x)) \vee \delta([y))} = \overline{\delta([x \wedge y))}$$
$$= \delta([x \wedge y)^o))$$
$$= \delta([x^o \vee y^o))$$
$$= \delta([x^o) \cap [y^o))$$
$$= \delta([x^o) \cap \delta([y^o))$$
$$= \overline{\delta([x))} \cap \overline{\delta([y))},$$
$$\overline{\delta([1))} = \delta([0)).$$

Therefore $M^o(L)$ is a de Morgan algebra. The remaining part can be easily observed. A characterization of δ-Ideals in terms of principal δ-Ideals is investigated in the following.

Theorem 3.13: For any ideal I in an *MS*-algebra L, then the following conditions are equivalent:

(1) I is a δ-Ideals

(2) $I=\cup_{a \in I} \delta([a^o))$

(3) For any x, y in L, $\delta([x^o))=\delta([y^o))$ and $x \in I$ imply $y \in I$.

Proof: (1) $\Rightarrow$ (2): Let I be a δ-Ideal. Then $I=\delta(F)$ for some filter F of L. Let $x \in I$. So we get

$$x \in I\ \delta(F) \Rightarrow x^o \in F$$
$$\Rightarrow x^{oo} \in \delta([x^o)) \subseteq \delta(F)$$
$$\Rightarrow x \in \delta([x^o)) \subseteq \bigcup_{a \in I} \delta([a^o)).$$

Then $I \subseteq \cup_{a \in I} \delta([a^o))$ Conversely, let $x \in \cup_{a \in I} \delta([a^o))$. Then we have,

$$x \in \cup_{a \in I} \delta([a^o)) \quad \Rightarrow x \in \delta([y^o)) \text{ for some } y \in I$$
$$\Rightarrow x \in (y^{oo}] \subseteq I \text{ as } y^{oo} \in I$$
$$\Rightarrow \bigcup_{a \in I} ([a^o)) \subseteq I.$$

Then $I = \cup_{a \in I} \delta([a^o))$.

(2) $\Rightarrow$ (3): Let $I = \cup_{i \in I} \delta([a^o))$. Suppose $\delta([x^o))=\delta([y^o))$ and $x \in I$. Then we get,

$$\delta([x^o))=\delta([y^o)) \text{ and } x \in I \Rightarrow \delta([y^o))=\delta([x^o)) \subseteq \bigcup_{a \in I} \delta([a^o))=I$$
$$\Rightarrow y^{oo} \subseteq I$$
$$\Rightarrow y^{oo} \in I \Rightarrow y \in I.$$

(3) $\Rightarrow$ (1): Assume the condition (3). Consider $F=\{x \in L: x^o \in I\}$. Let $x, y \in F$. Then $x^o, y^o \in I$. Hence $(x \wedge y)^o= x^o \vee y^o \in I$. Thus $x \wedge y \in F$. Now let $x \in F$ and $z \in L$ such that $z \ge x$. Then $z^o \le x^o \in I$ implies $z^o \in I$. Thus $z \in F$ and F is a filter of L. We claim that $I= \delta(F)$. Let $x \in \delta(F)$. Then we get,

$$x \in \delta(F) \Rightarrow x^o \in F$$

$\Rightarrow x^{\circ\circ} \in I$

$\Rightarrow x \in I \Rightarrow \delta(F) \subseteq I.$

For the converse, let $y \in I$. We have,

$y \in I$ and $\delta([y^{\circ})) = \delta([y^{\circ\circ\circ})) \Rightarrow y^{\circ\circ} \in I$ by (3)

$\Rightarrow y^{\circ} \in F$

$\Rightarrow y \in \delta(F)$

$\Rightarrow I \subseteq \delta(F).$

Therefore I is a δ-ideal.

δ-Ideals and Homomorphisms of *MS*-algebras

In this section, some properties of the homomorphic images and the inverse images of δ-Ideals are studied. By a homomorphism on an *MS*-algebra L, we mean a lattice homomorphism h satisfying $(h(x))^{\circ} = h(x^{\circ})$ for all $x \in L$.

Theorem 4.1

Let $h: L \to M$ be a homomorphism of an *MS*-algebra L onto an *MS*-algebra M. Then we have,

(1) for any $a \in L$, $h(\delta([a))) = \delta(h([a))$,

(2) for any δ-Ideal I of L, $h(I)$ is a δ-Ideal of M,

for any δ-Ideal I of L, $h(I) = \bigcup_{i\in I} \delta([((h(i))^{\circ})))$.

for any filter F of L, $h(\delta(F)) = \delta(h(F))$

Proof: (1) For all $a \in L$, we get,

$h(\delta([a))) = h((a^{\circ}]) = ((h(a))^{\circ}] = \delta([h(a)) = \delta(h([a))).$

(2) Let I be a δ-ideal of L. Then $I = \delta(F)$ for some filter F of L. Now

$h(I) = h(\delta(F)) = h\{x \in L: x^{\circ} \in F\}$

$= \{h(x) \in M: h(x^{\circ}) \in h(F)\}$

$= \{h(x) \in M: (h(x))^{\circ} \in h(F)\}$

$= \delta(h(F))$:

Then $h(I)$ is a δ-ideal of M as $h(F)$ is a filter of M.

(3) For any δ-ideal I of L, $I = \bigcup_{i\in I} \delta([i^{\circ}))$. Let $x \in h(I)$ then $x = h(i)$ for some $i \in I$. Then $(\delta([x^{\circ})) = \delta([(h(i))^{\circ})) \subseteq \bigcup_{i\in I} \delta([(h(i))^{\circ}))$. Conversely, let, $y \in \bigcup_{i\in I} \delta([(h(i))^{\circ}))$. Now,

$y \in \bigcup_{i\in I} \delta([(h(i))^{\circ})) \Rightarrow y \in \delta([(h(a))^{\circ})),\ a \in I$

$\Rightarrow y \in (((h(a))^{\circ\circ}]$

$\Rightarrow y \in \leq h(a)^{\circ\circ} \in h(I)$ as $a^{\circ\circ} \in I$

$\Rightarrow y \in h(I)$

$\Rightarrow \bigcup_{i\in I} \delta([(h(i))^{\circ})) \subseteq h(I)$

(4) Let $x \in \delta(h(F))$. Then we get,

$x \in \delta(h(F)) \quad \Rightarrow x^{\circ} \in h(F)$

$\Rightarrow x^{\circ} \in h(f), f \in F$

$\Rightarrow x = x^{\circ\circ} = h(f^{\circ})$

$\Rightarrow x \in \delta(h(F))$ as $f^{\circ} \in \delta(F)$ by lemma 3.3(3).

Then $\delta(h(F)) \subseteq h(\delta(F))$. For the converse we have,

$x \in \delta(h(F)) \quad \Rightarrow x = h(y),\ y \in \delta(F)$

$\Rightarrow x^{\circ} = h(y^{\circ}),\ y^{\circ} \in F$

$\Rightarrow x^{\circ} = h(y^{\circ}),\ y^{\circ} \in (h(F)$ as $y^{\circ} \in F$

$\Rightarrow x \in h(\delta(F)).$

Theorem 4.2

Let $f: L \to M$ be a homomorphism of an *MS*-algebra L into an *MS*-algebra M. Then we have,

(1) for any δ-ideal H of M, $f^{-1}(H)$ is a δ-ideal of L,

(2) *Ker* f is a δ-ideal of L.

Proof: (1) Since H is a δ-ideal of M, then $H = \delta(F)$ for some filter F of M. We claim $f^{-1}(H) = \delta(f^{-1}(F))$, where $f^{-1}(H)$ is an ideal of L. Now,

$x \in f^{-1}(H) \quad \Rightarrow f(x) = y,\ y \in H = \delta(F)$

$\Rightarrow (f(x))^{\circ} = f(x^{\circ}) = y,\ y^{\circ} \in F$

$\Rightarrow x^{\circ} \in \{f^{-1}(y^{\circ})\} \subseteq f^{-1}(F)$

$\Rightarrow x \in \delta(f^{-1}(F)).$

Conversely, $x \in \delta(f^{-1}(F))$. Then we have,

$x \in \delta(f^{-1}(F)) \quad \Rightarrow x^{\circ} \in f^{-1}(F)$

$\Rightarrow (f(x))^{\circ} = f(x^{\circ}) \in F$

$\Rightarrow f(x) \in \delta(F) = H$

$\Rightarrow x^{\circ} \in f^{-1}(H)$

$\Rightarrow \delta(f^{-1}(F)) \subseteq f^{-1}(H)$

Therefore $f^{-1}(H)$ is a δ-ideal of L.

(2) Since f is a homomorphism, then $Ker\ f = \{x \in L: f(x) = 0\}$ and $Coker\ f = \{x \in L: f(x) = 1$ are ideal and filter of L respectively. We claim $Ker\ f = (Coker\ f)$. Now

$x \in Ker\ f \Rightarrow f(x) = 0$

$\Rightarrow f(x^{\circ}) = f(x))^{\circ} = 1$

$\Rightarrow x^{\circ} \in Coker\ f$

$\Rightarrow x \in \delta(Coker\ f).$

Then $Ker\ f \subseteq \delta(Coker\ f)$. Conversely,

$x \in \delta(Coker\ f) \Rightarrow x^{\circ} \in Coker\ f$

$\Rightarrow f(x))^{\circ} = f(x^{\circ}) = 1$

$\Rightarrow f(x))^{\circ\circ} = f(x^{\circ\circ}) = 0$

$\Rightarrow x \in ker\ f$

Then $\delta(Coker\ f) \subseteq ker\ f$. Therefore $ker\ f$ is a δ-ideal of L.

Theorem 4.3

Let $h: L \to L_1$ be an onto homomorphism between *MS*-algebras $L = (L, \vee, \wedge, ^{\circ}, 0_L, 1_L)$ and $L_1 = (L_1, \vee, \wedge, ^{\circ}, 0_{L1}, 1_{L1})$. Then we have,

(1) $M^{\circ}(L)$ is homomorphic of $M^{\circ}(L_1)$,

(2) $I^{\delta}(L)$ is homomorphic of $I^{\delta}(L_1)$.

Proof: (1) Define $g: M^{\circ}(L) \to M^{\circ}(L_1)$ by $g(\delta([a))) = \delta([h(a)))$. Clearly, $g(\{0_L\}) = L_1$ and $g(L) = L_1$. For every $\delta([a))), \delta([b))) \in \in M^{\circ}(L)$ we get,

$$g(\delta([a)) \cap \delta([(b)))=g(\delta([a \wedge b)))$$
$$= \delta([h(a \wedge b))$$
$$= \delta([h(a) \wedge h(b)))$$
$$= \delta([h(a)) \cap \delta([h(b))$$
$$= g(\delta([a))) \cap g(\delta([b))),$$

and

$$g(\delta([a)) \vee \delta([(b))) = g(\delta([a \vee b)))$$
$$= \delta([h(a \vee b))$$
$$= \delta([h(a) \vee h(b)))$$
$$= \delta([h(a)) \vee \delta([h(b))$$
$$= g(\delta([a))) \vee g(\delta([b))),$$

also,

$$g\overline{(\delta([a))} = g(\delta([a^{o})))$$
$$= \delta([h[a^{o})))$$
$$= \overline{\delta[h[a)}$$
$$= \overline{g(\delta([a))}$$

Therefore g is a homomorphism of de Morgan algebras $M^{o}(L)$ and $M^{o}(L_1)$.

(2) Define the map π: $I^{\delta}(L) \to I^{\delta}(L_1)$ by $\pi(I)= \delta(h(F))$ where $I= \delta(F)$. It is clear that $\pi\{0_1\}=\{0_{L1}\}$ and $\pi(L)=L_1$. Let $I, J \in I^{\delta}(L)$. Then $I= \delta(F)$ and $J= \delta(G)$, where F and G are filters of L. Then we get,

$$\pi(I \vee J) = \delta(h(F \vee G))$$
$$= \delta(h(F) \vee h(G))$$
$$= \delta(h(F)) \vee \delta(h(G))$$
$$= \pi(I) \vee \pi(J),$$

and

$$\pi(I \cap J) = \delta(h(F \cap G))$$
$$= \delta(h(F) \cap h(G))$$
$$= \delta(h(F)) \cap \delta(h(G))$$
$$= \pi(I) \cap \pi(J)$$

Therefore π is a (0, 1)-lattice homomorphism and the proof is completed.

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Citation: Abd El-Mohsen B (2016) *δ*-Ideals in *MS*-Algebras. J Comput Sci Syst Biol 9: 028-032. doi:10.4172/jcsb.1000217