Ideals of the enveloping algebra $U(\mathrm{osp}(1,2))$ [1]

Č. BURDÍK [a], M. HAVLÍČEK [a], O. NAVRÁTIL [b], and S. POŠTA [a]

[a] Department of Mathematics, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, Trojanova 13, 120 00 Prague 2, Czech Republic
E-mail: burdik@kmlinux.fjfi.cvut.cz

[b] Department of Mathematics, Faculty of Transportation Sciences, Czech Technical University, Na Florenci 25, 110 00 Prague, Czech Republic
E-mail: navratil@fd.cvut.cz

Abstract

We explore the general form of two sided ideals of the enveloping algebra of the Lie superalgebra $\mathrm{osp}(1,2)$. We begin by disclosing the internal structure of $U(\mathrm{osp}(1,2))$ computing the decomposition of adjoint representation. The classification of the ideals we reach is done via presenting generators for the each ideal and by showing that each ideal is generated uniquely.

2000 MSC: 22D30

1 Introduction

Until today there exists no example of enveloping algebra with complete classification of all two-sided ideals with the exception of $U(\mathrm{sl}_2)$ [1]. In this article we present this complete classification for the enveloping algebra of Lie superalgebra $\mathrm{osp}(1,2)$. This Lie superalgebra contains Lie algebra sl_2 as subalgebra, but the classification of all both-sided ideals of $U(\mathrm{osp}(1,2))$ is richer and is not simply related to the classification of ideals of $U(\mathrm{sl}_2)$. The classification of all primitive ideals was done in the paper [2] and is contained in our more general result.

2 Structure of $U(\mathrm{osp}(1,2))$

The five dimensional Lie superalgebra $g = \mathrm{osp}(1,2)$ has enveloping algebra $U = U(g)$ which is complex associative algebra generated by elements $E^{\pm}$, H and $F^{\pm}$ satisfying the following commutation relations:

$$[H,E^{\pm}] = \pm E^{\pm}, \quad [E^+,E^-] = 2H, \quad [H,F^{\pm}] = \pm\tfrac{1}{2}F^{\pm}$$
$$\{F^+,F^-\} = \tfrac{1}{2}H, \quad [E^{\pm},F^{\mp}] = -F^{\pm}, \quad \{F^{\pm},F^{\pm}\} = \pm\tfrac{1}{2}E^{\pm}$$

Here $[x,y] = xy - yx$ denotes the commutator and $\{x,y\} = xy + yx$ the anti-commutator. It is well known that the Poincaré-Birkhoff-Witt theorem holds in this algebra and the basis of U can be taken as ordered monomials

$$E^{-\alpha}E^{+\beta}H^{\gamma}F^{-\delta}F^{+\varepsilon}, \quad \alpha,\beta,\gamma \geq 0,\ \delta,\varepsilon \in \{0,1\}$$

The adjoint representation of g, i.e the mapping $\mathrm{ad} : g \to L(U)$ defined for $a = a_1 \ldots a_k \in U,\ x, a_1, \ldots, a_k \in g$ by

$$\mathrm{ad}(x)a = [\![x,a]\!] = xa - (-1)^{\deg x \deg a}ax \tag{2.1}$$

[1] Presented at the 3rd Baltic-Nordic Workshop "Algebra, Geometry, and Mathematical Physics", Göteborg, Sweden, October 11–13, 2007.

where the degree is defined by the formulas

$$\begin{aligned}&\deg E^{\pm} = \deg H = 0, \quad \deg F^{\pm} = 1\\ &\deg a_1 \dots a_k = \deg a_1 + \dots + \deg a_k \quad \text{for } a_1, \dots, a_k \in g\end{aligned}$$

acts as a (super)derivation. It means that if we want to define adjoint representation on the whole space U we can take

$$\mathrm{ad}(u)a = \mathrm{ad}(x_1)\dots\mathrm{ad}(x_n)a = [\![x_1, [\![\dots, [\![x_n, a]\!]\dots]\!]]\!] \quad \text{for } u = x_1 \dots x_n, \quad x_i \in g$$

and proceed by linearity.

Now we compute the decomposition of U similarly as in the case of $U(\mathrm{sl}_2)$. The enveloping algebra U possess a natural filtration $\{U_n\}_{n\geq 0}$, $U_n \subset U_{n+1}$, given by degree n of elements in U_n. It is easily seen that adjoint representation has U_n as its invariant subspace (by applying supercommutator we can not obtain element of higher degree). It is completely reducible on each U_n [3] i. e. we can see U_n as a direct sum of invariant subspaces generated by certain highest weight vectors. The highest weight vector v of weight m satisfies relations

$$[\![E^+, v]\!] = 0, \quad [\![F^+, v]\!] = 0, \quad [\![H, v]\!] = mv$$

From these relations we can directly find that all highest weight vectors of small degree. Let us denote $[\![X]\!] = \mathrm{ad}(U)(X)$, where $X \subset U$ is any subset; for set containing only one element we shorten $[\![\{x\}]\!] = [\![x]\!]$. Commuting highest weight vectors of small degree we see that

$$\begin{aligned}&U_0 = [\![1]\!]\\ &U_1 = [\![E^+]\!] \oplus U_0\\ &U_2 = [\![E^{+2}]\!] \oplus [\![A]\!] \oplus [\![C]\!] \oplus U_1\\ &\dots\end{aligned}$$

where

$$C = 2E^-E^+ + 4F^-F^+ + 2H^2 + H$$

is Casimir element which generates the center of the algebra [3]; element

$$A = 3F^+ + 4E^+F^- - 4HF^+$$

has the weight $\frac{1}{2}$. Note that the element $C^\alpha E^{+\beta} A^\gamma$ has the weight $\beta + \frac{1}{2}\gamma$ and dimension of the space generated by the highest weight element with the weight m is $4m+1$. Because the elements F^+ and F^- satisfy the relations

$$F^{+2} = \tfrac{1}{4}E^+, \quad F^{-2} = -\tfrac{1}{4}E^- \tag{2.2}$$

we deduce that the highest weight element A can't be presented in the decomposition in the power greater than one. Thus we can claim that for any $n \geq 0$

$$U_n = \bigoplus_{\substack{\alpha,\beta\geq 0,\ \gamma\in\{0,1\}\\ 2\alpha+\beta+2\gamma\leq n}} [\![C^\alpha E^{+\beta} A^\gamma]\!] \tag{2.3}$$

The proof of this claim is based on dimensional check. It's not difficult to see that the sum $[\![v_1]\!] + [\![v_2]\!]$, where v_1, v_2 are highest weight vectors, is direct if and only if $v_1,\ v_2$ are linear independent. The vectors $C^\alpha E^{+\beta} A^\gamma$ are linear independent for different α, β, γ. The representation generated by the highest weight vector $C^\alpha E^{+\beta} A^\gamma$ has the dimension $4\beta + 2\gamma + 1$.

On the other hand the dimension of U_n is also easy to determine. The dimension of vector space of homogeneous polynomials of degree d in k variables is $\binom{k+d-1}{d}$. Due to the relation (2.2) we must consider only monomials which have zero or one factor equal to $F^{\pm}$. For the dimension of U_n we have the following recurrence relation:

$$\dim U_n = \dim U_{n-1} + \binom{n}{n-2} + 2\binom{n+1}{n-1} + \binom{n+2}{n}$$

(if we want to construct monomial of degree n we take into account monomials from three elements $E^{\pm}$, H of degree $n-2$ to which we append F^-F^+, monomials of degree $n-1$ to which we append F^- or F^+ and finally monomials of degree n). Simplifying we get

$$\dim U_n = \tfrac{1}{3}\,(2n^3 + 6n^2 + 7n + 3)$$

For the dimension of the space on the right hand side of (2.3) we get the same result:

$$\sum_{\substack{\alpha,\beta\geq 0,\ \gamma\in\{0,1\}\\ 2\alpha+\beta+2\gamma\leq n}} (4\beta + 2\gamma + 1) = \tfrac{1}{3}\,(2n^3 + 6n^2 + 7n + 3)$$

The following decomposition of adjoint action therefore takes place:

$$U = \bigoplus_{\substack{\alpha,\beta\geq 0\\ \gamma\in\{0,1\}}} [\![C^{\alpha} {E^+}^{\beta} A^{\gamma}]\!] \tag{2.4}$$

3 Important relations in $U(\mathrm{osp}(1,2))$

By direct calculation, we see that if I is any both-sided ideal of U, the following important implications hold in U:

$$ {E^+}^n \in I \Rightarrow \Big(C - \tfrac{1}{2}\,n(n-1)\Big){E^+}^{n-1} \in I \tag{3.1}$$

$$ {E^+}^n A \in I \Rightarrow \Big(C - \tfrac{1}{2}\,n(n+1)\Big){E^+}^{n-1} A \in I \tag{3.2}$$

$$ {E^+}^n A \in I \Rightarrow \Big(C + \tfrac{1}{8}\Big){E^+}^{n+1} \in I \tag{3.3}$$

Using these relations, the structure of ideals generated by the highest weight elements ${E^+}^n$ and ${E^+}^n A$ can be obtained. Let $s, n \in \{0, 1, 2, \ldots\}$ and denote

$$f_{n,s} = \prod_{k=s+1}^{n} \Big(C - \tfrac{1}{2}\,k(k-1)\Big)$$

Then

$$\Big({E^+}^n\Big) = \bigoplus_{s=0}^{\infty}\Big(f_{n,s}\mathbb{C}[C][\![{E^+}^s]\!] \oplus f_{n,s+1}\mathbb{C}[C][\![{E^+}^s A]\!]\Big)$$

$$\Big({E^+}^n A\Big) = \bigoplus_{s=0}^{\infty}\Big(\Big(C + \tfrac{1}{8}\Big) f_{n+1,s}\mathbb{C}[C][\![{E^+}^s]\!] \oplus f_{n+1,s+1}\mathbb{C}[C][\![{E^+}^s A]\!]\Big)$$

where (x) means ideal generated by element x.

Let's now have any both-sided ideal $I \subset U$. Because U is Noetherian ring, I is finitely generated, so we can write

$$I = (x_1, \ldots, x_n) \tag{3.4}$$

for some $x_i \in U$. Thanks to decomposition (2.4) we can replace x_i's by certain highest weight vectors. There exist numbers $n_i \geq 0$, $\gamma_i \in \{0,1\}$ and complex polynomials P_i, $i = 1, \ldots, m$ such that

$$I = (P_1(C)E^{+n_1}A^{\gamma_1}, P_2(C)E^{+n_2}A^{\gamma_2}, \ldots, P_m(C)E^{+n_m}A^{\gamma_m}) \tag{3.5}$$

Now we reduce the number of generators used in (3.5) by successive replacing the generators by more suitable ones.

First, if there are two generators having the same n's and same γ's, say $P_1(C)E^{+n}A^{\gamma}$ and $P_2(C)E^{+n}A^{\gamma}$, we can replace them by one generator $\gcd\{P_1(C), P_2(C)\}E^{+n}A^{\gamma}$.

Further simplification is possible due to the relations (3.1) and (3.2). Let there be, say, two generators, $P_1(C)E^{+n_1}$ and $P_2(C)E^{+n_2}$, where $n_1 < n_2$. First, we may assume without loss of generality that $P_2|P_1$. Next, we can replace the two generators by the suitable couple $P(C)Q(C)E^{+n_1}$ and $P(C)E^{+n_2}$, where $P(C) = \gcd\{P_1(C), P_2(C)\}$ and $Q(C)|f_{n_2,n_1}$. (The similar simplification applies to the generators having γ's$=1$.)

And thirdly, assume there are two generators in the list of the form $Q(C)E^{+n_1}$ and E^{+n_2}, $n_1 < n_2$, $Q(C)|f_{n_2,n_1}$. Then it is possible to replace them by new couple $Q(C)f_{n_1,0}$ and E^{+n_2}. (Again, the similar simplification applies to the generators having γ's$=1$.)

After finite number of steps, we are able to get the following: Every ideal I can be written of the form

$$I = \Big(P_1(C)E^{+n},\ P_1(C)Q_1(C),\ P_2(C)E^{+m}A,\ P_2(C)Q_2(C)A\Big) \tag{3.6}$$

where $n, m \geq 0$ and $P_1(C)$, $Q_1(C)$, $P_2(C)$ and $Q_2(C)$ are four polynomials such that

$$\Big(C - \tfrac{1}{2}\,n(n-1)\Big)\big|Q_1(C)\big|f_{n,0}\,, \qquad \Big(C - \tfrac{1}{2}\,m(m+1)\Big)\big|Q_2(C)\big|f_{m+1,1}$$

The form (3.6) can still be simplified using the relation (3.3). Finally we can reach the form

$$I = \Big(P(C)\Big(C+\tfrac{1}{8}\Big)^{\gamma}E^{+N+1}, P(C)\Big(C+\tfrac{1}{8}\Big)^{\gamma}Q(C)C^{\delta}, P(C)E^{+N}A, P(C)Q(C)A\Big) \tag{3.7}$$

where $M \geq 0$, $\gamma, \delta \in \{0,1\}$, P is some polynomial and Q is such that

$$\big(C - \tfrac{1}{2}\,M(M+1)\big)\big|Q(C)\big|f_{M+1,1}(C)$$

The uniqueness of ideal (3.7) is proven by exploring its internal structure. It can be shown that I having the form (3.7) decomposes as the following:

$$I = \bigoplus_{s=0}^{+\infty} \mathbb{C}[C]\llbracket I_s^0 E^{+s}\rrbracket \oplus \mathbb{C}[C]\llbracket I_s^1 E^{+s}A\rrbracket \tag{3.8}$$

where

$$I_s^0 = \gcd\Big\{\Big(C+\tfrac{1}{8}\Big)^{\gamma}Pf_{M+1,s}, \Big(C+\tfrac{1}{8}\Big)^{\gamma}C^{\delta}PQ\Big\}$$
$$I_s^1 = \gcd\Big\{Pf_{M+1,s+1}, PQ\Big\}$$

Using (3.8) we can state that the form (3.7) is unique, i. e. for different numbers M, γ, δ and polynomials P, Q we get different ideals.

4 Conclusion

We have shown that the most general form of every two sided ideal of the enveloping algebra $U(\mathrm{osp}(1,2))$ is given by formula (3.7). We have found that this form is unique for each ideal and thanks to this uniqueness we have obtained complete classification of both sided ideals of U. It would be nice to explore the origin of the fact that the surprisingly richer structure arises when we compare to the case of $U(\mathrm{sl}_2)$.

Acknowledgement

This work was partially supported by GACR 201/05/0857.

References

[1] S. Catoiu. Ideals of the Enveloping Algebra $U(\mathrm{sl}_2)$. J. Algebra, **202** (1998), 142-177.
[2] G. Pinczon. The Enveloping Algebra of the Lie Superalgebra $\mathrm{osp}(1,2)$. J. Algebra, **132** (1990), 219-242.
[3] M. Scheunert. *The Theory of Lie Superalgebras.* Springer-Verlag, New York, 1979.
[4] J. Dixmier. *Algebres Enveloppantes.* Gauthier-Villars Editeur, Paris, 1974.
[5] B. Konstant. Lie group representations on polynomial rings. Am. J. Math. **85** (1963), 327-404.
[6] A. Kirillov. *Elements of Representation Theory.* Nauka, Moscow, 1972 (in Russian).
[7] D. Flath. Decomposition of the enveloping algebra of sl_3. J. Math. Phys. **31** (1990), 1076-1077.

Received December 12, 2007
Revised March 08, 2008