Induced Riemannian Structures and Topology of Null Hypersurfaces in Lorentzian Manifold

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Abstract

Given a null hypersurface of a Lorentzian manifold, we induce a Riemannian metric on the null hypersurface using a normalizing field defined on some open set containing the null hypersurface. We establish links between the null geometry and basics invariants of the associated Riemannian metric. This allowed us, using some comparison theorems from Riemannian geometry, to get important results on the topology of the null hypersurface.

Keywords: Null hypersurface; Rigging; Closed normalization; Associated Riemannian metric

Introduction

In spite sharing common root, Lorentzian and Riemannian geometries diverge very fast. For example, in Lorentzian case, due to the causal character of three categorie of vector fields (namely, spacelike, timelike and null), the induced metric on a hypersuface is a non-degenerate metric tensor field or degenerate symmetric tensor field depending on whether the normal vector field is of the first two types or the third one. On no-degenerate hypersurfaces one can consider all the fundamental intrinsic and extrinsic geometric notions. In particular, a well defined (up to sign) of the unit orthogonal vector field is known to lead to a canonical splitting of the ambient tangent space into two factors: a tangent and an orthogonal one. Therefore by respective projections, one has fundamental equations such as the Gauss, the Codazzi, the Weingarten equations, along with the second fundamental form, shape operator, induced connection, etc. The case the normal vector field is null, the hypersurface is called null (or lightlike). Null hypersurfaces are then exclusive objects from Lorentzian manifolds, and have not Riemannian counterpart, making them interesting by their own from a geometric point of view, but also they are key objects for modern physics (quantum gravity effects). The geometry of null submanifolds is different and rather difficult since (contrary to the non-degenerate counterpart) the normal vector bundle intersects (non trivially) with the tangent bundle. Thus, one can not find natural projector (and hence there is no preferred induced connection such as Levi-Civita) to define induced geometric objects as usual. This degeneracy of the induced metric makes impossible to study them as part of standard submanifold theory, forcing to develop specific techniques and tools. For the most part, these tools are specific to a given problem, or sometimes with auxiliary non-canonical choices on which, unfortunately, depends the constructed null geometry. Indeed, Duggal and Bejancu introduced a non-degenerate screen distribution or equivalently a null transversal line vector bundle as we may see below so as to get a three factors splitting of the ambient tangent space and derive the main induced geometric objects such as second fundamental forms, shape operators, induced connection, curvature, etc. [1]. Unfortunately, the screen distribution is not unique and there is no preferred one in general. The least we can say is that for the above approach to be complete and consistent, we still need to build a distinguished normalization to accompany it. Most of the recent works of the first named author are indeed devoted to this normalization problem [2-4]. Given the collective expertise in Riemannian geometry, the ideal situation on could expect is that the developed tools could bring to a full reduction of problems in null geometry to purely Riemannian ones. In, the present first named author, after fixing a pair of normalization, constructed an associated Riemannian metric to the "normalized null structure" [5,6]. These ideas have been generalized and improved where authors used riggings defined on neighborhood of the null hypersurface. In the present paper, we first consider the associated Riemannian metric as [5] but arising from a null rigging defined on neighborhood of the null hypersurface, and establish links between the null geometry and basics invariants of the associated Riemannian metric. Also, note that one of the major issues of Riemannian geometry is how to obtain topological or differential properties of a manifold from some known properties of its curvatures. For example what can be said about a complete Riemannian manifold when some suitable estimates are known for the sectional or Ricci curvature? These considerations have been on much scrutiny with excellent results: Myers (compactness), Klingenberg (on the injectivity radius), Cheeger-Gromoll (splitting theorem), Shoen-Yau (3-manifolds that are diffeomorphic to the standard R^3, Gromov's estimate of the number of generators of the fundamental group and the Betti numbers when lower curvature bounds are given. For further background on this problem we refer to the excellent texts [7-11]. Since in the present paper we have established links between the null geometry and basics invariants of the associated Riemannian metric, it is reasonable to expect that the geometry of the null hypersurface provides insight informations on its topology. This constitutes our second and main goal. The plan of the article is as follows. Section (2) sets notations and definitions on riggings (normalizations) and review basics properties on null hypersurfaces. The associated Riemannian distance structure on the rigged (or normalized) null hypersurface are introduced and discussed. The relashionship between the null and the associated Riemannian geometry is considered in section (4) where we proceed to a connection of the main geometric objects (invariants) of both side involved in our analysis. In the last sections, thanks to some Riemannian comparison theorems we get some topological facts on the null hypersurfaces from its null geometry.

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Let $(M,\bar{g})$ be a $(\nu+2)$-dimensional Lorentzian manifold and $M$ a null hypersurface in $M$. This means that at each $p\in M$, the restriction $\bar{g}_{\nu\nu}|_{T_{\nu}p}$ is degenerate, that is there exists a non-zero vector $U\in T_{\nu}M$ such that $\bar{g}(U,\nu) = 0$ for $\nu\in T_{\nu}M$. Hence, in null setting, the normal bundle $TM^\perp$ of the null hypersurface $M$ is a rank $1$ vector subbundle of the tangent bundle $TM$, contrary to the classical theory of non-degenerate hypersurfaces for which the normal bundle has trivial intersection [0] with the tangent one plays an important role in the introduction of the main induced geometric objects on $M$. Let start with the usual tools involved in the study of such hypersurfaces according to Duggal [1]. They consist in fixing on the null hypersurface a geometric data formed by a lightlike section and sreen distribution. By screen distribution on $M$, we mean a complementary bundle of $TM^\perp$ in $TM$. In fact, there are infinitely many possibilities of choices for such a distribution provided the hypersurface $M$ be paracompact, but each of them is canonically isomorphic to the factor vector bundle $TM/ TM^\perp$. This means that at each $N$, $(\nu,2)$-tensor field $\phi$ on $M$ is nowhere vanishing, being an integrable distribution whenever it is closed. It is easy to check that $\mathcal{S}(N)\in\ker\phi$ and that the screen distribution $\mathcal{S}(N)$ is integrable whenever it is closed.

On normalized null hypersurface $(M,g,N)$, the local Gauss and Weingarten type formulas are given by:

\[
\nabla_{\nu} Y = \nabla_{\nu} Y + B^{\nu}(X,Y),
\]

\[
\nabla_{\nu} N = -A_{\nu}X + \tau^\nu(X),
\]

\[
\nabla_{\nu} P Y = \nabla_{\nu} P Y + C^\nu(X, PY)z_{\nu},
\]

\[
\nabla_{\nu} z = -A_{\nu}X + \tau^\nu(X)z_{\nu},
\]

for any $X,Y \in \Gamma(TM)$, where $\nabla$ denotes the Levi-Civita connection on $(M,g,N)$, $\nu$ denotes the connection on $M$ induced from $\nabla$ through the projection along the null rigging $N$, $\nu$ denotes the Levi-Civita connection on the screen distribution $\mathcal{S}(N)$ induced from $\nabla$ through the projection morphism $\rho$ of $\Gamma(TM)$ onto $\Gamma(\mathcal{S}(N))$ with respect to the decomposition. Now the $(0,2)$ tensor $B^{\nu}$ and $C^\nu$ are the second fundamental forms on $M$ and $\mathcal{S}(N)$ respectively, $A_{\nu}$ and $A^\nu$ are the shape operators on $TM$ and $\mathcal{S}(N)$ respectively and $z^\nu$ a $1$-form on $TM$ defined by $\tau^\nu(X) = \tau_{\nu}^{\nu}(X,N,\xi)$. For the second fundamental $B^{\nu}$ and $C^\nu$'s following hold:

\[
B^{\nu}(X,Y) = g(A_{\nu}X,Y) + C^\nu(X, PY) - g(A_{\nu}X, PY) + g(A_{\nu}X, N) - g(A_{\nu}X, N) = 0,
\]

\[
\forall X,Y \in \Gamma(TM),
\]

\[
B^{\nu}(X,\nu) = 0, A^\nu = 0.
\]

It follows from eqn. (10) that integral curves of $\xi$ are pregeodesic both $M$ and $M$ as consider these integral curves to be geodesics which means that $\tau^\nu(\xi) = 0$.

A null hypersurface $M$ is called totally umbilical (resp. geodesic) if there exists a smooth function $\rho$ on $M$ such that at each $p\in M$ and for all $u,v\in T_{\nu}M BN(u,v) = \rho g(u,v)$ resp $B^{\nu}$ vanishes identically on $M$. These are intrinsic notions on any null hypersurface in the following way. Note that $N$ being a null rigging for $M$, a vector field $N \in \Gamma(TM)$ is a null rigging for $M$ if and only if it is defined in an open set containing $M$ and there exist a function $\psi$ and section $\zeta$ of $TM$ with the properties that $\psi$, $i$ is nowhere vanishing, being $i$ the inclusion map and $2\nabla_{\nu} d\psi(\zeta)\|\zeta\|^2 = 0$.

Then we have for details on changes in normalizations) $B^{\nu} = \frac{1}{\psi}\nabla_{\nu}B^{\nu}$, which shows that total umbilicity and totally geodesibility are intrinsic properties for $M$ [3]. The total umbilicity and the total geodesibility conditions for $M$ can also be written respectively as $A^\nu = \rho P$ and $A^\nu = 0$. Also, the screen distribution $\mathcal{S}(N)$ is totally umbilical (resp. totally geodesic) if $C^\nu(X, PY) = \psi g(X, Y)$ for all $X,Y \in \Gamma(TM)$ resp. $C^\nu = 0$ which is equivalent to $A_{\nu} = \lambda P$ (resp. $A_{\nu} = 0$). It is noteworthy
to mention that the shape operators $A_{\nu}^{\mu}$ and $A_{\nu}$ are $\mathcal{S}(N)$-valued. The induced connection $\nabla$ is torsion-free, but not necessarily g-metric unless $M$ is totally geodesic. In fact we have for all tangent vector fields $X,Y$ and $Z$ in TM:

$$\langle \mathcal{V},gX,Y\rangle = B^\gamma(X,Y)\eta(Z) + B^\gamma(Z,X)\eta(Y).$$  \hspace{1cm} (12)

Let denote by $\mathcal{R}$ and $R$ the Riemann curvature tensors of $\mathcal{V}$ and $\nabla$, respectively. Recall the following Gauss-Codazzi equations for all $X,Y,Z \in \Gamma(TM)$, $N \in \tau TM$, $\xi \in \Gamma(TM)$.

$$\mathcal{g}(\mathcal{R}(X,Y)Z,\xi) = \langle \mathcal{V},B^\gamma(X,Y)\rangle - \langle \mathcal{V},B^\gamma(Y,Z)\rangle + B^\gamma(Y,Z)\tau^\gamma(X)$$

$$-B^\gamma(X,Z)\tau^\gamma(Y).$$  \hspace{1cm} (13)

$$\mathcal{g}(\mathcal{R}(X,Y)Z,\xi) = g(R(X,Y)Z,\xi) + B^\gamma(X,Z)C^\gamma(Y,\xi)$$

$$-B^\gamma(Y,Z)C^\gamma(X,\xi).$$  \hspace{1cm} (14)

$$\mathcal{g}(\mathcal{g}(X,Y)\xi,\eta) = \mathcal{g}(\mathcal{g}(X,Y)\xi,\eta)$$

$$= C^\gamma(Y,\xi)\eta - C^\gamma(X,\xi)\eta - 2d\tau^\gamma(X,Y).$$  \hspace{1cm} (15)

The shape operator $A_{\nu}^\mu$ is self-adjoint as the second fundamental form $B^\gamma$ is symmetric. However, this is not the case for the operator $A_{\nu}$ as shown in the following lemma.

**Lemma 2.1:** For all $X,Y \in TM$

$$\langle A_{\nu}^\mu(X,Y) - A_{\nu}(X,Y), Y \rangle = \tau^\gamma(Y)\eta(X) - \tau^\gamma(X)\eta(Y) - 2d\eta(Y,X)$$  \hspace{1cm} (16)

where (throughout) $\langle \cdot,\cdot \rangle = \mathcal{g}$ stands for the Lorentzian metric.

**Proof:** Recall that $\eta=\iota \theta$ where $\theta = (N,\iota)$ taking the differential of $\theta$ and using the weingarten formula, we have for all $X,Y \in TM$

$$2d\eta(Y,X) = 2d\theta(Y,X) = \langle \mathcal{V},sN,Y\rangle - \langle \mathcal{V},sN,X\rangle$$

$$= -\langle A_{\nu}^\mu(X,Y) + \tau^\gamma(Y)\eta(X) + \langle A_{\nu}^\mu(X,Y)\rangle - \tau^\gamma(Y)\eta(X)\rangle(X).$$

Hence

$$\langle A_{\nu}^\mu(X,Y) - A_{\nu}(X,Y), Y \rangle = \tau^\gamma(Y)\eta(X) - \tau^\gamma(X)\eta(Y) - 2d\eta(Y,X)$$  \hspace{1cm} (17)

as announced. In case the normalization is closed the (connection)1-form $\tau^\gamma$ is related to the shape operator $A_{\nu}$ as follows.

**Lemma 2.2:** Let $(M,\mathcal{g},N)$ be a closed normalization of a null hypersurface $M$ in a Lorentzian manifold such that $\tau^\gamma(t)=0$. Then

$$\tau^\gamma = -\langle A_{\nu}^\mu,\xi \rangle.$$  \hspace{1cm} (18)

**Proof:** Assume $\eta=\iota \theta$ closed and let $X,Y$ be tangent vector fields to $M$. The condition $\mathcal{X}\eta(X,Y)-\mathcal{Y}\eta(X,Y) = 0$ is equivalent to $\langle \mathcal{V},sN,Y\rangle = \langle \mathcal{V},sN,X\rangle$. Then by the weingarten formula, we get

$$\langle -A_{\nu}^\mu(X,Y) + \tau^\gamma(Y)\eta(X) + \langle A_{\nu}^\mu(X,Y)\rangle - \tau^\gamma(Y)\eta(X)\rangle(X).$$

In this relation, take $Y=\xi$ to get

$$\tau^\gamma(Y)\eta(X) = -\langle A_{\nu}^\mu,\xi \rangle + \tau^\gamma(\xi)\eta(X)$$

which gives the desired formula as $\tau^\gamma(\xi)=0$.

**Example 2.1**

In the pseudo-Euclidean space $\mathbb{R}^{n+1}_c(q \geq 1)$. The pseudo-Euclidean space $\mathbb{R}^{n+1}_c(q \geq 1)$ is the space of all piecewise smooth curves $\gamma: [0,1] \to M$ with $\gamma(0)=p$ and $\gamma(1)=q$. The ambient manifold being Lorentzian, the induced metric on $M$ has signature $(0,n)$. In particular, for each $t \in [0,1]$, we have $\|\gamma'(t)\|^2 + \eta(\gamma'(t)) = 0$ if and only if $\gamma' = 0$. We define the $\eta$-arc length of $\gamma \in \gamma M$ as follows.

$$L_\eta(\gamma) = \int_0^1 \|\gamma'(t)\|^2 + \eta(\gamma'(t)) dt.$$  \hspace{1cm} (19)

From the above facts and using standard techniques as in Riemannian setting, and noting that a tangent vector $X$ belongs to $\mathcal{S}(N)$ if and only if $\eta(X)=0$, one gets the following lemma.

**Lemma 3.1:** The map $d_\eta:M \times [0,\infty) \to \mathcal{S}(N)$ given by

$$d_\eta(x,q) = \inf_{\gamma \in \gamma M} L_\eta(\gamma)$$

is a distance function on $M$.  

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Definition 3.1
A normalized null hypersurface (M, g, N) is said to be \( \eta \)-complete if the metric space \((M, d^{\eta})\) is a complete space.

Notations 3.1: For \( x \in M \) we set
\[
S^i(1) = \{ X \in TM, \langle X, X \rangle = 1 - \eta(X)^2 \},
\]
\[
S^e(1) = \bigcup_{x \in M} S^i(1),
\]
and for all \( x \in \Gamma(TM) \), we set
\[
O_{\eta}(X) = \{ Y \in TM, \langle X, Y \rangle = -\eta(X)\eta(Y) \},
\]
where \( (\cdot) \) stands both for \( \overline{\cdot} \) or \( g \). Observe that, \( y \in O_{\eta}(X) \) iff \( X \in O_{\eta}(Y) \).

A remarkable fact is that due to the degeneracy of the induced metric \( g \) on the null hypersurface \( M \), it is not possible to define the natural dual (musical) isomorphisms \( * \) and \( \# \) between the tangent vector bundle \( TM \) and its dual \( T^*M \) following the usual Riemannian way. However, this construction is made possible by setting a rigging (normalization) \( N \) (we refer to ref. [5] for further details). Consider a normalized null hypersurface \((M, g, N)\) and we define one form define by:
\[
x_{\eta}(\Gamma(TM)) \rightarrow \Gamma(T^*M) \quad \quad \quad \quad \quad \quad \quad \quad (24)
\]
\[
x \rightarrow x^\eta = g(X, x) + \eta(X)g(Y, Y), \quad \forall Y \in \Gamma(TM), x \in \Gamma(T^*M) = g(X, x) + \eta(X)\eta(Y).
\]

Clarity, such a \( \# \) is an isomorphism of \( \Gamma(TM) \) on to \( \Gamma(T^*M) \) and can be used to generalize the usual non-degenerate theory. In the latter case, \( \Gamma(S(N)) \) coincides with \( \Gamma(TM) \), and as a consequence the \( 1 \)-form \( \eta \) vanishes identically and the projection morphism \( P \) becomes the identity map on \( \Gamma(TM) \). Let \( \# \) denote the inverse of the isomorphism \( \# \) given by eqn. (24). For \( x \in \Gamma(TM) \) (resp. \( w \in \Gamma(T^*M) \)), \( x^\eta \) (resp. \( w^\# \) ) is called the dual \( 1 \)-form of \( x \) (resp. the dual vector field of \( w \)) with respect to the degenerate metric \( g \). It follows from eqn. (24) that if \( w \) is a \( 1 \)-form on \( M \), we have for \( x \in \Gamma(TM) \),
\[
g_\eta(x, Y) = X^\eta(Y), \quad \forall \, X, Y \in \Gamma(TM),
\]
Define a \((0,2)\)-tensor \( g_\eta \) by \( g_\eta(x, Y) = X^\eta(Y), \forall \, X, Y \in \Gamma(TM) \). Clarity, \( g_\eta \) defines a non-degenerate metric on \( M \) which plays an important role in defining the usual differential operators gradient, divergence, laplacian with respect to degenerate metric \( g \) on null hypersurface \([5]\). It is called the associate metric to \( (M, g, N) \). Also, observe that in non-degenerate case, the two metrics \( g_\eta \) and \( g \) coincide. The \((0,2)\)-tensor \( g_\eta \)' inverse of \( g_\eta \) is called the pseudo-inverse of \( g \) with respect to the rigging \( N \). With respect to the quasiorthonormal local frame field \( \{ \xi_1, \xi_2, \ldots, \xi_n \} \) adapter to the decomposition eqn. (2) and (3) the following verifications are straightforward,
\[
g_\eta(\xi, x) = \eta(x), \quad \forall \, x \in \Gamma(TM)
\]
\[
g_\eta(x, Y) = g(X, Y), \quad \forall \, X, Y \in \Gamma(S(N)).
\]
In particular,
\[
g_\eta(\xi, \xi) = 1
\]
and the last equality in eqn. (26) is telling us that the restrict to \( S(N) \) the metrics \( g_\eta \) and \( g \) coincide. We know from eqn. (12) that the induced metric \( g_\eta \) is not compatible with the induced connection \( \nabla_\eta \) in general and this compatibility arises if and only the null hypersurface \( M \) is totally geodesic in \( \tilde{M} \). Let \( \nabla^\eta \) denote the Levi-Civita connection of the non-degenerate associate metric \( g_\eta \) on \( (M, g, N) \). We are now interested in characterizing the normalizations for which the Levi-Civita connection \( \nabla^\eta \) of \( g \) agrees with the induced connection \( \nabla \) due to \( N \), i.e \( \nabla = \nabla^\eta \) For this we recall the following.

Lemma 3.2: For all \( X, Y, Z \in \Gamma(TM) \) we have,
\[
\begin{align*}

\nabla_{\xi} g_\eta(Y, Z) &= \eta(Y)B^\eta(X, \xi) - C^\eta(X, \xi, Z) + \eta(Z)B^\eta(X, \xi, Y) - C^\eta(X, \xi, Y) + \eta(Z)C^\eta(X, \xi, Y) - C^\eta(X, \xi, Y) + 2\eta(Z, C^\eta(X, \xi, Y)) - \eta(Z)C^\eta(X, \xi, Y) \quad (27)
\end{align*}
\]

We derive the following result on the compatibility condition.

Theorem 3.1
Let \((M, g, N)\) be a normalized null hypersurface of a pseudo-Riemannian manifold \((\tilde{M}, g)\) [4]. The induced connections \( \nabla \) and the Levi-Civita connection \( \nabla^\eta \) of the associate metric \( g_\eta \) on \((M, g, N)\) agree if and only if for all \( X, Y, Z \in \Gamma(TM) \),
\[
\begin{align*}

B^\eta(X, Y, Z) &= C^\eta(X, Y, Z) = 0 \quad (27)
\end{align*}
\]

Definition 3.2
A normalized null hypersurface \((M, g, N)\) of a pseudo-Riemannian manifold \((\tilde{M}, g)\) is said to have a conformal screen if there exists a non-vanishing smooth function \( \varphi \) on \( M \) such that \( A_\varphi = \varphi A_\tau^\eta \) holds [4].

This is equivalent to saying that \( C^\eta(X, Y, Z) = \varphi B^\eta(X, Y) \) for all tangent vector fields \( X \) and \( Y \). The function \( \varphi \) is called the conformal factor. Theorem (3.1) asserts that the compatibility condition is fulfilled if and only if the normalization is screen conformal with constant conformal factor 1 and vanishing normalizing \( 1 \)-form \( \tau^\eta \).

Remark 3.1
Observe that in ambient Lorentzian case, the Riemannian distance \( d_{0}\eta \), associated to \((M, g)\) coincides with the metric \( d^{\eta} \) as given in section (3), i.e \( d^{\eta} = d_{0}\eta \). It follows the famous Hopf-Rinow theorem that the null hypersurface \((M, g, N)\) is \( d^{\eta} \)-complete if and only if the Riemannian manifold \((M, g)\) is complete. Also, for all \( x \in M \),
\[
S^i(1) = \{ \{ X \in TM, \langle X, X \rangle = 1 - \eta(X)^2 \} \} = \{ X \in \Gamma(TM), g_\eta(x, X) = 1 \},
\]
that is \( S^i(1) \) coincides the unit bundle of \( M \) with respect to the associated Riemannian metric \( g_\eta \) from the normalization. It also holds that for all \( X \in \Gamma(TM) \) \( O_{\eta}(X) = X^\eta \).

Relation Between the Null and Associated Riemannian Geometry
Connecting the covariant derivatives
Let \((M, g, N)\) be a normalized null hypersurface of pseudo-Riemannian manifold \((\tilde{M}, \tilde{g}) \). The induced connection on \( M \). In order to relate the main geometric objects of both null and associated non-degenerate geometry on the null hypersurface, we first need to relate the covariant derivatives \( \nabla^\eta \) and \( \nabla \). In this respect, we prove the following.

Proposition 4.1: Let \((M, g, N)\) be a normalized null hypersurface with rigged vector field \( \xi \). Then, for all \( X, Y \in \Gamma(TM) \), we have
\[
\nabla^\eta Y = \nabla Y + \frac{1}{2} \{ 2g(A_\varphi X, Y) = g(A_\varphi A_\tau^\eta X, Y) - g(A_\tau^\eta A_\varphi X, Y) \}
\]
\[
+ \eta(X)C^\eta(Y) + \eta(Y)C^\eta(X) + \eta(X)(i_d\eta)^{\xi} + \eta(Y)(i_d\eta)^{\xi} \quad (28)
\]
In particular for a closed normalization,
\[ V^a_x Y = V^a_y Y - \frac{1}{2}[2g(A'_x, X, Y) - g(A_x, X, Y) - g(A_y, Y, X) + \eta(Y)^r(X') + \eta(Y)^r(X')\xi] \]

**Proof.** Both connections \( \nabla^0 \) and \( \nabla \) are torsion free. We can write \( V^a_x Y = V^a_y Y + D(Y, X), V^a_x Y = \nabla(Y, X) \), with \( D \) a symmetric tensor. As \( V^a_x \) is \( g \)-metric, we have:

\[ g_x(D(Y, X), Z) + g_y(D(Y, X), Z) = X g_x(Y, Z) - g_x(Y, Z) - g_y(Y, Z) = (V_x g_x)(Y, Z). \]

Then using this and Lemma (3.2), we have:

\[ g_x(D(Y, X), Z) + g_x(Y, D(Y, X), Z) = \eta(Y)[B^\alpha (X, PY) - C^\alpha (X, PY)] \]

\[ + 2\eta(Y)^r(X') \eta(Y)^r(X'). \]

By circular permutation we get similar expression for \( g_x(D(Y, X), Z) + g_x(Y, D(Y, X), Z) \), and \( g_x(D(Y, X), Y) + g_x(D(Y, X), Y) \).

Summing the first two expressions minus the last one leads to:

\[ 2g_x(D(Y, X), Z) + \eta(x)[g_x(A_x, Y) - g_x(A_y, Y)] \]

\[ + \eta(Y)[g_x(A_x, Z) - g_x(A_y, Z)] \]

\[ + 2\eta(Y)^r(X') \eta(Y)^r(X') \eta(Y)[Z]. \]

Using eqns. (26) and (26), we get:

\[ 2g_x(D(Y, X), Z) = 2g_x[\eta(Y)(\alpha d\eta)^s + \eta(Y)(\beta d\eta)^s, Z]; \]

\[ + g_x[2\eta(Y)^r(X') + \eta(Y)^r(X')\xi], \]

\[ - g_x(A_x, Y) - g_x(A_y, Y, Z)]. \]

It follows the non-degeneracy of \( g \) that:

\[ D(Y, X) = \frac{1}{2}(2g_x[A_x, Y] + \eta(x)^r(X') + \eta(Y)^r(X') \xi - g_x(A_x, Y) - g_x(A_y, Y, Z)]. \]

(30)

Proposition 4.2: Let \((M, g, N)\) be a closed normalized null hypersurface with rigged vector field \( \xi \). Then, for all \( X, Y, \xi \in TM \) and \( U \in TM^\perp \) we have:

\[ g_x(R^{(X, Y)}_Z, U) = -g_x(R^{(Y, X)}_Z, U) + \frac{1}{2}[\phi(X, Y)_Z - \phi(Y, X)_Z]. \]

(33)

Then eqns. (37) and (38) consist on repeated applications of (30) in Proposition (4.1).

Remark 4.2: Note that by lemma (2.2), for a closed and conformal normalization (with factor \( \phi \) we have \( \xi = 0 \) and \( \phi(X, Z) = 2(1 - \phi)B^\alpha (X, Z) \)). Then eqns. (37) and (38) take the forms:

\[ g_x(R^{(X, Y)}_Z, U) = -g_x(R^{(Y, X)}_Z, U) \]

\[ + \frac{1}{2}[\phi(X, Y)_Z - \phi(Y, X)_Z]. \]

(34)

Proposition 4.3: Let \((M, g, N)\) be a closed normalized null hypersurface with rigged vector field \( \xi \). Then, for all \( X, Y, \xi \in TM \) and \( U \in TM^\perp \) we have:

\[ g_x(R^{(X, Y)}_Z, U) = -g_x(R^{(Y, X)}_Z, U) \]

\[ + \frac{1}{2}[\phi(X, Y)_Z - \phi(Y, X)_Z]. \]

(35)

**Proof.** The Riemann curvature tensor field \( R \) of type (1.3) is defined by:

\[ R^{(X, Y)}_Z = [\nabla^{(X, Y)}_Z - \nabla^{(Y, X)}_Z - \nabla^{(X, Y)}_Z + \nabla^{(Y, X)}_Z]. \]

(36)

Then eqns. (37) and (38) consist on repeated applications of (30) in Proposition (4.1).

Remark 4.2: Note that by lemma (2.2), for a closed and conformal normalization (with factor \( \phi \) we have \( \xi = 0 \) and \( \phi(X, Z) = 2(1 - \phi)B^\alpha (X, Z) \)). Then eqns. (37) and (38) take the forms:

\[ g_x(R^{(X, Y)}_Z, U) = -g_x(R^{(Y, X)}_Z, U) \]

\[ + \frac{1}{2}[\phi(X, Y)_Z - \phi(Y, X)_Z]. \]

(34)

In the following we let \( Ric \) and \( Ric \) denote the Ricci curvature of \( V^0 \) and \( V \) respectively. Recall that \( Ric^{(X, Y)}_Z = trace X \rightarrow R^{(X, Y)}_Z, \forall X, Y, \xi \in TM \). This is a symmetric (0,2)-tensor on TM. Unfortunately, the corresponding quantity \( Ric(X, Y) \) obtained from \( V \) is no longer symmetric in general, due to the fact that the induced Riemann curvature \( R \) on the normalized null hypersurface \((M, g, N)\) fails to have the usual algebraic curvature symmetries in general. Precisely, the induced Ricci tensor \( Ric \) is given by:

\[ Ric^{(X, Y)}_Z = Ric^{(X, Y)}_Z + B^{(X, Y)}_Zg^{\alpha}_{\beta}A_{\alpha} - \theta(\overrightarrow{\xi}, Y)X - g(A_x, A_y, Z). \]

(37)

The Ricci curvature of the ambient manifold is defined by:

\[ Ric^{(X, Y)}_Z = \frac{1}{2}[Ric^{(X, Y)}_Z + Ric^{(Y, X)}_Z]. \]

(38)

In the following we let \( Ric \) and \( Ric \) denote the Ricci curvature of \( V^0 \) and \( V \) respectively. Recall that \( Ric^{(X, Y)}_Z = trace X \rightarrow R^{(X, Y)}_Z, \forall X, Y, \xi \in TM \). This is a symmetric (0,2)-tensor on TM. Unfortunately, the corresponding quantity \( Ric(X, Y) \) obtained from \( V \) is no longer symmetric in general, due to the fact that the induced Riemann curvature \( R \) on the normalized null hypersurface \((M, g, N)\) fails to have the usual algebraic curvature symmetries in general. Precisely, the induced Ricci tensor \( Ric \) is given by:

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(37)

The Ricci curvature of the ambient manifold is defined by:

\[ Ric^{(X, Y)}_Z = \frac{1}{2}[Ric^{(X, Y)}_Z + Ric^{(Y, X)}_Z]. \]

(38)
for all $X, Y, T M$.

**Theorem 4.1**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigged vector field $\xi$ and $t^\alpha(\xi) = 0$ in a $(n+2)$-pseudo-Riemannian manifold. Then,

$$Ric^\alpha(X, Y) = Ric(X, Y) - \langle [A_\xi^\alpha X, Y] - \langle A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle \varepsilon_{ij}^\alpha$$

$$+ (\langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle \varepsilon_{ij}^\alpha)

+ (\langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(39)

**Proof.** Let $p \in M$ and $(E_1, \ldots, E_n)$ be a quasiorthonormal basis for $(T_p^\alpha M, g)$ with $\text{Span}(E_1, \ldots, E_n) = S(N)$. When dealing with indices, we adopt the following conventions: $i, j, k, \ldots = \{1, \ldots, n\}$, $\alpha, \beta, \gamma = \{0, \ldots, n\}$, and $a, b, \ldots, \in \{0, \ldots, n+1\}$. Then we have:

$$Ric^\alpha(X, Y) = \sum_{i=1}^n g_{ij}^\alpha R^\alpha(E_i, X)Y, E_i).$$

Thus from eqn. (37) and (38), we get,

$$R^\alpha_{\xi}(X, Y) = g_{ij}^\alpha R^\alpha(E_i, X)Y, E_i) + \sum_{i=1}^n g_{ij}^\alpha R^\alpha(E_i, X)Y, E_i).$$

Then, $Ric^\alpha(X, Y) = g_{ij}^\alpha R^\alpha(E_i, X)Y, E_i) + \sum_{i=1}^n g_{ij}^\alpha R^\alpha(E_i, X)Y, E_i).$

$$= -g(A_\xi^\alpha X, A_\xi^\alpha X) + \frac{1}{2} [g(A_\xi^\alpha X, Y) + g(A_\xi^\alpha Y, X) - \tau^\alpha(X)\eta(Y)]$$

$$+ - (\tau^\alpha(Y)\eta(X) \langle A_\xi^\alpha X, Y \rangle - (\langle V_\alpha Z_\xi^\alpha (X), Y \rangle + \langle V_\alpha A_\xi^\alpha X, Y \rangle + \langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(40)

**Corollary 4.2**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigid vector field $\xi$ and $t^\alpha(\xi) = 0$ in a $(n+2)$-pseudo-Riemannian manifold with constant curvature $k$. Then,

$$Ric^\alpha(X, Y) = nk(X, Y) + (A_\xi^\alpha X, Y)\text{tr}A_\xi^\alpha - (A_\xi^\alpha X, A_\xi^\alpha Y)$$

$$- \langle [A_\xi^\alpha X, Y] - \langle A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle \varepsilon_{ij}^\alpha$$

$$- (\langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle \varepsilon_{ij}^\alpha)

+ (\langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(41)

**Theorem 4.4**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigid vector field $\xi$ and $t^\alpha(\xi) = 0$ in a pseudo-Riemannian manifold. Then

$$r^\alpha = -\langle [A_\xi^\alpha X, A_\xi^\alpha Y] + (A_\xi^\alpha X, Y)\text{tr}A_\xi^\alpha,

+ \langle (V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(42)

**Proof.** To get eqn. (45), take $Y = (\xi)$ in the (41). Recall that from a geometric point of view and in practice, one gets the scalar curvature by contracting with a (non-degenerate) metric the (symmetric) Ricci curvature. It turns out that in the null geometry setting, such a scalar quantity cannot be calculated by the usual way (degeneracy of the induced metric and the failure of symmetry in the induced Ricci curvature) [12]. This justifies introducing a symmetrized Ricci curvature as an associated non-degenerate metric $g_{\xi}$ in calculating this scalar quantity. More precisely, the extrinsic scalar curvature $r^\alpha$ on the null hypersurface $(M, g, N)$ is given by $g_{\xi}$ trace of the symmetrized Ricci curvature $Ric^\alpha$. With respect to a local quasiorthonormal frame $(e_0 = \xi, e_1, \ldots, e_n)$ for $(M, g, N)$ we have

$$r^\alpha = g_{A_\xi^\alpha} Ric^\alpha_{\xi}. (43)$$

Now let $r^\alpha$ denote the scalar curvature of the non-degenerate metric $g_{\xi}$ on $M$ that is the contraction of $Ric^\alpha$ with respect to $g_{\xi}$. In the following, we state a formula relating the extrinsic scalar curvature $r^\alpha$ to the associated scalar curvature $r^\alpha$.

**Theorem 4.5**

Let $(M, g, N)$ be a closed normalized null hypersurface with rigid vector field $\xi$ and $t^\alpha(\xi) = 0$ in a pseudo-Riemannian manifold. Then

$$r^\alpha = -\langle [A_\xi^\alpha X, A_\xi^\alpha Y] + (A_\xi^\alpha X, Y)\text{tr}A_\xi^\alpha,

+ \langle (V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(44)

**Proof.** We have

$$r^\alpha = g_{A_\xi^\alpha} Ric^\alpha_{\xi}. (45)$$

in a local quasiorthonormal frame field $(e_0 = \xi, e_1, \ldots, e_n)$ for $(M, g, N)$ with span $(e_0, \ldots, e_n) = S(N)$. But

$$Ric^\alpha_{\xi} - Ric^\alpha_{\xi} = -\langle [A_\xi^\alpha X, A_\xi^\alpha Y] + (A_\xi^\alpha X, Y)\text{tr}A_\xi^\alpha,

+ \langle (V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \langle V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(46)

Hence, by contracting each side with $g_{\xi}$ and taking into account Proposition (4.1) along with the following facts:

$$g_{\xi} (V_\alpha Z_\xi^\alpha (X), Y) = \text{tr} (\nabla_{A_\xi^\alpha} + g_{\xi} (V_\alpha Z_\xi^\alpha (X), Y) = 0;$$

$$g_{\xi} ((V_\alpha A_\xi^\alpha X, Y) + \langle (V_\alpha Z_\xi^\alpha (X), Y \rangle - (\langle V_\alpha A_\xi^\alpha X, Y \rangle + \tau^\alpha(X)\eta(Y) \rangle).$$

(47)
Corollary 4.3
Let \((M,g,N)\) be a closed normalized null hypersurface of a \((n+2)\)-dimensional Lorentzian space form and

\[
\tau(\xi) = 0.\quad \text{Then,}\quad r^\nu = -n^k + 2r^\nu_tr^k - (r^\nu_A)_t.\quad \text{(48)}
\]

Suppose \(\pi\) is a non-degenerate plane (for \(g\)) in \(T_pM\). The real number

\[
K_\pi = \frac{g(A^\nu,U^\nu,V^\nu) - (g(A,U,V))}{g(U,U)g(V,V) - (g(U,V))^2}
\]

is the sectional curvature of \((\text{with respect to } g)\). A similar definition holds if \(\pi\) is non-degenerate with respect to \(g\), and we denote the corresponding quantity by \(K_\pi\), as we know, it is easy to check that the right hand side of eqn. (49) does not depend on the basis of \(\pi\). Let \(M\) and \(H\) be a null plane of \(T_pM\) direct by \(\pi = T_pM\). The null sectional curvature of \(H\) with respect to \(\xi\) is the real number

\[
K_\pi = \frac{g(A^\nu,U^\nu,V^\nu) - (g(A,U,V))}{g(U,U)g(V,V) - (g(U,V))^2}
\]

is the sectional curvature of \(H\) (for \(g\)). Below, \(\pi(X, Y) = \text{span}(X, Y)\) denotes a null plane directed by \((X, Y)\) in \(\text{span}(X, Y)\). Now, we show the following.

Lemma 4.1: Let \((M,g,N)\) be a closed normalized null hypersurface with a rigget vector field \(X\) and \(r(\xi) = 0\) in a Lorentzian manifold. Then for all \(p\in M\) and \(\pi\in S(N)\) we have:

\[
K_\pi = K(X) = B^\nu(X,Y)^\nu - B^\nu(X,Y)B^\nu(Y,Y) + B^\nu(X,Y)C^\nu(X,Y) - B^\nu(X,Y)C^\nu(Y,Y),\quad \text{(51)}
\]

Where \(X\) and \(Y\) are orthogonal in \(S(N)\) and \(\pi = \text{span}(X, Y)\)

Proof. Observe that a plane \(\pi\in S(N)\) is both non-degenerate with respect to \(g\) and \(\pi\) (simultaneously) or not. Now, eqn. (51) is a direct use of eqn. (37) in the eqn. (49), taking into account the fact without loss of generality, we have assumed \(X\) and \(Y\) \(\gamma_n\)-unit and orthogonal in \(N\) (and hence also for \(g\)).

Theorem 4.5
Let \((M,g,N)\) an \((n+1)\)-dimensional be a closed normalized null hypersurface of a Lorentzian space form \(M(k)\) and \(r^\nu(\xi) = 0\). Then for a non-degenerate plane \(\pi = \text{span}(X, Y)\subset T_pM, (p\in M)\)

\[
K_\pi = K(X) = B^\nu(X,Y)^\nu - B^\nu(X,Y)B^\nu(Y,Y) + B^\nu(X,Y)C^\nu(X,Y) - B^\nu(X,Y)C^\nu(Y,Y) + 2[\eta(X)C^\nu(Y,Y) + \eta(Y)C^\nu(X,Y)]B^\nu(X,Y) - 2[\eta(X)C^\nu(Y,Y) + \eta(Y)C^\nu(X,Y)]B^\nu(X,Y).\quad \text{(52)}
\]

Where \(X, Y\in S^0(1), Y\in O_p(X)\).

Proof. From \(X, Y\in S^0(1), Y\in O_p(X)\), we infer that \(g(X, X) = g(Y, Y) = 1\) and \(g(X, Y) = 0\). It follows that \(g(X, Y)g(Y, Y) - g(X, Y)Y = 1\) and \(K(X,Y) = -g(R(Y,Y)X,Y)\). Here, for a vector field \(X\), we brief \(X\) and \(X^\nu\) for \(PX\) and \(\eta(X)\) respectively, where \(P\) is the morphism projection of TM onto \(S(N)\) and then, \(X^\nu = X^\nu X^\nu\)

\[
K_\pi = K(X) = B^\nu(X,Y)^\nu - B^\nu(X,Y)B^\nu(Y,Y) + B^\nu(X,Y)C^\nu(X,Y) - B^\nu(X,Y)C^\nu(Y,Y) + 2[\eta(X)C^\nu(Y,Y) + \eta(Y)C^\nu(X,Y)]B^\nu(X,Y) - 2[\eta(X)C^\nu(Y,Y) + \eta(Y)C^\nu(X,Y)]B^\nu(X,Y).\quad \text{(52)}
\]
in eqn. (56). For a totally geodesic null hypersurface, the second fundamental form \( B^Y \) vanishes identically and we get the following.

**Corollary 4.4**

Let \((M,g,N)\) be a closed normalized a totally geodesic null hypersurface with \( \rho \neq 0 \) in a Lorentzian space form \((\overline{M},\overline{g})\) Then of a Lorentzian manifold \((\overline{M},\overline{g})\) that shows that \( S(N) \) is totally umbilical in null hypersurface \( M \).

\[
K_{\rho}(X,Y) = k[1 - \eta^2(X) - \eta^2(Y)].
\]

**Corollary 4.5**

Let \((M,g,N)\) be a closed normalized compact null hypersurface with \( \rho \neq 0 \) in a Lorentzian space form \((\overline{M},\overline{g})\) Then of a Lorentzian manifold \((\overline{M},\overline{g})\) that shows that \( S(N) \) is totally umbilical in null hypersurface \( M \).

\[
K_{\rho}(X,Y) = k[1 - \eta^2(X) - \eta^2(Y)].
\]

**Proof.**

set \( \eta^2 = 0 \) in eqn. (56). Now from Remark (4.2) and definition (3.2) the last claim follows.

A section \( N \) is called conformal Killing (CKV in short) or conformal collineation on \((\overline{M},\overline{g})\) if \( \rho \in C^\infty(M) \) for some \( \rho \in C^\infty(M) \) In case \( \rho \) vanishes identically \( N \) is called a Killing vector field and \( L_\xi \overline{g} = 0 \).

**Fact 4.1:** Assume \( N \) is a closed conformal collineation rigging of \((M,g)\). Then, the normalizing one-form \( \tau^N \) vanishes identically on \( M \). Moreover, the screen distribution \( S(N) \) is integrable and totally umbilical.

**Proof.**

Let \( X,Y \) be tangent to \( M \). We have:

\[
2\rho g(X,Y) = 2\rho g(X,Y) = (L_\xi \overline{g})(X,Y) = (\overline{\nabla}_X N,Y) + (X,\overline{\nabla}_Y N),
\]

that is

\[
-(A_\xi X,Y) + \tau^N(X)\eta(Y) + \tau^N(Y)\eta(X) = 2\rho g(X,Y).
\]

Hence, set \( \eta = \xi \) to get:

\[
2\tau^N(X) = 0
\]

i.e \( \tau^N(X) = 0 \), which shows that \( \tau^N \) vanishes on \( M \). Then we get by the closed assumption

\[
0 = (\overline{\nabla}_X N,Y) - (X,\overline{\nabla}_Y N) = -C^\infty(X,PY) + C^\infty(Y,PX) \quad \text{(as } \tau^N = 0).\]

So, for \( X,Y \in S(N) \) we have

\[
C^\infty(X,Y) = C^\infty(Y,X),
\]

that is the screen distribution is integrable. Now, return to eqn. (56) to get

\[
-C^\infty(X,PY) - C^\infty(Y,PX) = 2\rho g(X,Y), \quad X,Y \in \Gamma(TM).
\]

Hence, for \( X,Y \in S(N) \),

\[
C^\infty(X,Y) = -\rho g(X,Y).
\]

But, as

\[
C^\infty(\xi, PY) = (A_\xi, PY) = -\tau^N(PY) = 0.
\]

We deduce that

\[
C^\infty(X,PY) = -\rho g(X,Y)
\]

for all \( X,Y \in \Gamma(TM) \) which shows that \( S(N) \) is totally umbilical in null hypersurface \( M \).

**Theorem 4.6**

Let \((M,g,N)\) be a null hypersurface of a Lorentzian space form \((\overline{M},\overline{g})\) with a closed and conformally Killing (but not Killing) normalization. Then \((M,g)\) is totally umbilical in \((\overline{M},\overline{g})\). Moreover, \( \rho \) being the nowhere vanishing conformal factor of \( N \), we have for all \( X,Y \in \mathcal{S}_n(1) \), \( Y \in O_0(X) \),

\[
K_{\rho}(X,Y) = (3k - 2\xi(\rho))[1 - \eta^2(X) - \eta^2(Y)] + \frac{(\xi(\rho) - k)}{\rho^2} \eta^2(X)\eta^2(Y).
\]

**Proof.**

From the previous fact, \( S(N) \) is integrable and totally umbilical with umbilicity factor-\( \rho \). So, in this case, it is a well known fact that the following holds (MEN14, p.110)

\[
\{\xi(\rho) + \rho g^2(X)\} = -\rho B^N(X,Y).
\]

Hence, as \( \tau^N(0) = 0 \) and \( \rho \) is everywhere non zero, we get

\[
B^N(X,Y) = \frac{(\xi(\rho) - k)}{\rho^2} g(X,Y), \quad X,Y \in \Gamma(TM),
\]

that is \((M,g)\) is totally umbilical. Now, using previous expressions of \( B^N \) and \( C^\infty \) eqn. (57), we get for all \( X,Y \in \mathcal{S}_n(1) \), \( Y \in O(X) \),

\[
K_{\rho}(X,Y) = k[1 - \eta^2(X) - \eta^2(Y)] + \frac{(\xi(\rho) - k)}{\rho^2} \eta^2(X)\eta^2(Y).
\]

For all \( X,Y \in \mathcal{S}_n(1) \), \( Y \in O(X) \),

\[
g(X,Y) = 1 - \eta(X)^2, \quad g(Y,Y) = 1 - \eta(Y)^2
\]

and

\[
g(X,Y) = -\eta(X)\eta(Y),
\]

which after substitution leads to the desired expression of \( K_{\rho}(X,Y) \).

**Relationship between Curvature and Topology of Null Hypersurface**

In this section, we study the null geometry of manifolds and link their invariants to those of the induced associated Riemannian metric on them through the normalization. Thereafter, we use some comparison theorems from Riemannian geometry to get informations on the underline manifold topology.

**Theorem 5.1**

Let \((M,g,N)\) be a closed normalized compact null hypersurface of a Lorentzian manifold \((\overline{M},\overline{g})\) and \( \tau = 0 \).

If

\[
\text{Ric}(X,X) \geq \{(A_\xi X,X) - (A_\xi X,X)\} + \rho \eta(X)^2
\]

\[
-(\nabla_{\xi} A_{\xi})(X,X) + (\nabla_{\xi} A_{\xi})(X,X)
\]

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holds for all $X,T,M$, then the universal Riemannian covering of the associated Riemannian manifold $(M,g)$ is isometric to a product $(M \times \mathbb{R}^\delta, g \times \delta)$, where $\delta$ stands for the usual euclidean metric on $\mathbb{R}^\delta$ and $(M,g)$ is a compact simply connected Riemannian manifold with non negative Ricci curvature.

**Proof.** This is a direct application of a splitting result by Cheeger and Gromoll on compact Riemannian manifolds with non negative Ricci curvature [9,10]. Indeed, the ambient manifold being Lorentzian, we know that $M$ endowed with the associated metric $g$ is Riemannian. Also, conditions $\tau=0$ eqn. (64) and the expression (47) show that the Ricci curvature $\text{Ric}^g$ of the Riemannian metric $g$ satisfies $\text{Ric}^g \geq 0$ that is non negative, as the hypersurface is compact, the claim follows [11,12].

**Theorem 5.2**

Let $(M^{m+1}, g, N)$ be a closed normalized compact null hypersurface of dimension $\mathbb{L}$ and $\tau=0$, and

$$\text{Ric}(X,X) \leq \langle (\nabla \cdot A) X, X \rangle - \langle A X, X \rangle \text{pr}A_i$$

$$= -\langle (\nabla \cdot A_i) X, X \rangle + \langle (\nabla \cdot A_i) X, X \rangle).$$

Then every Killing vector field on $(M,g)$ is identically zero and group of isometry is finite.

**Proof.** Using Theorem (4.6), the assumption eqn. (64) is equivalent to saying that the hypersurface $M$ endowed with the associated metric $g$ has a negative sectional curvature. As it is compact, we conclude by theorem of Bochner [13] that every Killing vector field on null hypersurface is identically zero.

In dimension $3$, Schoen and Yau proved in that a complete non-compact manifold with positive Ricci curvature is diffeomorphic to the standard euclidean space $\mathbb{R}^3$. Using this and eqn. (47) lead to the following [8].

**Theorem 5.3**

Let $(M,g,N)$ be a closed normalized $d\delta$-complete non-compact null hypersurface of $\mathbb{L}$ and $\tau=0$, then

$$\text{Ric}(X,X) \leq \langle (\nabla \cdot A) X, X \rangle - \langle A X, X \rangle \text{pr}A_i$$

$$= -\langle (\nabla \cdot A_i) X, X \rangle + \langle (\nabla \cdot A_i) X, X \rangle).$$

Then the manifold structure of the null hypersurface $(M,g)$ is diffeomorphic to $\mathbb{R}^3$.

**Proof.** The ambient lorentzian manifold has dimension $4$, so the null hypersurface is 3-dimensional. The inequality ensures that the associated metric $g$ has a positive Ricci curvature. By the Shoen-Yau above quoted theorem [8], the claim follows.

**Theorem 5.4**

Let $(M,g,N)$ be a closed $d\delta$-complete null hypersurface of a Lorentzian space form $(M,k,g)$ with conformally Killing (but not Killing) normalization. Assume that for all $X,Y \in \mathcal{S}(1) \in \Omega(\mathbb{L})$, we have:

$$\frac{\langle \zeta(X), \zeta(Y) \rangle}{\rho^2} \eta(X) \eta(Y) \leq (3k - 2\zeta^2(\rho))|\eta(X)|^2 + |\eta(Y)|^2 - 1.$$ 

Then the universal covering of the null hypersurface is diffeomorphic to $\mathbb{R}^3$.

**Proof.** Now, $(M^{m+1},g,N)$ being dcomplete, it follows Remark 3.1 that the Riemannian manifold $(M,g,N)$ is complete. Also, using Theorem (4.6), the assumption eqn. (64) is equivalent to saying that the hypersurface $M$ endowed with the associated metric $g$ has a nonpositive sectional curvature. We conclude by Hadmar theorem that the universal of null hypersurface is diffeomorphic to $\mathbb{R}^3$.

The authors proved, using a correspondence for isometric immersions into product spaces that, on a complete Riemannian manifold $M$ with negative Ricci curvature, and whose scalar curvature is bounded above by a negative constant, the standard Euclidean space $(\mathbb{R}^n,\text{Euc})$ cannot be isometrically immersed into the Lorentzian product space $M \times \mathbb{L}$. A direct consequence of this fact is the following [14].

**Theorem 5.5**

Let $(M,g,N)$ be a $d\delta$-complete null hypersurface of a $(n+1)$-dimensional Lorentzian manifold $(M, g)$ with a closed normalization and $\tau=0$.

Assume that

$$\text{Ric}(X,X) \leq \langle (\nabla \cdot A_i) X, X \rangle - \langle A_i X, X \rangle \text{pr}A_i$$

$$= -\langle (\nabla \cdot A_i) X, X \rangle + \langle (\nabla \cdot A_i) X, X \rangle),$$

and

$$r \leq \langle \text{pr}A_i - \text{pr}A_j, \text{pr}A_i - \text{pr}A_j \rangle + \langle (\nabla \cdot A_i), X \rangle < c,$$

for some positive constant $c$. Then the standard Euclidean space $(\mathbb{R}^n,\text{Euc})$ cannot be isometrically immersed into the Lorentzian product space $M \times \mathbb{L}$ where the underline manifold $M$ is endowed with the Riemannian associated metric $g \times \mathbb{L}$ being $\mathbb{R}$ with the negative definite metric $-dt^2$.

**Proof.** The null hypersurface $M$ being $d\delta$-complete, it follows that $M$ endowed with the associated metric $g$ is a complete Riemannian manifold. Also, as $\tau=0$ from eqns. (62) and (47) we infer that the Ricci curvature of the associated Riemannian metric $g$ is negative, and by eqns. (51) and (63) its scalar curvature is bounded above by the negative constant $-c$. This completes the proof.

**Theorem 5.6**

Let $(M, g)$ a Lorentzian manifold and $(M,g,N)$ be a closed normalized complete null hypersurface in $\mathbb{L}$ with

$$\langle \zeta(X), \zeta(Y) \rangle = (3k - 2\zeta^2(\rho))|\eta(X)|^2 + |\eta(Y)|^2 - 1.$$ 

Then no complete Riemannian surface $(\sum, \sum)$ of constant curvature $c>0$ can be isometrically immersed into $M \times \mathbb{L}$.

**Proof.** The null hypersurface $M$ being $d\delta$-complete, it follows that $M$ endowed with the associated metric $g$ is a complete Riemannian manifold. Also, as $\tau=0$, from eqns. (62) and (47).

**Conclusion**

We infer that the sectional curvature of the associated Riemannian metric $g$ is negative, by using theorem José [14,15] on correspondence for isometric immersions into product spaces, we complete the proof with the research result of Through the research establishment of links between the null geometry and basics invariants of the associated Riemannian explained successfully.

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