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Induced Riemannian Structures and Topology of Null Hypersurfaces in Lorentzian Manifold

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Abstract

Given a null hypersurface of a Lorentzian manifold, we induce a Riemannian metric on the null hypersurface using a normalizing field defined on some open set containing the null hypersurface. We establish links between the null geometry and basics invariants of the associated Riemannian metric. This allowed us, using some comparison theorems from Riemannian geometry, to get important results on the topology of the null hypersurface.

Keywords: Null hypersurface; Rigging; Closed normalization; Associated Riemannian metric

Introduction

In spite sharing common root, Lorentzian and Riemannian geometries diverge very fast. For example, in Lorentzian case, due to the causal character of three categorie of vector fields (namely, spacelike, timelike and null), the induced metric on a hypersurface is a non-degenerate metric tensor field or degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On no-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) of the unit orthogonal vector field is known to lead to a canonical splitting of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations, along with the second fundamental form, shape operator, induced connection, etc. The case the normal vector field is null, the hypersurface is called null (or lightlike). Null hypersurfaces are then exclusive objects from Lorentzian manifolds, and have not Riemannian counterpart, making them interesting by their own from a geometric point of view, but also they are key objects for modern physics (quantum gravity effects). The geometry of null submanifolds is different and rather difficult since (contrary to the non-degenerate conterpart) the normal vector bundle intersects (non trivially) with the tangent bundle. Thus, one can not find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects as usual. This degenerancy of the induced metric makes impossible to study them as part of standard submanifold theory, forcing to develop specific techniques and tools. For the most part, these tools are specific to a given problem, or sometimes with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Indeed, Duggal and Bejancu introduced a non-degenerate screen distribution or equivalently a null transversal line vector bundle as we may see below so as to get a three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, shape operators, induced connection, curvature, etc. [1]. Unfortunately, the screen distribution is not unique and there is no preferred one in general. The least we can say is that for the above approach to be complete and consistent, we still need to build a distinguished normaization to accompany it. Most of the recent works of the first named author are indeed devoted to this normalization problem [2-4]. Given the collective expertise in Riemannian geometry, the ideal situation on could expect is that the developed tools could bring to a full reduction of problems in null geometry to purely Riemannian ones. In, the present first named author, after fixing a pair of normalization, constructed an associated Riemannian metric to the "normalized null structure" [5,6]. These ideas have been generalized and improved where authors used riggings defined on neighborhood of the null hypersurface. In the present paper, we first consider the associated Riemannian metric as [5] but arising from a null rigging defined on neighborhood of the null hypersurface, and establish links between the null geometry and basics invariants of the associated Riemannian metric. Also, note that one of the major issues of Riemannian geometry is how to obtain topological or differential properties of a manifold from some known properties of its curvatures. For example what can be said about a complete Riemannian manifold when some suitable estimates are know for the sectional or Ricci curvature? These considerations have been on much scrutinity with excellent results: Myers (compactness), Klingenberg (on the injectivity radius), Cheeger-Gromoll (splitting theorem), Shoen-Yau (3-manifolds that are diffeomorphic to the standard R^3, Gromov's estimate of the number of generators of the fundamental group and the Betti numbers when lower curvature bounds are given. For further background on this problem we refer to the excellent texts [7-11]. Since in the present paper we have established links between the null geometry and basics invariants of the associated Riemannian metric, it is reasonnable to expect that the geometry of the null hypersurface provides insight informations on its topology. This constitutes our second and main goal. The plan of the article is as follows. Section (2) sets notations and definitions on riggings (normalizations) and review basics properties on null hypersurfaces. The associated Riemannian distance structure on the rigged (or normalized) null hypersurface are introduced and discussed. The relashionship between the null and the associated Riemannian geometry is considered in section (4) where we proceed to a connection of the main geometric objects (invariants) of both side involved in our analysis. In the last sections, thanks to some Riemannian comparison theorems we get some topological facts on the null hypersurfaces from its null geometry.

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Riggins and Preliminaries of Null Hypersurfaces

Let $(\overline{M}, \overline{g})$ be a $(n+2)$-dimensional Lorentzian manifold and M a null hypersurface in $\overline{M}$. This means that at each $p \in M$, the restriction $\overline{g}_{p_{|T_pM}}$ is degenerate, that is there exists a non-zero vector $U \in T_pM$ such that $\overline{g}(U, X) = 0$ for $X \in T_pM$. Hence, in null setting, the normal bundle $TM^{\perp}$ of the null hypersurface M^{n+1} is a rank 1 vector subbundle of the tangent bundle TM, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection $\{0\}$ with the tangent one plays an important role in the introduction of the main induced geometric objects on M. Let start with the usual tools involved in the study of such hypersurfaces according to Duggal [1]. They consist in fixing on the null hypersurface a geometric data formed by a lightlike section and sreen distribution. By screen distribution on M^{n+1}, we mean a complementary bundle of $TM^{\perp}$ in TM. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface M be paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM / TM^{\perp}$. For reasons that will become obious in few lines below, let denote such a distribution by $S(N)$. We then have,

$$TM = \mathcal{S}(N) \oplus_{orth} TM^{\perp}. \tag{1}$$

Where $\oplus_{orth}$ denotes orthogonal direct sum. From ref. [1], it is know that for a null hypersurface equipped with aa screen distribution, there exists a unique rank 1 vector subbundle $tr(TM)$ of $T\overline{M}$ over M, such that for any non-zero section ξ of $TM^{\perp}$ on coordinate neighborhood $\mathcal{U} \subset M$ there exists a unique section N of $tr(TM)$ on $\mathcal{U}$ satisfying:

$$\overline{g}(N, \xi) = 1, \overline{g}(N, N) = \overline{g}(N, W) = 0 \ \forall W \in \Gamma(\mathcal{S}N)_{|\mathcal{U}}. \tag{2}$$

Then $T\overline{M}$ is decomposed as following:

$$T\overline{M} = \mathcal{S}(N) \oplus_{orth} \left(TM^{\perp} \oplus tr(TM)\right) = TM \oplus tr(TM). \tag{3}$$

We call $tr(TM)$ a (null) transversal vector bundle along M. In fact, from eqns. (2) and (3) one shows that, conversely, a choice of a transversal bundle $tr(TM)$ determines uniquely the distribution $\mathcal{S}(N)$. A vector field N as in eqn. (2) is called a null transversal vector field of M. It is then noteworthy that the choice of null transversal vector field N along M determines both the null transversal vecto r bundle, the screen distribution $\mathcal{S}$ and a unique radical vector field, say satisfying eqn. (2). Now, to continue our discussion, we need to clarify the concept of rigging for our null hypersurface.

Definition 2.1

Let M be a null hypersurface of a Lorentzian manifold. A rigging for M is a vector field L defined on some open set containing M such that $L_p \notin T_pM$ *for each* $p \in M$.

We say that we have a null rigging in case the restriction of L to the null hypersurface is a null vector field. From now on we fix a null rigging N for M. In particular this rigging fixes a unique null vector field $\xi \in \Gamma(TM^{\perp})$ called the rigged vector field, all of them defined in an open set containing M (hence globally on M) such that eqns. (1)-(3) hold. Whence, from now on, by a normalized (or rigged) null hypersurface we mean a triplet (M,g,N) where $g = \overline{g}_{\mathcal{M}}$ is the induced metric on M and N a null rigging for M. In fact, in case the ambient manifold $\overline{M}$ has Lorentzian signature, at an arbitrary point p in M, a real null cone C_p is invarianty defined in ambient tangent space $T_p\overline{M}$ and is tangent to M along a generator emanating from p. this generator is exactly the radical fiber $\Delta_p = T_pM^{\perp}$ and for each null rigging N for M and each $p \in M$ we have $N_p \in C_p \setminus \Delta_p$. Actually, a lightlike hypersurface M of a Lorentzian manifold is a hypersurface which is tangent to the lightlike cone C_p at each point $p \in M$. Recall that a spacetime $(\overline{M}, \overline{g})$ is a connected Lorentzian manifold which is "time-oriented", i.e a causal cone at each $T_p\overline{M}, p \in \overline{M}$ (the "future"causal cone) has been continuously chosen. Hence, null hypersurfaces in spacetimes can be naturally given an orientation by such a continuous distribution of causal cones c_p. Let N be a null rigging of a null hypersurface of a Lorentzian manifold $(\overline{M}, \overline{g})$ and $\theta = \overline{g}(N, \hat{A}\cdot)$ the 1-form metrically equivalent to N. Then, take

$$\eta = i^{\star}\theta \tag{4}$$

to be its restriction to M, the map $i : M \to \overline{M}$ being the inclusion map. The normalized null hypersurface (M,g,N) will said to be closed if θ is closed on M. It is easy to check that $\mathcal{S}(N)=ker$ and that the screen distribution $\mathcal{S}(N)$ is integrable whenever is closed.

On normalized null hypersurface (M,g,N), the local Gauss and Weingarten type formulas are given by:

$$\overline{\nabla}_X Y = \nabla_X Y + B^N(X, Y)N, \tag{5}$$

$$\overline{\nabla}_X N = -A_N X + \tau^N(X)N, \tag{6}$$

$$\nabla_X PY = \nabla^*_X PY + C^N(X, PY)\xi, \tag{7}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau^N(X)\xi, \tag{8}$$

for any $X, Y \in \Gamma(TM)$, where $\overline{\nabla}$ denotes the Levi-civita connection on $(\overline{M}, \overline{g}), \nabla$ denotes the connection on M induced from $\overline{\nabla}$ through the projection along the null rigging N, ∇^* denotes the Levi-Civita connection on the screen distribution $\mathcal{S}(N)$ induced from∇ through the projection morphism P of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}(N))$ with respect to the decomposition. Now the (0,2) tensor B^N and C^N are the second fundamental forms on TM and $\mathcal{S}(N)$ respectively, A_N and $A^{\star}_{\xi}$ are the shape operators on TM and $\mathcal{S}(N)$ respectively and ^{N} a 1-form on TM defined by $\tau^N(X) = \overline{g}(\overline{\nabla}_X N, \xi)$. For the second fundamental $B^N$and $C^N$the following hold:

$$B^N(X,Y) = g(A^*_{\xi}X, Y), C^N(X, PY) = g(A_N X, PY), \overline{g}(A_N Y, N) = 0. \overline{g}(A^*_{\xi}X, N) = 0 \tag{9}$$

$\forall X, Y \in \Gamma(TM)$, and

$$B^N(X, \xi) = 0, A^{\star}_{\xi} = 0. \tag{10}$$

It follows from eqn. (10) that integral curves of ξ are pregeodesicin both $\overline{M}$ and M as consider these integral curves to be geodesics which means that

$$\tau^N(\xi) = 0 \tag{11}$$

A null hypersurface M is called totally umbilical (resp. geodesic) if there exists a smooth function ρ on M such that at each $p \in M$ and for all $u,v \in T_pM$ $BN(u,v)_p=(p)g_p(u,v)$resp B^Nvanishes identicallynon M). These are intrinsic notions on any null hypersurface in the following way. Note that N being a null rigging for M, a vector field $\widetilde{N} \in \Gamma(T\overline{M})$ is a null rigging for M if and only if it is defined in an open set containing M and there exist a function ψ and section ζ of TM with the properties that $\psi \circ i$ is nowhere vanishing, being i the inclusion map and $2\psi \tilde{A}\theta(\zeta) \| \zeta \|^2 = 0$.

Then we have for details on changes in normalizations) $B^{\widetilde{N}} = \frac{1}{\psi \circ i} B^N$ which shows that total umbilicity and totally geodesibility are intrinsic properties for M [3]. The total umbilicity and the total geodesibility conditions for M can also be written respectively as $A^{\star}_{\xi} = \rho P$ and $A^{\star}_{\xi} = 0$. Also, the screen distribution $\mathcal{S}(N)$ is totally umbilical (resp. totally geodesic) if $C^N(X, PY) = \lambda g(X, Y)$ for all $X,Y \in \Gamma(TM)$ resp. $C^N=0$) which is equivalent to $A_N = \lambda P$ (resp $A_N=0$). It is noteworthy

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to mention that the shape operators A_ξ^{star} and A_N are $\mathcal{S}(N)$-valued. The induced connection ∇ is torsion-free, but not necessarily g-metric unless M be totally geodesic. In fact we have for all tangent vector fields X,Y and $Z\in TM$:

$$(\nabla_X g)(Y,Z) = B^N(X,Y)\eta(Z) + B^N(X,Z)\eta(Y). \quad (12)$$

Let denote by $\overline{R}$ and R the Riemann curvature tensors of $\overline{\nabla}$ and ∇, respectively. Recall the following Gauss-Codazzi equations for all $X,Y,Z\in\Gamma(TM)$, $N\in tr(TM)$ $\xi\in\Gamma(TM^\perp)$.

$$\overline{g}(\overline{R}(X,Y)Z,\xi) = (\nabla_X B^N)(Y,Z) - (\nabla_Y B^N)(X,Z) + B^N(Y,Z)\tau^N(X)$$

$$-B^N(X,Z)\tau^N(Y). \quad (13)$$

$$\overline{g}(\overline{R}(X,Y)Z,PW) = g(R(X,Y)Z,PW) + B^N(X,Z)C^N(Y,PW)$$

$$-B^N(Y,Z)C^N(X,PW) \quad (14)$$

$$\overline{g}(\overline{R}(X,Y)\xi,N) = \overline{g}(R(X,Y)\xi,N)$$

$$= C^N(Y,A_\xi^* X) - C^N(X,A_\xi^* Y) - 2d\tau^N(X,Y). \quad (15)$$

The shape operator $A_\xi^\star$ is self-adjoint as the second fundamental form B^N is symmetric. However, this is not the case for the operator A_N as show in the following lemma.

Lemma 2.1: *For all X,YTM*

$$\langle A_N X,Y\rangle - \langle A_N Y,X\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X) - 2d\eta(X,Y) \quad (16)$$

where (throughout) $\langle,\rangle = \overline{g}$ stands for the Lorentzian metric.

Proof. Recall that $\eta=i^*\theta$ where $\theta=\langle N,.\rangle$ taking the differential of θ and using the weingarten formula, we have for all $X,Y\in (TM)$

$$2d\eta(X,Y) = 2d\theta(X,Y) = \langle\overline{\nabla}_X N,Y\rangle - \langle\overline{\nabla}_Y N,X\rangle$$

$$= -\langle A_N X,Y\rangle + \tau^N(X)\eta(Y) + \langle A_N Y,YX\rangle - \tau(Y)\eta(X).$$

Hence

$$\langle A_N X,Y\rangle - \langle A_N Y,X\rangle = \tau^N(X)\eta(Y) - \tau^N(Y)\eta(X). \quad (17)$$

as anounced. In case the normalization is closed the (connection)1-form τ^N is related to the shape operator of M as follows.

Lemma 2.2: *Let (M,g,N) be a closed normalization of a null hypersurface M in a Lorentzian manifold such that τ^N()=0. Then*

$$\tau^N = -\langle A_N\xi,.\rangle. \quad (18)$$

Proof. Assume $\eta=i^*\theta$ closed and let X,Y be tangent vector fields to M. The condition $X.\eta(Y) - Y.\eta(X) - \eta([X,Y]) = 0$ is equivalent to $\langle\overline{\nabla}_X N,Y\rangle = \langle\overline{\nabla}_Y N,X$. Then by the weingarten formula, we get $\langle -A_N X,Y\rangle + \tau^N(X)\eta(Y) = \langle -A_N Y,YX\rangle + \tau(Y)\eta(X)$. In this relation, take $Y=\xi$ to get $\tau^N(X) = -\langle A_N\xi,X\rangle + \tau^N(\xi)\eta(X)$ which gives the desired formula as $\tau^N(\xi)=0$

Exemple 2.1

In the pseudo-Euclidean space $R_q^{n+2}(q\geq 1)$. The pseudo-Euclidean space $R_q^{n+2}(q\geq 1)$ $\langle,\rangle = \overline{g} = -\sum_{i=1}^{q} dx^i\otimes dx^i + \sum_{a=q+1}^{n+1} dx^a\otimes dx^a$ where ($x^0,\ldots x^{n+1}$) stands for the usual rectangular coordinates of R^{n+2} Consider the Monge hypersurface

$$M = \{x^0,\ldots,x^{n+1}\in R_q^{n+2}, x^0 = F(x^1,\ldots,x^{n+1}\}) \quad (19)$$

Where $F:\Omega\to R$ is a smooth function defined on Ω an open set of R^{n+1}. Throughout, V denote a the constant vector field $\frac{\partial}{\partial_0}$. It is easy to check that such a hypersurface is null if and only if F is a solution of the partial differential equation

$$1 + \sum_{i=1}^{q-1}(F'_{x^i})^2 = \sum_{a=1}^{n+1}(F'_{x^a})^2. \quad (20)$$

(Which we assume from now on) and that the rank one normal bundle $TM^\perp$ is spanned by the global vector field

$$\xi = \frac{\partial}{\partial_0} - \sum_{i=1}^{q-1} F'_{x^i}\frac{\partial}{\partial_{x^i}} + \sum_{a=1}^{n+1} F'_{x^a}\frac{\partial}{\partial_{x^a}}. \quad (21)$$

Then, the vector field defined by

$$N = -V + \frac{1}{2}\xi = \frac{1}{2}[-\frac{\partial}{\partial_{x^0}} - \sum_{i=1}^{q-1} F'_{x^i}\frac{\partial}{\partial_{x^i}} + \sum_{a=1}^{n+1} F'_{x^a}\frac{\partial}{\partial_{x^a}}] \quad (22)$$

is null rigging for the null Monge hypersurface M. For this null rigging, we have τ^N=0Indeed, let $\overline{\nabla}$ denote the Levi-Civita connection of R_q^{n+2} and $X\Gamma(TM)$ a tangent vector field. We have:

$$\tau^N(X) = \langle\overline{\nabla}_X N,\xi\rangle = \langle\overline{\nabla}_X(-V + \frac{1}{2}\xi),\xi\rangle = \frac{1}{2}\langle\overline{\nabla}_X\xi,\xi\rangle 0. \quad (23)$$

Now, we show that the normalization given by eqn. (22) is closed (and in particular, the distribution $\mathcal{S}(N)$ is integrable). The 1-forme θ metrically equivalent to N is given by $\theta = \langle -V + \frac{1}{2}\xi,.$ and $\eta=i^*\theta$ (i the inclusion map). Let X Y be two vector fields on M smoothly exetended in two vector fields on R_q^{n+2} (we also denote these extensions by X and Y). We have

$$2d\eta(X,Y) = X.\langle N,Y\rangle - Y.\langle N,X\rangle - \langle N,[X,Y]\rangle$$

$$= \langle\overline{\nabla}_X N,Y\rangle - \langle\overline{\nabla}_Y N,X\rangle$$

$$= \langle\overline{\nabla}_X(-V + \frac{1}{2}\xi,Y\rangle - \langle\overline{\nabla}_Y(-V + \frac{1}{2}\xi,X\rangle$$

$$= \frac{1}{2}\langle\overline{\nabla}_X\xi,Y\rangle - \frac{1}{2}\langle\overline{\nabla}_Y\xi,X\rangle$$

$$= \frac{1}{2}\langle -A_\xi^\star X - \tau^N(X)\xi,Y\rangle - \frac{1}{2}\langle A_\xi^\star Y - \tau^N(Y)\xi,X\rangle$$

$$= 0$$

(as $A_\xi^\star$ is symmetric and τ^N=0) which shows that η is closed. On can observe that this rigging induces a conformal screen $\mathcal{S}(N)$ and that $A_N = \frac{1}{2}A_\xi^\star$.

Riemannian Distance Associated

Let (M^{n+1},g,N) be a normalized null hypersurface, of a Lorentzian manifold. For $p,q\in M$ let $\Omega_{p,q}$ denote the space of all piecewise smooth curves $\gamma:[0,1]\to M$ with $\gamma(0)$=pand $\gamma(1)=q$ The ambient manifold being Lorentzian, the induced metric on M has signature $(0,n)$. In particular, for each $t\in[0,1]$, we have $\|\gamma'\|^2 + \eta(\gamma'(t))^2 = 0$ if and only if $\gamma' = 0$. We define the η-arc length of $\gamma\in\Omega_{p,q}$ by

$$L_\eta(\gamma) = \int_0^1 [\|\gamma'\|^2 + \eta(\gamma'(t))^2]^{\frac{1}{2}}dt.$$

From the above facts and using standard techniques as in Riemannian setting, and noting that a tangent vector X belongs to $\mathcal{S}(N)$ if and only if $\eta(X)$=0, one gets the following lemma.

Lemma 3.1: *The map $d^\eta:M\times M\to[0,\infty]$ given by*

$$d^\eta(p,q) = \inf_{\gamma\in\Omega_{p,q}} L_\eta(\gamma)$$

is a distance function on M.

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Definition 3.1

A normalized null hypersurface (M,g,N) *is said to be* η*-complete if the metric espace* (M,d^{η}) *is a complet space.*

Notations 3.1: *For* $x\in M$ *we set*

$$\mathcal{S}^0_x(1)=\{X\in T_xM,\langle X,X\rangle=1-\eta(X)^2\},$$
$$\mathcal{S}^0(1)=\bigcup_{x\in M}\mathcal{S}^0_x(1),$$

and for all $X\in\Gamma(TM)$, we set

$$O_\eta(X)=\{Y\in TM,\langle X,Y\rangle=-\eta(X)\eta(Y)\},$$

where $\langle,\rangle$ stands for both $\overline{g}$ or g. Observe that, $Y\in O_\eta(X)$ iff $X\in O_\eta(Y)$.

A remarkable fact is that due to the degenerancy of the induced metric g on the null hypersurface M, it is not possible to define the natural dual (musical) isomorphisms # and # between the tangent vector bundle TM and its dual T^*M following the usual Riemannian way. However, this construction is made possible by setting a rigging (normalization) N (we refer to ref. [5] for further details). Consider a normalized null hypersurface (M,g,N) and we define one form define by:

$$\begin{array}{llll} \#_\eta & :\Gamma(TM) & \to & \Gamma(TM^*) \\ & X & \mapsto & X^{\#_\eta}=g(X,.)+\eta(X)\eta(.),\forall Y\in\Gamma(TM),X^{\#_\eta}(Y)=g(X,Y)+\eta(X)\eta(Y). \end{array} \tag{24}$$

Cleary, such a # is an isomorphism of $\Gamma(TM)$ on to $\Gamma(T^*M)$ and can be used to generalize the usual non-degenerate theory. In the latter case, $\Gamma(\mathcal{S}(N))$ coincides with $\Gamma(TM)$, and as a consequence the 1-forme vanishes identically and the projection morphism P becomes the identity map on $\Gamma(TM)$. Let $\#_\eta$ denote the inverse of the isomorphism # given by eqn. (24). For $X\in\Gamma(TM)$ (resp. $w\in T^*M$)), $X^{\#_\eta}$ (resp. $w^{\#_\eta}$) is called the dual 1-form of X (resp. the dual vector field of w) with respect to the degenerate metric g. It follows from eqn. (24) that if w is a 1-form on M, we have for $X\in\Gamma(TM)$,

$$g_\eta(X,Y)=X^{\#_\eta}(Y),\forall X,Y\in\Gamma(TM). \tag{25}$$

Define a (0,2)-tensor g_η by $g_\eta(X,Y)=X^{\#_\eta}(Y),\forall X,Y\in\Gamma(TM)$. Cleary, g_η defines a non-degenerate metric on M which plays an important role in defining the usual differential operators gradient, divergence, Laplacien with respect to degenerate metric g on null hypersurface [5]. It is called the associate metric to g on (M,g,N). Also, observe that innon-degenerate case, the two metrics g_η and g coincide. The (0,2)-tenseur g_η^{-1}, inverse of g_η is called the pseudo-inverse of g with respect to the rigging N. With respect to the quasi orhonormal local frame field $\{\partial_0:=\xi,\partial_1,\dots,\partial_n,N\}$ adapter to the decomposition eqn. (2) and (3) the following verifications are straighforward,

$$g_\eta(\xi,X)=\eta(X),\ \forall X\in\Gamma(\Gamma(TM)$$
$$g_\eta(X,Y)=g(X,Y)\ \forall X,Y\in\Gamma(\mathcal{S}(N)). \tag{26}$$

In particular,

$$g_\eta(\xi,\xi)=1$$

and the last equality in eqn. (26) is telling us that the restrict to $\mathcal{S}(N)$ the metrics g and g coincide. We know from eqn. (12) that the induced metric g is not compatible with the induced connection in general and this compatibility arises if and ony the null hypersurface M is totally geodesic in $\overline{M}$. Let ∇^η denote the Levi-Civita connection of the non-degenerate associate metric g_η on (M,g,N). We are now interested in characterizing the normalizations for which the Levi-Civita connection ∇^η of g agrees with the induced connection ∇ due to N, i.e $\nabla^\eta=\nabla$ For this we recall the following.

Lemma 3.2: For all $X,Y,Z\in\Gamma(TM)$ we have,

$$(\nabla_X g_\eta)(Y,Z)=\eta(Y)[B^N(X,PZ)-C^N(X,PZ)]$$
$$+\eta(Z)[B^N(X,PY)-C^N(X,PY)]$$
$$+2\tau^N(X)\eta(Y)\eta(Z).$$

We derive the following result on the compatibility condition.

Theorem 3.1

Let (M,g,N) *be a normalized null hypersurface of a pseudo-Riemannian manifold* $(\overline{M},\overline{g})$ *[4]. The induced connexions* ∇ *and the Levi-Civita connexion* ∇^η *of the associate metric* g_η *on* (M,g,N) *agree if and only if for all*

$X,Z\in\Gamma(TM)$,

$$\begin{cases} B^N(X,PY) & = & C^N(X,PY) \\ \tau^N(X) & = & 0 \end{cases} \tag{27}$$

Definition 3.2

A normalized null hypersurface (M,g,N) *of a pseudo-Riemannian manifold* $(\overline{M},\overline{g})$ *is said to have a conformal screen if there exists a non vanishing smooth function* φ *on M such that* $A_N=\varphi A^*_\xi$ *holds* [4].

This is equivalent to saying that $C^N(X,PY)=\varphi B^N(X,Y)$ for all tangent vector fields X and Y. The function φ is called the conformal factor. Theorem (3.1) asserts that the compatibility condition is fulfilled if and only if the normalization is screen conformal with constant conformal factor 1 and vanishing normalizing 1-form τ^N.

Remark 3.1

Observe that in ambient Lorentzian case, the Riemannian distance d_{g_η} associted to (M,g) coincides with the metric d^η as given in section (3), i.e $d^\eta=d_{g_\eta}$. It follows the famous Hopf-Rinow theorem that the null hypersurface (M,g,N) is d^η-complete if and only if the Riemannian manifold (M,g_η) is complete. Also, for all $x\in M$,

$$\mathcal{S}^0_x(1)=\{X\in T_xM,\langle X,X\rangle=1-\eta(X)^2\}=\{X\in T_xM,g_\eta(X,X)=1\},$$

that is $\mathcal{S}^0(1)$ coincides the unit bundle of M with respect to the associated Riemannian metric g_η from the normalization. It also hods that for all $X\in TM$, $O_\eta(X)=X^{\perp_{g_\eta}}$.

Relation Between the Null and Associated Riemannian Geometry

Connecting the covariant derivatives

Let (M,g,N) be a normalized null hypersurface of pseudo-Riemannian manifold $(\overline{M}^{n+2},\overline{g})$, ∇ the induced connection on M. In order to relate the main geometric objects of both null and associated non-degenerate geometry on the null hypersurface, we first need to relate the covariant derivatives ∇^η and ∇. In this respect, we prove the following.

Proposition 4.1: *Let* (M,g,N) *be a normalized null hypersurface with rigged vector field . Then, for all* $X,Y\in\Gamma(TM)$*,we have*

$$\nabla^\eta_X Y=\nabla_X Y+\frac{1}{2}[2g(A^*_\xi X,Y)-g(A_N X,Y)-g(A_N Y,X)$$
$$+\eta(X)\tau^N(Y)+\eta(Y)\tau^N(X)]\xi+\eta(X)(i_Y d\eta)^{\#_\eta}+\eta(Y)(i_X d\eta)^{\#_\eta}. \tag{28}$$

In particular for a closed normalization,

$$\nabla^{\eta}_{X}Y = \nabla_{X}Y + \frac{1}{2}[2g(A^{*}_{\xi}X,Y) - g(A_{N}X,Y) - g(A_{N}Y,X)$$
$$+\eta(X)\tau^{N}(Y) + \eta(Y)\tau^{N}(X)]\xi$$

Proof. Both connections$\nabla^{\eta}$and ∇ are torsion free. the we can write

$$\nabla^{\eta}{}_{X}Y = \nabla_{X}Y + \mathcal{D}(X,Y), \forall X,Y \in \Gamma(TM),$$

with $\mathcal{D}$ is a symmetric tensor. As ∇^{η} is g_{η}-metric, we have

$$g_{\eta}(\mathcal{D}(X,Y),Z) + g_{\eta}(Y,\mathcal{D}(X,Z)) = X.g_{\eta}(Y,Z) - g_{\eta}(\nabla_{X}Y,Z) - g_{\eta}(Y,\nabla_{X}Z)$$
$$= (\nabla_{X}g_{\eta})(Y,Z).$$

Then using this and Lemma (3.2), we have

$$g_{\eta}(\mathcal{D}(X,Y),Z) + g_{\eta}(Y,\mathcal{D}(X,Z)) = \eta(Y)[B^{N}(X,PZ) - C^{N}(X,PZ)]$$
$$+\eta(Z)[B^{N}(X,PY) - C^{N}(X,PY)]$$
$$+2\tau^{N}(X)\eta(Y)\eta(Z).$$

By circular permutation we get similar expression for $g_{\eta}(\mathcal{D}(Y,Z),X) + g_{\eta}(\mathcal{D}(Y,X),Z)$, and $g_{\eta}(\mathcal{D}(Z,X),Y) + g_{\eta}(\mathcal{D}(Z,Y),X)$. Summing the first two expressions minus the last one leads to

$$2g_{\eta}(\mathcal{D}(X,Y),Z) = \eta(X)[g_{\eta}(A_{N}Z,Y) - g_{\eta}(A_{N}Y,Z)]$$
$$+\eta(Y)[g_{\eta}(A_{N}Z,X) - g_{\eta}(A_{N}X,Z)]$$
$$+\eta(Z)[2g_{\eta}(A^{*}_{\xi}Y,X)) - g_{\eta}(A_{N}X,Y) - g_{\eta}(A_{N}Y,X)$$
$$+2[\tau^{N}(Y)\eta(X)\eta(Z) + \tau^{N}(X)\eta(Y)\eta(Z) - \tau^{N}(Z)\eta(X)\eta(Y)]. \quad (29)$$

Now using eqn. (26) and (16), we get

$$2g_{\eta}(\mathcal{D}(X,Y),Z) = 2g_{\eta}[\eta(X)(i_{Y}d\eta)^{\#\eta} + \eta(Y)(i_{X}d\eta)^{\#\eta}, Z].$$
$$+g_{\eta}[2g_{\eta}(A^{*}_{\xi}X,Y)) + \eta(X)\tau^{N}(Y) + \eta(Y)\tau^{N}(X)$$
$$-g_{\eta}(A_{N}X,Y) - g_{\eta}(A_{N}Y,X)\xi, Z]. \quad (30)$$

It follows the non-degenerancy of g_{η}that

$$\mathcal{D}(X,Y) = \frac{1}{2}(2g_{\eta}(A^{*}_{\xi}X,Y)) + \eta(X)\tau^{N}(Y) + \eta(Y)\tau^{N}(X$$
$$-g_{\eta}(A_{N}X,Y) - g_{\eta}(A_{N}Y,X))\xi$$
$$+\eta(X)(i_{Y}d\eta)^{\#\eta} + \eta(Y)(i_{X}d\eta)^{\#\eta}, \quad (31)$$

which, using the fact that the operators A^{*}_{ξ} and A_{N} are $\mathcal{S}(N)$-valued and $g_{\eta|\mathcal{S}(N)} = g_{|\mathcal{S}(N)}$ gives the desired formulas. with $d\eta(X,Y) = \frac{1}{2}[X.\eta(Y) - Y.\eta(X) - \eta([X,Y])]$.

Remark 4.1

From above proposition (4.1), it follows that if the screen distribution is integrable (which is equivalent to the symmetry of C^{N}*on* $\mathcal{S}$ (*N*)× (*N*) *we have for all X,Y*∈$\mathcal{S}$ (*N*)

$$\nabla^{\eta}_{X}Y = \nabla_{X}Y + [B^{N}(X,Y) - C^{N}(X,Y)]\xi. \quad (32)$$

Throughout, the normalization wil be assumed closed. Beyond its technical aspect this assumption guarentee integrability of thee screen distribution $\mathcal{S}$ (*N*).

Some curvature relations

In this section we relate various curvature tensors of the null geometry of (*M*,*g*,*N*) to those of the associated non-degenerate metric g_{η} on *M*. Let *R* and *R* denote the Riemann curvature tensors of $\nabla^{\eta}$and ∇ respectively. Using proposition (4.1) we prove the following.

Proposition 4.2: *Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ. Then, for all X,Y,ZΓ(TM), the following hold.*

$$R^{\eta}(X,Y)Z = R(X,Y)Z + \frac{1}{2}\{\phi_{(X,Z)}A^{\star}_{\xi}Y - \phi_{(Y,Z)}A^{\star}_{\xi}X\} - \frac{1}{2}\{g((\nabla_{Y}A_{N})(Z),X) + (\nabla_{Y}\eta)(Z)\tau^{N}(X)$$
$$+\eta(X)[(\nabla_{Y}\tau^{N})(Z) - g(A^{\star}_{\xi}Y, A_{N}Z) + g_{\eta}(A_{N}Y, A_{N}Z)] - 2g_{\eta}(\nabla_{X}A^{\star}_{\xi})(Y),Z)$$
$$-\eta(Z)[g(A^{\star}_{\xi}X, A_{N}Y) - g(A_{N}X, A^{\star}_{\xi}Y) + d\tau^{N}(X,Y)] + g(\nabla_{X}A_{N})(Y),Z)$$
$$+g((\nabla_{X}A_{N})(Z),Y) + \eta(Y)[g(A^{\star}_{\xi}X, A_{N}Z) - g(A_{N}X, A_{N}Z) + (\nabla_{X}\tau^{N})(Z)]$$
$$-(\nabla_{X}\eta)(Z)\tau^{N}(Y) + 2g((\nabla_{Y}A^{\star}_{\xi})(X),Z) - g(\nabla_{Y}A_{N})(X),Z) + d\eta(X,Y)\tau^{N}(Z)\}\xi$$

Proposition 4.3: *Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ. Then, for all* $X,Y,W \in \Gamma(TM)$ *and* $U \in TM^{\perp}$ *we have,*

$$g_{\eta}(R^{\eta}(X,Y)Z, PW) = g(R(X,Y)Z, PW)$$
$$+\frac{1}{2}\{\phi_{(X,Z)}B^{N}(Y,W) - \phi_{(Y,Z)}B^{N}(X,W)\} \quad (33)$$

$$g_{\eta}(R^{\eta}(X,Y)Z, U) = -g(R(X,Y)U, PZ)$$
$$-\frac{1}{2}[g(A_{N}U,Y)B^{N}(X,Z) - g(A_{N}U,X)B^{N}(Y,Z)]$$
$$-\frac{1}{2}[\tau^{N}(X)B^{N}(Y,Z) - \tau^{N}(Y)B^{N}(X,Z)]\eta(U), \quad (34)$$

Where ϕ(*X*, *Z*) is given by:

$$\phi_{(X,Z)} = 2B^{N}(X,Z) - g(A_{N}X,Z) - g(A_{N}Z,X)$$
$$+\tau^{N}(X)\eta(Z) + \tau^{N}(Z)\eta(X). \quad (35)$$

Proof. The Riemann curvature tensor field R^{η} of type (1.3) is defined by

$$R^{\eta}(X,Y)Z = [\nabla^{\eta}_{X}, \nabla^{\eta}_{Y}]Z - \nabla^{\eta}_{[X,Y]}Z. \quad (36)$$

Then eqns. (37) and (38) consist on repeated applications of (30) in Proposition (4.1).

Remark 4.2

Note that by lemma (2.2), for a closed and conformal normalization (with factor φ we *have* $\tau^{N} = 0$ *and* $\phi(X,Z) = 2(1-\varphi)B^{N}(X,Z)$. *Then eqns. (37) and (38) take the forms*

$$g_{\eta}(R^{\eta}(X,Y)Z, PW) = g(R(X,Y)Z, PW)$$
$$+(1-\varphi)[B^{N}(X,Z)B^{N}(Y,W)$$
$$-B^{N}(Y,Z)B^{N}(X,W)] \quad (37)$$
$$g_{\eta}(R^{\eta}(X,Y)Z, U) = -g(R(X,Y)U, Z). \quad (38)$$

In the following we let Ric^{η} and *Ric* denote the Ricci curvature of ∇^{η} and ∇ respectively. Recall that $Ric^{\eta}(X,Y) = trace\, Z \mapsto R^{\eta}(Z,X)Y, \forall X,Y \in TM$. this is a symmetric (0,2)-tensor on *TM*. unfortunately, the corresponding quantity *Ric*(*X*,*Y*) obtained from ∇ is no longer symmetric in general, due to the fact that the induced Riemann curvature *R* on the normalized null hypersurface (*M*,*g*,*N*) fails to have the usual algebraic curvature symmetries in general. Precisely, the induced Ricci tensor *Ric* is given by

$$Ric(X,Y) = \overline{Ric}(X,Y) + B^{N}(X,Y)trA_{N} - \theta(\overline{R}(\xi,Y)X) - g(A_{N}X, A^{\star}_{\xi}Y),$$

where $\overline{Ric}(X,Y)$ denotes the Ricci curvature of the ambient manifold. We define the (0,2)-symmetrized Ricci tensor Ric^{0} on the null hypersurface by

$$Ric^{0}(X,Y) = \frac{1}{2}[Ric(X,Y) + Ric(Y,X)]$$

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for all X,YTM.

Theorem 4.1

Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ and $\tau^N(\xi)=0$ *in a* $(n+2)$â *pseudo-Riemannian manifold. Then,*

$$Ric^\eta(X,Y)=Ric(X,Y)-[\langle A_\xi^*X,Y\rangle-\langle A_NX,Y\rangle+\tau^N(X)\eta(Y)\,]trA_\xi^*$$
$$+\langle(\nabla_\xi A_\xi^*)(X),Y\rangle-\langle(\nabla_\xi A_N)(X),Y\rangle$$
$$+(\nabla_\xi\tau^N)(X)\eta(Y)-(\nabla_X\tau^N)(Y). \quad (39)$$

Proof. Let $p\in M$ and $(E_0:=\xi,E_1,\cdots,E_n)$ be a quasiorthonormal basis for (T_pM,g_p) with Span $(E_1,\cdots,E_n)=\mathcal{S}(N)_p$. When dealing with indices, we adopt the following conventions: $i,j,k,\cdots\in\{1,\cdots,n\},\alpha,\beta,\gamma\in\{0,\cdots,n\}$, and $a,b,\cdots,\in\{0,\cdots,n+1\}$. Then we have:

$$Ric^\eta(X,Y)=\sum_{\alpha=0}^{n}g_\eta^{\alpha\alpha}R^\eta(E_\alpha,X)Y,E_\alpha). \quad (40)$$

Thus from eqn. (37) and (38), we get,

$$R^\eta ic(X,Y)=g_\eta(R^\eta(\xi,X)Y,\xi)+\sum_{i}^{n}g_\eta(R^\eta(E_i,X)Y,E_i)$$
$$=Ric(X,Y)+g(A_\xi^*X,A_\xi^*Y)-g(A_\xi^*X,Y)-g(A_\xi^*X,Y)trA_\xi^*$$
$$-g(A_\xi^*X,A_NY)+\frac{1}{2}[g(A_NX,Y)+g(A_NY,X)-\tau^N(X)\eta(Y)$$
$$-\tau^N(Y)\eta(X)\,]trA_\xi^*-g(R(\xi,X)\xi,Y)-\overline{g}(\overline{R}(\xi,X)Y,N).$$
$$R^\eta ic(X,Y)=Ric(X,Y)-[\langle A_\xi^*X,Y\rangle-\langle A_NX,Y\rangle\,]trA_\xi^*$$
$$-\tau^N(X)\eta(Y)trA_\xi^*+\langle X,(\nabla_\xi A_\xi^*)(Y)\rangle-(\nabla_X\tau^N)(Y)$$
$$-\langle(\nabla_\xi A_N)(X),Y\rangle+2d\tau^N(\xi,X)\eta(Y)+\tau^N(A_\xi^*X)\eta(Y).$$

Where

$$Ric(X,Y)=\sum_{i=1}^{n}\varepsilon_i g(R(E_i,X)Y,E_i)+\overline{g}(R(\xi,X)Y,N), \quad (41)$$

is induced Ricci tensor curvature on a null hypersurface. But $\langle X,(\nabla_\xi A_\xi^*)(Y)\rangle=\langle(\nabla_\xi A_\xi^*)(X),Y\rangle$ and $2d\tau^N(\xi,X)=(\nabla_\xi\tau^N)(X)-(\nabla_X\tau^N)(\xi)$ and $(\nabla_X\tau^N)(\xi)=\tau^N(A_\xi^*X)$. Also $g(R(\xi,X)\xi,Y)=g(A_\xi^*X,A_\xi^*Y)-g(X,(\nabla_\xi A_\xi^*)(Y))$ and

$$\overline{g}(\overline{R}(\xi,X)Y,N)=(\nabla_X\tau^N)(Y)+\langle\nabla_\xi A_N)(X),Y\rangle$$
$$-2d\tau^N(\xi,X)\eta(Y)-\langle A_NA_\xi^*X,Y\rangle.$$

By substituting previous terms in the above expression of $Ric^\eta(X,Y)$ we get the desired formula.

Corollary 4.1

Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ an $\tau^N(\xi)=0$. *in a* $(n+2)$â *pseudo-Riemannian manifold with constant curvature k. Then*

$$Ric^\eta(X,Y)=nk\langle X,Y\rangle+\langle A_\xi^*X,Y\rangle trA_N-\langle A_NX,A_\xi^*Y\rangle$$
$$-[\langle A_\xi^*X,Y\rangle-\langle A_NX,Y\rangle+\tau^N(X)\eta(Y)\,]trA_\xi^*+\langle(\nabla_\xi A_\xi^*)(X),Y\rangle$$
$$-\langle(\nabla_\xi A_N)(X),Y\rangle+(\nabla_\xi\tau^N)(X)\eta(Y)-(\nabla_X\tau^N)(Y). \quad (42)$$

Theorem 4.2

Let (M,g,N) be a closed normalized null hypersurface with rigged vector fieldξ and (0,2)-symmetrized Ricci tensor Ric^0 *on null hypersurface and* $\tau^N(\xi)=0$ *in a* $(n+2)$â *pseudo-Riemannian manifold. Then*

$$Ric^\eta(X,Y)=Ric^0(X,Y)-[\langle A_\xi^*X,Y\rangle-\langle A_NX,Y\rangle+\tau^N(X)\eta(Y)\,]trA_\xi^*$$
$$+\langle(\nabla_\xi A_\xi^*)(X),Y\rangle-\langle(\nabla_\xi A_N)(X),Y\rangle$$
$$+(\nabla_\xi\tau^N)(X)\eta(Y)-(\nabla_X\tau^N)(Y). \quad (43)$$

Corollary 4.2

Let (M,g,N) be a closed normalized of a null hypersurface with rigged vector field and $\tau^N(\xi)=0$ *and (0,2)-symmetrized Ricci tensor* Ric^0 *on null hypersurface in a* $(n+2)$â *pseudo-Riemannian manifold with constant curvature k. Then,*

$$Ric^\eta(X,Y)=nk\langle X,Y\rangle+\langle A_\xi^*X,Y\rangle trA_N-\frac{1}{2}[\langle A_NX,A_\xi^*Y\rangle+\langle A_NY,A_\xi^*X\rangle$$
$$-\langle(\nabla_\xi A_N)(X),Y\rangle+(\nabla_\xi\tau^N)(X)\eta(Y)-(\nabla_X\tau^N)(Y).$$
$$-\langle(\nabla_\xi A_N)(X),Y\rangle+(\nabla_\xi\tau^N)(X)\eta(Y)-(\nabla_X\tau^N)(Y). \quad (44)$$

Theorem 4.3

Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ and $\tau^N(\xi)=0$ *in a* $(n+2)$â *pseudo-Riemannian manifold. Then*

$$Ric^\eta(X,\xi)=Ric(X,\xi)-\tau^N(X)trA_\xi^*+2d\tau^N(\xi,X). \quad (45)$$

Proof. To get eqn. (45), take Y= in the (4.1). Recall that from a geometric point of view and in practice, one gets the scalar curvature by contracting with a (non-degenerate) metric the (symmetric) Ricci curvature. It turns out that in the null geometry setting, such a scalar quantity cannot be calculated by the usual way (degenerancy of the induced metric and the failure of symmetry in the induced Ricci curvature) [12]. This justify introduction of a symmetrized Ricci curvature tensor Ric^0 and our use of the associated non-degenerate metric g_η in calculating this scalar quantity. More precisely, the extrinsic scalar curvature r^0on the rigged null hypersurface (M,g,N) is given by g_η-trace of the symmetrized Ricci curvature Ric^0. With respect to a local quasiorthonormal frame $(e_0:=\xi,e_1,\cdots,e_n)$ for (M,g_η) we have

$$r0=g_\eta^{\alpha\beta}Ric_{\alpha\beta}^0. \quad (46)$$

Now let r^η denote the scalar curvature of the non-degenerate metric g_η on M that is the contraction of $Ric^\eta$with respect to g. In the following, we state a formula relating the extrinsic scalar curvature r^0 to the associated scalar curvature r^η

Theorem 4.4

Let (M,g,N) be a closed normalized null hypersurface with rigged vector field ξ and $\tau^N(\xi)=0$ *in a pseudo-Riemannian manifold. Then*

$$r^\eta=r^0-[trA_\xi^*-trA_N\,]trA_\xi^*$$
$$+[tr(\nabla_\xi A_\xi^*)-tr(\nabla_\xi A_N)-div^g\tau^{N\#}. \quad (47)$$

Proof. We have

$$r^\eta=g_\eta^{\alpha\alpha}Ric_{\alpha\alpha}^\eta$$

in a local quasiorthonormal frame field $(e_0:=\xi,e_1,\cdots,e_n)$ for $(M,g_\eta$ with span $(e_1,\cdots,e_n)=\mathcal{S}(N)$. But

$$Ric_{\alpha\alpha}^\eta=Ric_{\alpha\alpha}^0-[\langle A_\xi^*e_\alpha,e_\alpha\rangle-\langle A_Ne_\alpha,e_\alpha\rangle+\tau^N(e_\alpha)\eta(e_\alpha)]trA_\xi^*]$$
$$+[\langle(\nabla_\xi A_\xi^*)(e_\alpha,e_\alpha\rangle-\langle(\nabla_\xi A_N)(e_\alpha),e_\alpha\rangle]$$
$$+[(\nabla_\xi\tau^N)(e_\alpha)\eta(e_\alpha)-(\nabla_{e_\alpha}\tau^N)(e_\alpha)].$$

Hence, by contracting each side with $g_\eta^{\alpha\alpha}$ and taking into account Proposition (4.1) along with the following facts: $(\nabla_{e_i}\tau)(e_i)=\eta(\tau^\#)g(A_\xi^\star e_i,e_i)+g(\nabla_{e_i}\tau^\#,e_i)=g(\nabla_{e_i}\tau^\#,e_i)$, $g_\eta^{\alpha\alpha}(\nabla_\xi\tau^N)(e_\alpha)\eta(e_\alpha)=0$,

$g_\eta^{\alpha\alpha}\langle(\nabla_\xi A_N)(e_\alpha),e_\alpha\rangle=tr(nabla_\xi A_N)+g_\eta(\nabla_\xi(\tau^{N\#_\eta}$ and

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$g_\eta^{\alpha\alpha}\tau^N(e_\alpha)\eta(e_\alpha)=0, g_\eta^{\alpha\alpha}\langle(nabla_\xi A^*)(e_\alpha),e_\alpha\rangle = tr(\nabla_\xi A_\xi^*)$, we get relation in eqn. (51).

For null hypersurface in (n+2)-dimensional ambient space form $\overline{M}(k)$, it is easy to see that the extrinsic scalar curvature is given by $r^0=n^2k+trA_\xi^* trA_N - tr(A_\xi^* A_N)$. Then, we have the following [1].

Corollary 4.3

Let (M,g,N) be a closed normalized null hypersurface of a (n+2)-dimensional Lorentzian space and

τ(ξ)=0. Then

$$r^\eta = n^2k + 2trA_x^* itrA_N - tr(A_\xi^* A_N) - (trA_\xi^*)^2$$
$$+[tr(\nabla_\xi A_\xi^*) - tr(\nabla_\xi A_N) - div^g \tau^{N\#} \tag{48}$$

Suppose π is a non-degenerate plane (for g_η) in T_pM. The real number

$$K_\eta(\pi)=\frac{g_\eta(R^\eta(U,V)V,U)}{g_\eta(U,U)g_\eta(V,V)-(g_\eta(U,V))^2} \tag{49}$$

is the sectional curvature of (with respect to g). A similar definition holds if π is non-degenerate with respect to g, and we denote the corresponding quantity by $K(\pi)$, as we know,it is easy to check that the right hand side of eqn. (49) does not depend on the basis of π Lep $pM$and H be a null plane of T_pM direct by $\xi_p \in T_pM^\perp$. The null sectional curvature of H with respect to $\xi_p$is the real number

$$K_{\xi_p}(H)=\frac{g_p(R(W_p,\xi_p)\xi_p,W_p)}{g_p(W_p,W_p)} \tag{50}$$

for an arbitrary non-null vector W_p in H. Obviously, this quantity is independent of W_p but depends in a quadratic fashion on the rigged vector ξ_p. Below, $\pi(\xi,X)=span\{\xi,X\}$ denotes a null plane directed by (the null vector) ξ and X that is $\pi(\xi,X)=span\{\xi,X\}$. Now, we show the following.

Lemma 4.1: *Let (M,g,N) be a closed normalized null hypersurface with rigged vector field and $\tau^N(\xi)=0$ in a Lorentzian manifold. Then for all p∈M and π⊂∈𝒮(N) we have:*

$$K_\eta(\pi)=K(\pi)+B^N(X,Y)^2-B^N(X,X)B^N(Y,Y)$$
$$+B^N(X,X)C^N(Y,Y)-B^N(X,Y)C^N(X,Y), \tag{51}$$

Where X and Y are ortogonal in 𝒮 (N) and π=span{X,Y}

Proof. Observe that a plane π⊂𝒮(N) is both non-degenerate with respect to g and g (simultaneously) or not. Now, eqn. (51) is a direct use of eqn. (37) in the eqn. (49), taking into account the fact without loss of generality, we have assumed X and Y g_η-unit and orthogonal in (N) (and hence also for g).

Theorem 4.5

Let (M,g,N) an (n+1)*-dimensional be a closed normalized null hypersurface of a Lorentzian space form M(k) and $\tau^N(\xi)=0$. Then for a non-degenerate plane* $\pi=span\{X,Y\}\subset T_pM,(p\in M)$

$$K_\eta(X,Y)=k[1-\eta^2(X)-\eta^2(Y)]+B^N(X,Y)^2$$
$$-2B^N(X,Y)C^N(PX,PY)+B^N(Y,Y)C^N(PX,PX)-B^N(X,X)B^N(Y,Y)$$
$$+B^N(X,X)C^N(PY,PY)+2[\eta(X)\tau^N(Y)+\tau^N(X)\eta(Y)]B^N(X,Y)$$
$$-2[\eta(X)\tau^N(X)B^N(Y,Y)+\eta(Y)\tau^N(Y)B^N(X,X)]. \tag{52}$$

Where $X,Y\in\mathcal{S}_p^0(1)$, $Y\in O_\eta(X)$.

Proof. From $X,Y\in\mathcal{S}_p^0(1), Y\in O_\eta(X)$, we infer that $g_\eta(X,X)=g_\eta(Y,Y)=1$ and $g(X,Y)=0$. It follows that $g_\eta(X,X)g_\eta(Y,Y)-g_\eta(X,Y)^2=1$ and $K_\eta(X,Y)=g_\eta(R^\eta(X,Y)Y,X)$. Here, for a vector field X, we brief X^s and X^0 for PX and $\eta(X)\xi$ respectively, where P is the morphism projection of TM onto $\mathcal{S}(N)$ and then, $X=X^0+X^5$

$$K_\eta = K_\eta(X^0+X^s,Y^0+Y^s)$$
$$=g_\eta(R^\eta(X^0+X^s,Y^0+Y^s)(Y^{å\circ}+Y^s,X^0+X^s)$$
$$=g_\eta(R^\eta(X^0,Y^s)Y^s,X^0)+2g_\eta(R^\eta(X^s,Y^0)Y^s,X^0)+2g_\eta(R^\eta(X^s,Y^s)Y^s,X^0)$$
$$+g_\eta(R^\eta(X^s,Y^0)Y^0,X^s)+2g_\eta(R^\eta(X^s,Y^0)Y^s,X^s)+g_\eta(R^\eta(X^s,Y^s)Y^s,X^s).$$

Using eqn. (38), we get

$$g_\eta(R^\eta(X^0,Y^s)Y^s,X^0)=-g(R(X^0,Y^{\text{Å}})X^0,Y^s)-\frac{1}{2}[g(A_NX^0,Y^s)g(A^*X^0,Y^s)$$
$$-g(A_NX^0,X^0)g(A_\xi^*Y^s,Y^s)]-\frac{1}{2}[\tau^N(X^0)g(A_\xi^*Y^s,Y^s)$$
$$-\tau^N(Y^s)g(A_\xi^*X^0,Y^s)]\eta(X^0)$$
$$=-g(R(X^0,Y^s)X^0,Y^s)$$
$$=g(R(Y^s,X^0)X^0,Y^s)$$
$$=\eta(X)^2g(R(Y^s,\xi)\xi,Y^s)$$
$$=[1-\eta(Y)^2]\eta(X)^2K_\xi(\pi(\xi,Y^s)).$$

Similarly, we have $g_\eta(R^\eta(X^s,Y^0)Y^0,X^s)=[1-\eta(X)^2]\eta(Y)^2K_\xi(\pi(\xi,X^s))$,

$$g_\eta(R^\eta(X^s,Y^s)Y^s,X^0)=-\eta(X)[g(R(X^s,Y^s)\xi,Y^s)-\tau^N(X)B^N(X,Y)$$
$$+\tau^N(X)B^N(Y,Y)],$$
$$g_\eta(R^\eta(X^s,Y^0)Y^s,X^s)=\eta(Y)[g(R(X^s,\xi)Y^s,X^s)-\tau^N(Y)B^N(X^s,X^s)$$

$g_\eta(R^\eta(X^s,Y^s)Y^s,X^s)=-g(R(X^s,Y^0),X^0,Y^s)$ and

$$g_\eta(R^\eta(X^s,Y^s)Y^s,X^s)=g(R(X^s,Y^s)Y^s,X^s)+B^N(X,Y)^2-B^N((X,X)B^N(Y,Y)$$
$$+B^N(X,X)C^N(Y^s,Y^s)-B^N(X,Y)C^N(X^s,Y^s).$$

Hence

$$K_\eta(X,Y)=[1-\eta^2(Y)]\eta^2(X)K_\xi(\pi(\xi,Y^s))+[1-\eta^2(X)]\eta^2(Y)K_\xi(\pi(\xi,X^s))$$
$$-2g(R(X^s,Y^0)X^0,Y^s)-2\eta(X)g(R(X^s,Y^s)\xi,Y^s)+2\eta(X)\tau^N(Y)B^N(X,Y)$$
$$-2\eta(X)\tau^N(X)B^N(Y,Y)+2g(R(X^s,Y^0)Y^s,X^s)+B^N(X,Y)^2$$
$$-2\tau^N(Y)\eta(Y)B^N(X,X)+g(R(X^s,Y^s)Y^s,X^s-B^N(X,X)B^N(Y,)Y)$$
$$+B^N(X,X)C^N(Y^s,Y^s)-B^N(X,Y)C^N(X^s,Y^s)$$

Now, we recall that the ambient is Lorentzian space form with curvature (=k). Then, by Gauss-Codazzi equations an being in mind $X,Y\in\mathcal{S}_p^0(1), Y\in O_\eta(X)$, it is easy to chek that $g(R(X^s,Y^s)X^0,Y^s)=0\; g(R(X^s,Y^0)X^0,Y^s)=0$

$$g(R(X^S(Y^0)Y^s,X^s)=\eta(Y)\tau(X)B(X,Y)$$
$$(R(X^s,Y^s)Y^s,X^s)=k\{\langle Y^s,Y^s\rangle\langle X^s,X^s\rangle-\langle X^s,Y^s\rangle\langle Y^s,X^s\rangle$$
$$-B^N(X^s,Y^s)C^N(Y^s,X^s)+B^N(Y^s,Y^s)C^N(X^s,X^s).$$
$$=k[1-\eta^2(X)-\eta^2(Y)]$$
$$-B^N(X,Y)C^N(PX,PY)+B^N(Y,Y)C^N(PX,PX).$$

and $K_\xi(\pi(\xi,.))=0$. By substitution, we get the announced relation

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in eqn. (56). For a totally geodesic null hypersurface, the second fundamental form B^N vanishes identically and we get the following.

Corollary 4.4

Let (M,g,N) *an* $(n+1)$*-dimensional be a closed normalized a totally geodesic null with rigged field* ξ *and* $\tau^N(\xi)=0$ *in a Lorentzian space form* $\overline{M}(k)$. *Then*

$$K_\eta(X,Y)=k\,[1-\eta^2(X)-\eta^2(Y)\,].$$

Corollary 4.5

Let (M,g,N) *be a closed normalized null hypersurface with rigged field* ξ *and* $\tau^N(\xi)=0$ *in a Lorentzian space form* $\overline{M}(k)$ *Then of a Lorentzian space form* $(\overline{M}(k),\overline{g})$. *Then*

$$K_\eta(X,Y)=k\,[1-\eta^2(X)-\eta^2(Y)\,]+B^N(X,Y)^2-2B^N(X,Y)C^N(X,PY)$$

$$+B^N(Y,Y)C^N(X,PX)-B^N(X,X)B^N(Y,Y)+B^N(X,X)C^N(Y,PY). \quad (53)$$

In particular, for if the screen distribution is conformal with conformal factor , then

$$K_\eta(X,Y)=k[1-\eta(X)^2-\eta(Y)^2]$$

$$+(2\varphi-1)[B^N(X,X)B^N(Y,Y)-B^N(X,Y)^2]. \quad (54)$$

Proof. obtain relation eqn. (57) by setting $\tau^N=0$ in eqn. (56). Now from Remark (4.2) and definition (3.2) the last claim follows.

A section N is called conformal Killing (CKV in short) or conformal collineation on $(\overline{M},\overline{g})$ if $\rho\in C^\infty(M)$ for some $\rho\in C^\infty(M)$. In case ρ vanishes identically N is called a Killing vector field and $L_N\overline{g}=0$.

Fact 4.1: *Assume N is a closed conformal collineation rigging of* (M,g). *Then, the normalizing one-form* τ^N *vanishes identicaly on M. Morever, the screen distribution* $\mathcal{S}(N)$ *is integrable and totally umbilical.*

Proof. Let X,Y be tangent to M. We have:

$$2\rho g(X,Y)=2\rho\overline{g}(X,Y)=(L_N\overline{g})(X,Y)=\langle\overline{\nabla}_X N,Y\rangle+\langle X,\overline{\nabla}_Y N\rangle, \quad (55)$$

that is

$$-\langle A_N X,Y\rangle-\langle A_N Y,X\rangle+\tau^N(X)\eta(Y)+\tau^N(Y)\eta(X)=2\rho g(X,Y). \quad (56)$$

Hence, set $Y=\xi$ to get:

$2\tau^N(X)=0$

i.e $\tau^N(X)=0$, which shows that τ^N vanishes on M. Then we get by the closed assumption

$$0=\langle\overline{\nabla}_X N,Y\rangle-\langle X,\overline{\nabla}_Y N\rangle$$

$$=-C^N(X,PY)+C^N(Y,PX)\ (as\ \tau^N=0).$$

So, for $X,Y\mathcal{S}(N)$ we have

$$C^N(X,Y)=C^N(Y,X),$$

that is the screen distribution is integrable. Now, return to eqn. (56) to get

$$-C^N(X,PY)-C^N(Y,PX)=2\rho g(X,Y),\ X,Y\in\Gamma(TM).$$

Hence, for $X,Y\in\mathcal{S}(N)$,

$$C^N(X,Y)=-\rho g(X,Y).$$

But, as

$$C^N(\xi,PY)=\langle A_N\xi,PY\rangle=-\tau^N(PY)=0.$$

we deduce that

$$C^N(X,PY)=-\rho g(X,Y)$$

for all $X,Y\in\Gamma(TM)$ which shows that $\mathcal{S}(N)$ is totally umbilical in null hypersurface M.

Theorem 4.6

Let (M,g,N) *be a null hypersurface of a Lorentzian space form* $(\overline{M}(k),\overline{g})$ *with a closed and conformally Killing (but not Killing) normalization. Then* (M,g) *is totally umbilical in* $(\overline{M},\overline{g})$. *Moreover,* ρ *being the nowhere vanishing conformal factor of N, we have for all* $X,Y\in\mathcal{S}_x^0(1)$, $Y\in O_\eta(X)$

$$K_\eta(X,Y)=(3k-2\xi(\rho))\,[1-\eta^2(X)-\eta^2(Y)\,]+\frac{(\xi(\rho)-k)^2}{\rho^2}\eta^2(X)\eta^2(Y). \quad (57)$$

Proof. From the previous fact, $\mathcal{S}(N)$ is integrable and totally umbilical with umbilicity factor-ρ. So, in this case, it is a well know fact that the following holds (MEN14, p.110)

$$\{\xi(-\rho)+\rho\tau^N(\xi)+k)\}g(Y,X).=-\rho\,B^N(Y,X).$$

Hence, as $\tau^N()=0$ and ρ is everywhere non zero, we get

$$B^N(Y,X)=(\frac{\xi(\rho)-k}{\rho})g(Y,X),\ X,Y\in\Gamma(TM),$$

that is (M,g) is totally umbilical. Now, using previous expressions of $B^N$and C^N eqn. (57), we get for all $X,Y\in\mathcal{S}_x^0(1)$, $Y\in O(X)$

$$K_\eta)=k\,[1-\eta^2(X)-\eta^2(Y)\,]+(\frac{\xi(\rho)-k)}{\rho})^2g^2(X,Y)$$

$$-2(\frac{\xi(\rho)-k}{\rho})(-\rho)g^2(X,Y)+2\frac{\xi(\rho)-k}{\rho}(-\rho)g(X,X)g(Y,Y)$$

$$=k\,[1-\eta^2(X)-\eta^2(Y)\,]+(\frac{\xi(\rho)-k)}{\rho})^2\eta^2(X)\eta^2(Y)$$

$$-2\rho\frac{\xi(\rho)-k}{\rho}[1-\eta^2(X)-\eta^2(Y)\,].$$

But for all $X,Y\in\mathcal{S}_x^0(1)$, $Y\in O(X)$

$$g(X,X)=1-\eta(X)^2,\ g(Y,Y)=1-\eta(Y)^2$$

and

$$g(X,Y)=-\eta(X)\eta(Y).$$

and which after substitution leads to the desired expression of

$K_\eta(X,Y)$ (58)

Relationship between Curvature and Topology of Null Hypersurface

In this section, we study the null geometry of manifolds and link their invariants to those of the induced associated Riemannian metric on them through the normalization. Thereafter, we use some comparison theorems from Riemannian geometry to get informations on the underline manifold topology.

Theorem 5.1

Let (M,g,N) *be a closed normalized compact null hypersurface of a Lorentzian manifold* $(\overline{M}^{n+2},\overline{g})$ *and* $\tau=0$

If

$$Ric(X,X)\geq[\langle A_\xi^*X,X\rangle-\langle A_N X,X\rangle]trA_\xi^*$$

$$-\langle(\nabla_\xi A_\xi^*)(X),X\rangle+\langle(\nabla_\xi A_N)(X),X\rangle$$

holds for all XTM, then the universal Riemannian covering of the associated Riemannian manifold (M,g_η) is isometric to a product $(\widehat{M}\times\mathbb{R}^q,\widehat{g}\times\delta)$ where δ stands for the usual eucidean metric on $\mathbb{R}^q$ and $(\widehat{M},\widehat{g})$ is a compact simply connected Riemannian manifold with non negative Ricci curvature.

Proof. This is a direct application of a splitting result by Cheeger and Gromoll on compact Riemannian manifolds with non negative Ricci curvature [9,10]. Indeed, the ambient manifold being Lorentzian, we know that M endowed with the associated metric g_η is Riemannian. Also, conditions τ=0 eqn. (64) and the expression (47) show that the Ricci curvature Ric^η of the Riemannian metric g satisfies Ric^η 0 that is non negative. as the hypersurface is compact, the claim follows [11,12].

Theorem 5.2

Let (M^{n+1},g,N) be a closed normalized compact null hypersurface of dimensional Lorentzian manifold $(\overline{M},\overline{g})$ and τ=0, and

$$Ric(X,X)<[\langle A_\xi^*X,X\rangle-\langle A_NX,X\rangle]trA_\xi^*$$
$$-\langle(\nabla_\xi A_\xi^*)(X),X\rangle+\langle(\nabla_\xi A_N)(X),X\rangle. \tag{59}$$

Then every Killing vector field on (M,g_η) is identically zero and group of isometry is finite.

Proof. Using Theorem (4.6), the assumption eqn. (64) is equvalent to saying that the hypersurface M,endowed with the associated metric g_η has a negative sectional curvature. As it is compact, we conclude by theorem of Bochner [13] that every Killing vector field on null hypersurface is identically zero.

In dimension 3, Schoen and Yau proved in that a complete non-compact manifold with positive Ricci curvature is diffeoomorphic to the standard euclidean space $\mathbb{R}^3$. Using this and eqn. (47) lead to the folowing [8].

Theorem 5.3

Let (M,g,N) be a closed normalized d^η-complete non-compact null hypersurface of 4-dimensional Lorentzian manifold $(\overline{M},\overline{g})$ and τ=0 and

$$Ric(X,X)>[\langle A_\xi^*X,X\rangle-\langle A_NX,X\rangle]trA_\xi^*$$
$$-\langle(\nabla_\xi A_\xi^*)(X),X\rangle+\langle(\nabla_\xi A_N)(X),X\rangle. \tag{60}$$

Then the manifold structure of the null hypersurface (M,g,N) is diffeomorphic to $\mathbb{R}^3$.

Proof. The ambient lorentzian manifold has dimension 4, so the null hypersurface is 3-dimensional. The inequality ensures that the associted metric g_η has a positive Ricci curvature. By the Shoen-Yau above quoted theorem [8], the claim follows.

Theorem 5.4

Let (M,g,N) be a closed d^η-complete null hypersurface of a Lorentzian space form $(\overline{M}(k),\overline{g})$ with conformally Killing (but not Killing) normalization. Assume that for all $X,Y\in\mathcal{S}_x^0(1)$, $Y\in O_\eta(X)$ we have:

$$\frac{[\xi(\rho)-k]^2}{\rho^2}\eta(X)^2\eta(Y)^2\le(3k-2\xi(\rho))[\eta(X)^2+\eta(Y)^2-1]. \tag{61}$$

Then the universal covering of the null hypersurface is diffeomorphic to $\mathbb{R}^n$.

Proof. Now, (M^{n+1},g,N) being dcomplete, it follows Remark 3.1 that the Riemannian manifold (M,g_η) is complete. Also, using Theorem (4.6), the assumption eqn. (64) is equivalent to saying that the hypersurface M endowed with the associated metric g has a nonpositive sectional curvature. we conclude by Hadmar theorem that the universal of null hypersurface is diffeomorphic to $\mathbb{R}^n$.

The authors proved, using a correspondence for isometric immersions into product spaces that, on a complete Riemannian manifold M with negative Ricci curvature, and whose scalar curvature is bounded above by a negative constant, the standard Euclidean space (Rn,Euc) cannot be isometrically immersed into the Lorentzian product space M×𝕃. A direct consequence of this fact is the following [14].

Theorem 5.5

Let (M,g,N) be a d^η-complete null hypersurface of a $(n+1)$-dimensional $(n\ge2)$ Lorentzian manifold $(\overline{M},\overline{g})$ with a closed normalization and τ=0. Assume that

$$Ric(X,X)<[\langle A_\xi^*X,X\rangle-\langle A_NX,X\rangle]trA_\xi^* \tag{62}$$
$$-\langle(\nabla_\xi A_\xi^*)(X),X\rangle+\langle(\nabla_\xi A_N)(X),X\rangle,$$

and

$$r\le[trA_\xi^*-trA_N]trA_\xi^*-tr(\nabla_\xi A_\xi^*)+tr(\nabla_\xi A_N)<c, \tag{63}$$

for some positive constant c. Then the standard Euclidean space $(\mathbb{R}^n,Euc)$ cannot be isometrically immmersed into the Lorentzian product space M^nL where the underline manifold M is endowed with the Riemannian associated metric g_η. 𝕃 being $\mathbb{R}$ with the negative definite metric $-dt^2$.

Proof. The null hypersurface M being d^η-complete, it follows that M endowed with the associated metric g_η is a complete Riemannian manifold. Also, as τ=0 from eqns. (62) and (47) we infer that the Ricci curvature of the associated Riemannian metric g_η is negative, and by eqns. (51) and (63) its scalar curvature is bounded above by the negative constant $-c$. This completes the proof.

Theorem 5.6

Let $(\overline{M}^3,\overline{g})$ a Lorentzian manifold and (M,g,N) be a closed normalized complete null hypersurface in $\overline{M},\overline{g})$ with

$$\frac{[\xi(\rho)-k]^2}{\rho^2}\eta(X)^2\eta(Y)^2>(3k-2\xi(\rho))[\eta(X)^2+\eta(Y)^2-1]. \tag{64}$$

Then no complete Riemannian surface $(\sum,\langle,\rangle\sum)$ of constant curvature $c>K_M$ can be isometrically immersed into $M^2\times$𝕃

Proof. The null hypersurface M being d^η -complete, it follows that M endowed with the associated metric g_η is a complete Riemannian manifold. Also, as τ=0, from eqns. (62) and (47).

Conclusion

We infer that the sectional curvature of the associated Riemannian metric g_η is negative, by using theorem José [14,15] on correspondence for isometric immersions into product spaces, we completes the proof wih the research result of Through the research establishment of links between the null geometry and basics invariants of the associated Riemannian metric is explained successfully.

References

1. Duggal KL, Bejancu A (1996) Lightlike submanifolds of semi-riemannian Manifolds and applications. Kluwer Acad Publishers, USA.
2. Atindogbe C (2010) Normalisation and prescribed extrinsic scalar curvature on light like hypersurfaces. Journal of Geometry and Physics 60: 1762-1770.
3. Atindogbe C, Berard-Bergery L (2013) Distinguished Normalization on Non-

Minimal Null Hypersurfaces. Mathemetical Sciences and Applications E-notes 1: 18-35.

4. Atindogbe C, Duggal KL (2004) Conformal sreen on lightlike hypersurfaces. Int J Pure and Appied Math 11: 421-442.
5. Atindogbe C, Ezin JP, Tossa J (2003) Pseudo-inversion of degenerate metrics. Int J Mathematics and Mathematical Sciences 55: 3479-3501.
6. Gutiérrez M, Olea B (2012) Lightlike hypersurfaces in Lorentzian manifolds. Cornel University Library.
7. Hebey H (1997) Introduction to Nonlinear Variety Analysis. Diderot Publisher, France.
8. Schoen R, Yau ST (1982) Complete three dimensional manifold with positive curvature, in Seminar on Differential Geometry, Princeton University Press, USA.
9. Cheeger J, Gromoll D (1971) The splitting theorem for manifold of nonnegative Ricci curvature. Journal of Differential Geometry 6: 119-128.
10. Cheeger J, Gromoll D (1972) On the structure of complete manifolds of nonnegative Ricci curvature. Annals of Mathematics 96: 413-443.
11. Eschenburg JH (1994) Comparison theorems in Riemannian geometry. Augsburg.
12. Atindogbe C (2009) Scalar curvature on lightlike hypersurfaces. Applied Sciences 11: 9-18.
13. Bochner S (1946) Vector fields and Ricci curvature. Bull Amer Maths Sco 52: 776-797.
14. José AG, Jiménez A, Mira P (2010) A correspondence for isometric immersions into product spaces and its applications. Journal of Geometry and Physics 60: 1819-1832.
15. Katsuno K (1980) Null Hypersurfaces in Lorentzian Manifolds. Math Proc Cab Phil Soc 88: 175-182.