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Research Article

Infinitesimal Deformations of the Model $\mathbb{Z}_3$-Filiform Lie Algebra

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Abstract In this work, it is considered that the vector space is composed by the infinitesimal deformations of the model $\mathbb{Z}_3$-filiform Lie algebra $L^{n,m,p}$. By using these deformations, all the $\mathbb{Z}_3$-filiform Lie algebras can be obtained, hence the importance of these deformations. The results obtained in this work, together with those obtained by Khakimdjanov and Navarro (*J. Geom. Phys.* 2011 and 2012), lead to compute the total dimension of the mentioned space of deformations.

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1 Introduction

The concept of filiform Lie algebras was firstly introduced by Vergne [18]. This type of nilpotent Lie algebra has important properties; in particular, every filiform Lie algebra can be obtained by a deformation of the model filiform algebra L_n. In the same way as filiform Lie algebras, all filiform Lie superalgebras can be obtained by infinitesimal deformations of the model Lie superalgebra $L^{n,m}$ [1,4,8,9].

Continuing with the work of Vergne, we have generalized the concept and the properties of the filiform Lie algebras into the theory of color Lie superalgebras. Thus, *filiform G-color Lie superalgebras* and the model *filiform G-color Lie superalgebra* were obtained in a previous study [10].

In the present, paper the focus of interest are color Lie superalgebras with a $\mathbb{Z}_3$-grading vector space (i.e., $G=\mathbb{Z}_3$, due to its physical applications) [3,7,6,13,16,17]. Due to the fact that the one admissible commutation factor for $\mathbb{Z}_3$ is exactly $\beta(g,h)=1\ \forall g,h$, $\mathbb{Z}_3$-color Lie superalgebras are indeed $\mathbb{Z}_3$-color Lie algebras or $\mathbb{Z}_3$-graded Lie algebras. Thus, we have studied the infinitesimal deformations of the model $\mathbb{Z}_3$-color Lie superalgebra (i.e., the model $\mathbb{Z}_3$-filiform Lie algebra $L^{n,m,p}$). By means of these deformations, all $\mathbb{Z}_3$-filiform Lie algebras can be obtained, hence the importance of these deformations.

Khakimdjanov and Navarro [11,12] decomposed the space of these infinitesimal deformations, noted by $Z^2(L;L)$, into six subspaces of deformations:

$$\begin{aligned}Z^2(L;L)\cap\mathrm{Hom}\left(L_0\wedge L_0,L_0\right)&\oplus Z^2(L;L)\cap\mathrm{Hom}\left(L_0\wedge L_1,L_1\right)\oplus Z^2(L;L)\\ &\cap\mathrm{Hom}\left(L_0\wedge L_2,L_2\right)\oplus Z^2(L;L)\cap\mathrm{Hom}\left(L_1\wedge L_1,L_2\right)\oplus Z^2(L;L)\\ &\cap\mathrm{Hom}\left(L_1\wedge L_2,L_0\right)\oplus Z^2(L;L)\cap\mathrm{Hom}\left(L_2\wedge L_2,L_1\right)=A\oplus B\oplus C\oplus D\oplus E\oplus F.\end{aligned}$$

In the present paper, a method is given that will allow to determine the dimension of the subspaces A, B, and C, giving explicitly the total dimension of all of them (Theorems 19, 23, and 24). This result, together with those obtained by Khakimdjanov and Navarro [11,12], leads to obtain the total dimension of the infinitesimal deformations of the model $\mathbb{Z}_3$-filiform Lie algebra $L^{n,m,p}$ (Main theorem).

We do assume that the reader is familiar with the standard theory of Lie algebras. All the vector spaces that appear in this paper (and thus, all the algebras) are assumed to be $\mathbb{F}$-vector spaces ($\mathbb{F}=\mathbb{C}$ or $\mathbb{R}$) with finite dimension.

2 Preliminaries

The vector space V is said to be $\mathbb{Z}_n$-graded if it admits a decomposition in direct sum, $V=V_0\oplus V_1\oplus\cdots V_{n-1}$. An element X of V is called homogeneous of degree γ ($\deg(X)=d(X)=\gamma$), $\gamma\in\mathbb{Z}_n$, if it is an element of V_γ.

Let $V = V_0 \oplus V_1 \oplus \cdots V_{n-1}$ and $W = W_0 \oplus W_1 \oplus \cdots W_{n-1}$ be two graded vector spaces. A linear mapping $f : V \to W$ is said to be homogeneous of degree γ ($\deg(f) = d(f) = \gamma$), $\gamma \in \mathbb{Z}_n$, if $f(V_\alpha) \subset W_{\alpha+\gamma \,(\mathrm{mod}\, n)}$ for all $\alpha \in \mathbb{Z}_n$. The mapping f is called a homomorphism of the $\mathbb{Z}_n$-graded vector space V into the $\mathbb{Z}_n$-graded vector space W, if f is homogeneous of degree 0. Now it is evident how we define an isomorphism or an automorphism of $\mathbb{Z}_n$-graded vector spaces.

A superalgebra $\mathfrak{g}$ is just a $\mathbb{Z}_2$-graded algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. That is, if we denote by $[\ ,\ ]$ the bracket product of $\mathfrak{g}$, we have $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta \,(\mathrm{mod}\, 2)}$ for all $\alpha, \beta \in \mathbb{Z}_2$.

Definition 1 (see [14])**.** Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a superalgebra whose multiplication is denoted by the bracket product $[\ ,\ ]$. We call $\mathfrak{g}$ a Lie superalgebra if the multiplication satisfies the following identities:

(1) $[X,Y] = -(-1)^{\alpha\cdot\beta}[Y,X], \forall X \in \mathfrak{g}_\alpha, \forall Y \in \mathfrak{g}_\beta$.
(2) $(-1)^{\gamma\cdot\alpha}[X,[Y,Z]] + (-1)^{\alpha\cdot\beta}[Y,[Z,X]] + (-1)^{\beta\cdot\gamma}[Z,[X,Y]] = 0\ \forall X \in \mathfrak{g}_\alpha, Y \in \mathfrak{g}_\beta, Z \in \mathfrak{g}_\gamma$ with $\alpha,\beta,\gamma \in \mathbb{Z}_2$.

Identity (2) is called the graded Jacobi identity, and it will be denoted by $J_g(X,Y,Z)$.

We observe that if $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ is a Lie superalgebra, we have that $\mathfrak{g}_0$ is a Lie algebra and $\mathfrak{g}_1$ has the structure of a $\mathfrak{g}_0$-module.

Color Lie (super)algebras can be seen as a direct generalization of Lie (super)algebras. Indeed, the latter are defined through antisymmetric (commutator) or symmetric (anticommutator) products, although for the former, the product is neither symmetric nor antisymmetric and is defined by means of a commutation factor. This commutation factor is equal to ± 1 for (super)Lie algebras and more general for arbitrary color Lie (super)algebras. As happened for Lie superalgebras, the basic tool to define color Lie (super)algebras is a grading determined by an abelian group.

Definition 2. Let G be an abelian group. A commutation factor β is a map $\beta : G \times G \to \mathbb{F} \setminus \{0\}$, ($\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$), satisfying the following constraints:

(1) $\beta(g,h)\beta(h,g) = 1\ \forall g,h \in G$
(2) $\beta(g,h+k) = \beta(g,h)\beta(g,k)\ \forall g,h,k \in G$
(3) $\beta(g+h,k) = \beta(g,k)\beta(h,k)\ \forall g,h,k \in G$.

The definition above implies, in particular, the following relations:

$$\beta(0,g) = \beta(g,0) = 1, \quad \beta(g,h) = \beta(-h,g), \quad \beta(g,g) = \pm 1 \quad \forall g,h \in G,$$

where 0 denotes the identity element of G. In particular, fixing g one element of G, the induced mapping $\beta_g : G \to \mathbb{F} \setminus \{0\}$ defines a homomorphism of groups.

Definition 3. Let G be an abelian group and β a commutation factor. The (complex or real) G-graded algebra

$$L = \bigoplus_{g \in G} L_g$$

with bracket product $[\ ,\ ]$, is called a (G,β)-color Lie superalgebra if for any $X \in L_g$, $Y \in L_h$, and $Z \in L$, we have:

(1) $[X,Y] = -\beta(g,h)[Y,X]$ (anticommutative identity)
(2) $[[X,Y],Z] = [X,[Y,Z]] - \beta(g,h)[Y,[X,Z]]$ (Jacobi identity).

Corollary 4. *Let $L = \bigoplus_{g \in G} L_g$ be a (G,β)-color Lie superalgebra. Then we have:*

(1) *L_0 is a (complex or real) Lie algebra where 0 denotes the identity element of G.*
(2) *For all $g \in G \setminus \{0\}$, L_g is a representation of L_0. If $X \in L_0$ and $Y \in L_g$, then $[X,Y]$ denotes the action of X on Y.*

Examples. For the particular case $G = \{0\}$, $L = L_0$ reduces to a Lie algebra. If $G = \mathbb{Z}_2 = \{0,1\}$ and $\beta(1,1) = -1$, we have *ordinary Lie superalgebras*; that is, a *Lie superalgebra* is a $(\mathbb{Z}_2,\beta)$-color Lie superalgebra where $\beta(i,j) = (-1)^{ij}$ for all $i,j \in \mathbb{Z}_2$.

Definition 5. A representation of a (G,β)-color Lie superalgebra is a mapping $\rho : L \to \mathrm{End}(V)$, where $V = \bigoplus_{g \in G} V_g$ is a graded vector space such that:

$$\big[\rho(X), \rho(Y)\big] = \rho(X)\rho(Y) - \beta(g,h)\rho(Y)\rho(X)$$

for all $X \in L_g$, $Y \in L_h$.

We observe that for all $g,h \in G$ we have $\rho(L_g)V_h \subseteq V_{g+h}$, which implies that any V_g has the structure of a L_0-module. In particular considering the adjoint representation ad_L we have that every L_g has the structure of a L_0-module.

Two (G,β)-color Lie superalgebras L and M are called *isomorphic* if there is a linear isomorphism $\varphi : L \to M$ such that $\varphi(L_g) = M_g$ for any $g \in G$ and also $\varphi([x,y]) = [\varphi(x),\varphi(y)]$ for any $x,y \in L$.

Let $L = \bigoplus_{g\in G} L_g$ be a (G,β)-color Lie superalgebra. The *descending central sequence* of L is defined by

$$\mathcal{C}^0(L) = L, \quad \mathcal{C}^{k+1}(L) = \left[\mathcal{C}^k(L), L\right] \quad \forall k \geq 0.$$

If $\mathcal{C}^k(L) = \{0\}$ for some k, the (G,β)-color Lie superalgebra is called *nilpotent*. The smallest integer k such as $\mathcal{C}^k(L) = \{0\}$ is called the *nilindex* of L.

Also, we are going to define some new *descending sequences of ideals*, see [10]. Let $L = \bigoplus_{g\in G} L_g$ be a (G,β)-color Lie superalgebra. Then, we define the new descending sequences of ideals $\mathcal{C}^k(L_0)$ (where 0 denotes the identity element of G) and $\mathcal{C}^k(L_g)$ with $g \in G \setminus \{0\}$, as follows:

$$\mathcal{C}^0(L_0) = L_0, \quad \mathcal{C}^{k+1}(L_0) = \left[L_0, \mathcal{C}^k(L_0)\right], \quad k \geq 0$$

and

$$\mathcal{C}^0(L_g) = L_g, \quad \mathcal{C}^{k+1}(L_g) = \left[L_0, \mathcal{C}^k(L_g)\right], \quad k \geq 0,\ g \in G \setminus \{0\}.$$

Using the descending sequences of ideals defined above, we give an invariant of color Lie superalgebras called *color-nilindex*. We are going to particularize this definition for $G = \mathbb{Z}_3$.

Definition 6 (see [11])**.** If $L = L_0 \oplus L_1 \oplus L_2$ is a nilpotent $(\mathbb{Z}_3,\beta)$-color Lie superalgebra, then L has *color-nilindex* (p_0, p_1, p_2), if the following conditions hold:

$$\left(\mathcal{C}^{p_0-1}(L_0)\right)\left(\mathcal{C}^{p_1-1}(L_1)\right)\left(\mathcal{C}^{p_2-1}(L_2)\right) \neq 0$$

and

$$\mathcal{C}^{p_0}(L_0) = \mathcal{C}^{p_1}(L_1) = \mathcal{C}^{p_2}(L_2) = 0.$$

Definition 7 (see [10])**.** Let $L = \bigoplus_{g\in G} L_g$ be a (G,β)-color Lie superalgebra. L_g is called a *L_0-filiform module* if there exists a decreasing subsequence of vectorial subspaces in its underlying vectorial space V, $V = V_m \supset \cdots \supset V_1 \supset V_0$, with dimensions $m, m-1, \ldots 0$, respectively, $m > 0$, and such that $[L_0, V_{i+1}] = V_i$.

Remark 8. The definition of filiform module is also valid for G-graded Lie algebras.

Definition 9 (see [10])**.** Let $L = \bigoplus_{g\in G} L_g$ be a (G,β)-color Lie superalgebra. Then L is a *filiform color Lie superalgebra* if the following conditions hold:

(1) L_0 is a filiform Lie algebra where 0 denotes the identity element of G.
(2) L_g has structure of L_0-filiform module, for all $g \in G \setminus \{0\}$

Definition 10. Let $L = \bigoplus_{g\in G} L_g$ be a G-graded Lie algebra. Then L is a *G-filiform Lie algebra* if the following conditions hold:

(1) L_0 is a filiform Lie algebra where 0 denotes the identity element of G.
(2) L_g has structure of L_0-filiform module, for all $g \in G \setminus \{0\}$

It is not difficult to see that for $G = \mathbb{Z}_3$, there is only one possibility for the commutation factor β, that is:

$$\beta(g,h) = 1 \quad \forall g,h \in \mathbb{Z}_3 = \{0,1,2\}.$$

From now on, we will consider this commutation factor, and we will write "$\mathbb{Z}_3$-color" instead of "$(\mathbb{Z}_3,\beta)$-color". We will note by $\mathcal{L}^{n,m,p}$, the variety of all $\mathbb{Z}_3$-color Lie superalgebras $L = L_0 \oplus L_1 \oplus L_2$ with $\dim(L_0) = n+1$, $\dim(L_1) = m$ and $\dim(L_2) = p$. $\mathcal{N}^{n,m,p}$ will be the variety of all nilpotent $\mathbb{Z}_3$-color Lie superalgebras, and $\mathcal{F}^{n,m,p}$ is the subset of $\mathcal{N}^{n,m,p}$ composed of all filiform color Lie superalgebras.

Remark 11. If $G = \mathbb{Z}_3$, then $\beta(g,h) = 1\ \forall g,h$. Thus, *$\mathbb{Z}_3$-color Lie superalgebras are effectively $\mathbb{Z}_3$-graded Lie algebras, and filiform $\mathbb{Z}_3$-color Lie superalgebras are $\mathbb{Z}_3$-filiform Lie algebras*.

In the particular case of $G=\mathbb{Z}_3$, the theorem of adapted basis rests as follows for $L=L_0\oplus L_1\oplus L_2\in\mathcal{F}^{n,m,p}$:

$$\begin{cases}[X_0,X_i]=X_{i+1}, & 1\le i\le n-1,\\ [X_0,X_n]=0, & \\ [X_0,Y_j]=Y_{j+1}, & 1\le j\le m-1,\\ [X_0,Y_m]=0 & \\ [X_0,Z_k]=Z_{k+1}, & 1\le k\le p-1,\\ [X_0,Z_p]=0. & \end{cases}$$

with $\{X_0,X_1,\dots,X_n\}$ a basis of L_0, $\{Y_1,\dots,Y_m\}$ a basis of L_1, and $\{Z_1,\dots,Z_p\}$ a basis of L_2. The model $\mathbb{Z}_3$-filiform Lie algebra, $L^{n,m,p}$, is the simplest $\mathbb{Z}_3$-filiform Lie algebra; and it is defined in an adapted basis $\{X_0,X_1,\dots,X_n,Y_1,\dots,Y_m,Z_1,\dots,Z_p\}$ by the following non-null bracket products:

$$L^{n,m,p}=\begin{cases}[X_0,X_i]=X_{i+1}, & 1\le i\le n-1\\ [X_0,Y_j]=Y_{j+1}, & 1\le j\le m-1\\ [X_0,Z_k]=Z_{k+1}, & 1\le k\le p-1.\end{cases}$$

3 Cocycles and infinitesimal deformations

Recall that a *module* $V=V_0\oplus V_1\oplus V_2$ of the $\mathbb{Z}_3$-color Lie superalgebra L is a bilinear map of degree 0, $L\times V\to V$ satisfying:

$$\forall X\in L_g,\ Y\in L_h,\ v\in V: X(Yv)-Y(Xv)=[X,Y]v$$

color Lie superalgebra cohomology is defined in the following well-known way (see, e.g., [15]): in particular, the superspace of *q-dimensional cocycles* of the $\mathbb{Z}_3$-color Lie superalgebra $L=L_0\oplus L_1\oplus L_2$ with coefficients in the L-module $V=V_0\oplus V_1\oplus V_2$ will be given by:

$$C^q(L;V)=\bigoplus_{q_0+q_1+q_2=q}\operatorname{Hom}\left(\wedge^{q_0}L_0\otimes\wedge^{q_1}L_1\otimes\wedge^{q_2}L_2,V\right).$$

This space is graded by $C^q(L;V)=C^q_0(L;V)\oplus C^q_1(L;V)\oplus C^q_2(L;V)$ with

$$C^q_p(L;V)=\bigoplus_{\substack{q_0+q_1+q_2=q\\ q_1+2q_2+p\equiv r\bmod 3}}\operatorname{Hom}\left(\wedge^{q_0}L_0\otimes\wedge^{q_1}L_1\otimes\wedge^{q_2}L_2,V_r\right)$$

The *coboundary operator* $\delta^q: C^q(L;V)\to C^{q+1}(L;V)$, with $\delta^{q+1}\circ\delta^q=0$ is defined in general, with L an arbitrary (G,β)-color Lie superalgebra and V an L-module, by the following formula for $q\ge 1$:

$$\begin{aligned}\left(\delta^q g\right)\left(A_0,A_1,\dots,A_q\right)=&\sum_{r=0}^{q}(-1)^r\beta\left(\gamma+\alpha_0+\cdots+\alpha_{r-1},\alpha_r\right)A_r\cdot g\left(A_0,\dots,\hat{A}_r,\dots,A_q\right)\\ &+\sum_{r<s}(-1)^s\beta\left(\alpha_{r+1}+\cdots+\alpha_{s-1},\alpha_s\right)g\left(A_0,\dots,A_{r-1},[A_r,A_s],A_{r+1},\dots,\hat{A}_s,\dots,A_q\right),\end{aligned}$$

where $g\in C^q(L;V)$ of degree γ, and $A_0,A_1,\dots,A_q\in L$ are homogeneous with degrees $\alpha_0,\alpha_1,\dots,\alpha_q$, respectively. The sign ˆ indicates that the element below must be omitted, and empty sums (like $\alpha_0+\cdots+\alpha_{r-1}$ for $r=0$ and $\alpha_{r+1}+\cdots+\alpha_{s-1}$ for $s=r+1$) are set equal to zero. In particular, for $q=2$, we obtain:

$$\begin{aligned}\left(\delta^2 g\right)\left(A_0,A_1,A_2\right)=&\,\beta\left(\gamma,\alpha_0\right)A_0\cdot g\left(A_1,A_2\right)-\beta\left(\gamma+\alpha_0,\alpha_1\right)A_1\cdot g\left(A_0,A_2\right)+\beta\left(\gamma+\alpha_0+\alpha_1,\alpha_2\right)A_2\cdot g\left(A_0,A_1\right)\\ &-g\left(\left[A_0,A_1\right],A_2\right)+\beta\left(\alpha_1,\alpha_2\right)g\left(\left[A_0,A_2\right],A_1\right)+g\left(A_0,\left[A_1,A_2\right]\right).\end{aligned}$$

Let $Z^q(L;V)$ denote the kernel of δ^q and let $B^q(L;V)$ denote the image of δ^{q-1}, then we have that $B^q(L;V)\subset Z^q(L;V)$. The elements of $Z^q(L;V)$ are called *q-cocycles*; the elements of $B^q(L;V)$ are the *q-coboundaries*. Thus, we can construct the so-called *cohomology groups*:

$$H^q(L;V)=Z^q(L;V)/B^q(L;V),\quad H^q_p(L;V)=Z^q_p(L;V)/B^q_p(L;V),\quad \text{if } G=\mathbb{Z}_3 \text{ then } p=0,1,2.$$

Two elements of $Z^q(L;V)$ are said to be *cohomologous* if their residue classes modulo $B^q(L;V)$ coincide, that is, if their difference lies in $B^q(L;V)$.

We will focus our study in the 2-cocycles $Z_0^2(L^{n,m,p};L^{n,m,p})$ with $L^{n,m,p}$, the model filiform $\mathbb{Z}_3$-color Lie superalgebra. Thus, $G=\mathbb{Z}_3$ and the only admissible commutation factor is exactly $\beta(g,h)=1$. Under all these restrictions, the condition that have to verify $\psi\in C_0^2(L^{n,m,p};L^{n,m,p})$ to be a 2-cocycle rests

$$\begin{aligned}(\delta^2\psi)(A_0,A_1,A_2) &= [A_0,\psi(A_1,A_2)]-[A_1,\psi(A_0,A_2)]+[A_2,\psi(A_0,A_1)]\\ &\quad -\psi([A_0,A_1],A_2)+\psi([A_0,A_2],A_1)+\psi(A_0,[A_1,A_2])=0\end{aligned}$$

for all $A_0,A_1,A_2\in L^{n,m,p}$. We observe that $L^{n,m,p}$ has the structure of a $L^{n,m,p}$-module via the adjoint representation.

We consider a homogeneous basis of $L^{n,m,p}=L_0\oplus L_1\oplus L_2$, in particular an adapted basis $\{X_0,X_1,\dots,X_n,Y_1,\dots,Y_m,Z_1,\dots,Z_p\}$ with $\{X_0,X_1,\dots,X_n\}$ a basis of L_0, $\{Y_1,\dots,Y_m\}$ a basis of L_1 and $\{Z_1,\dots,Z_p\}$ a basis of L_2.

Under these conditions, we have the following lemma.

Lemma 12 (see [11,12]). *Let ψ be such that $\psi\in C_0^2(L^{n,m,p};L^{n,m,p})$, then ψ is a 2-cocycle, $\psi\in Z_0^2(L^{n,m,p};L^{n,m,p})$, iff the 10 conditions below hold for all $X_i,X_j,X_k\in L_0$, $Y_i,Y_j,Y_k\in L_1$ and $Z_i,Z_j,Z_k\in L_2$*

(1) $[X_i,\psi(X_j,X_k)]-[X_j,\psi(X_i,X_k)]+[X_k,\psi(X_i,X_j)]-\psi([X_i,X_j],X_k)+\psi([X_i,X_k],X_j)+\psi(X_i,[X_j,X_k])=0$

(2) $[X_i,\psi(X_j,Y_k)]-[X_j,\psi(X_i,Y_k)]+[Y_k,\psi(X_i,X_j)]-\psi([X_i,X_j],Y_k)+\psi([X_i,Y_k],X_j)+\psi(X_i,[X_j,Y_k])=0$

(3) $[X_i,\psi(X_j,Z_k)]-[X_j,\psi(X_i,Z_k)]+[Z_k,\psi(X_i,X_j)]-\psi([X_i,X_j],Z_k)+\psi([X_i,Z_k],X_j)+\psi(X_i,[X_j,Z_k])=0$

(4) $[X_i,\psi(Y_j,Y_k)]-[Y_j,\psi(X_i,Y_k)]+[Y_k,\psi(X_i,Y_j)]-\psi([X_i,Y_j],Y_k)+\psi([X_i,Y_k],Y_j)+\psi(X_i,[Y_j,Y_k])=0$

(5) $[X_i,\psi(Y_j,Z_k)]-[Y_j,\psi(X_i,Z_k)]+[Z_k,\psi(X_i,Y_j)]-\psi([X_i,Y_j],Z_k)+\psi([X_i,Z_k],Y_j)+\psi(X_i,[Y_j,Z_k])=0$

(6) $[X_i,\psi(Z_j,Z_k)]-[Z_j,\psi(X_i,Z_k)]+[Z_k,\psi(X_i,Z_j)]-\psi([X_i,Z_j],Z_k)+\psi([X_i,Z_k],Z_j)+\psi(X_i,[Z_j,Z_k])=0$

(7) $[Y_i,\psi(Y_j,Y_k)]-[Y_j,\psi(Y_i,Y_k)]+[Y_k,\psi(Y_i,Y_j)]-\psi([Y_i,Y_j],Y_k)+\psi([Y_i,Y_k],Y_j)+\psi(Y_i,[Y_j,Y_k])=0$

(8) $[Y_i,\psi(Y_j,Z_k)]-[Y_j,\psi(Y_i,Z_k)]+[Z_k,\psi(Y_i,Y_j)]-\psi([Y_i,Y_j],Z_k)+\psi([Y_i,Z_k],Y_j)+\psi(Y_i,[Y_j,Z_k])=0$

(9) $[Y_i,\psi(Z_j,Z_k)]-[Z_j,\psi(Y_i,Z_k)]+[Z_k,\psi(Y_i,Z_j)]-\psi([Y_i,Z_j],Z_k)+\psi([Y_i,Z_k],Z_j)+\psi(Y_i,[Z_j,Z_k])=0$

(10) $[Z_i,\psi(Z_j,Z_k)]-[Z_j,\psi(Z_i,Z_k)]+[Z_k,\psi(Z_i,Z_j)]-\psi([Z_i,Z_j],Z_k)+\psi([Z_i,Z_k],Z_j)+\psi(Z_i,[Z_j,Z_k])=0.$

Proposition 13 (see [11]). *ψ is an infinitesimal deformation of $L^{n,m,p}$ iff ψ is a 2-cocycle of degree 0, $\psi\in Z_0^2(L^{n,m,p};L^{n,m,p})$.*

Theorem 14 (see [10]). (1) *Any filiform (G,β)-color Lie superalgebra law μ is isomorphic to $\mu_0+\varphi$, where μ_0 is the law of the model filiform (G,β)-color Lie superalgebra, and φ is an infinitesimal deformation of μ_0 verifying that $\varphi(X_0,X)=0$ for all $X\in L$, with X_0 the characteristic vector of model one.*

(2) *Conversely, if φ is an infinitesimal deformation of a model filiform (G,β)-color Lie superalgebra law μ_0 with $\varphi(X_0,X)=0$ for all $X\in L$, then the law $\mu_0+\varphi$ is a filiform (G,β)-color Lie superalgebra law iff $\varphi\circ\varphi=0$.*

Thus, any $\mathbb{Z}_3$-filiform Lie algebra (filiform $\mathbb{Z}_3$-color Lie superalgebra) will be a linear deformation of the model $\mathbb{Z}_3$-filiform Lie algebra (the model $\mathbb{Z}_3$-color Lie superalgebra); that is, $L^{n,m,p}$ is the model $\mathbb{Z}_3$-filiform Lie algebra, and another arbitrary $\mathbb{Z}_3$-filiform Lie algebra will be equal to $L^{n,m,p}+\varphi$, with φ an infinitesimal deformation of $L^{n,m,p}$, hence the importance of these deformations. So, in order to determine all the $\mathbb{Z}_3$-filiform Lie algebras, it is only necessary to compute the infinitesimal deformations or so-called 2-cocycles of degree 0, that vanish on the characteristic vector X_0. Thanks to the following lemma, these infinitesimal deformations can be decomposed into six subspaces.

Lemma 15 (see [11,12]). *Let $Z^2(L;L)$ be the 2-cocycles $Z_0^2(L^{n,m,p};L^{n,m,p})$ that vanish on the characteristic vector X_0. Then $Z^2(L;L)$ can be divided into six subspaces; that is, if $L^{n,m,p}=L=L_0\oplus L_1\oplus L_2$, we will have:*

$$\begin{aligned}Z^2(L;L) &= Z^2(L;L)\cap\operatorname{Hom}(L_0\wedge L_0,L_0)\oplus Z^2(L;L)\cap\operatorname{Hom}(L_0\wedge L_1,L_1)\oplus Z^2(L;L)\\ &\quad\cap\operatorname{Hom}(L_0\wedge L_2,L_2)\oplus Z^2(L;L)\cap\operatorname{Hom}(L_1\wedge L_1,L_2)\oplus Z^2(L;L)\\ &\quad\cap\operatorname{Hom}(L_1\wedge L_2,L_0)\oplus Z^2(L;L)\cap\operatorname{Hom}(L_2\wedge L_2,L_1)=A\oplus B\oplus C\oplus D\oplus E\oplus F.\end{aligned}$$

In order to obtain the dimension of A, B, and C, we are going to adapt *the $\mathfrak{sl}_2(\mathbb{C})$-module method* that we have already used for Lie superalgebras [1,4,8] and for color Lie superalgebras [11,12]. Next, we will do it explicitly for $A=Z^2(L;L)\cap\operatorname{Hom}(L_0\wedge L_0,L_0)$.

4 Dimension of $A = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_0, L_0)$

In general, any cocycle $a \in Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_0, L_0)$ will be any skew-symmetric bilinear map from $L_0 \wedge L_0$ to L_0 such that:

$$\begin{aligned}&[X_i, a(X_j,X_k)] - [X_j, a(X_i,X_k)] + [X_k, a(X_i,X_j)] - a([X_i,X_j],X_k)\\ &\quad + a([X_i,X_k],X_j) + a(X_i,[X_j,X_k]) = 0 \quad \forall X_i,X_j,X_k \in L_0\end{aligned} \tag{4.1}$$

with $a(X_0,X) = 0\ \forall X \in L$. As $X_0 \notin \mathrm{Im}\, a$ and taking into account the bracket products of L, then (4.1) can be rewritten as follows:

$$\big[X_0, a\big(X_j,X_k\big)\big] - a\big(\big[X_0,X_j\big],X_k\big) - a\big(X_j,\big[X_0,X_k\big]\big) = 0, \quad 1 \le j < k \le n. \tag{4.2}$$

In order to obtain the dimension of the space of cocycles for A, we apply an adaptation of the $\mathfrak{sl}(2,\mathbb{C})$-module method that we used in a previous study [11].

Recall the following well-known facts about the Lie algebra $\mathfrak{sl}(2,\mathbb{C})$ and its finite-dimensional modules, see, for example, [2,5]:

$\mathfrak{sl}(2,\mathbb{C}) = \langle X_-, H, X_+\rangle$ with the following commutation relations:

$$\begin{cases}\big[X_+,X_-\big] = H,\\ \big[H,X_+\big] = 2X_+,\\ \big[H,X_-\big] = -2X_-.\end{cases}$$

Let V be a n-dimensional $\mathfrak{sl}(2,\mathbb{C})$-module, $V = \langle e_1,\dots,e_n\rangle$. Then, up to isomorphism, there exists a unique structure of an irreducible $\mathfrak{sl}(2,\mathbb{C})$-module in V given in a basis $\{e_1,\dots,e_n\}$ as follows [2]:

$$\begin{cases}X_+\cdot e_i = e_{i+1}, & 1 \le i \le n-1,\\ X_+\cdot e_n = 0,\\ H\cdot e_i = (-n+2i-1)e_i, & 1 \le i \le n.\end{cases}$$

It is easy to see that e_n is the maximal vector of V; and its weight, called the highest weight of V, is equal to $n-1$.

Let $W_0, W_1, \dots, W_k$ be $\mathfrak{sl}(2,\mathbb{C})$-modules, then the space $\mathrm{Hom}(\otimes_{i=1}^k W_i, W_0)$ is a $\mathfrak{sl}(2,\mathbb{C})$-module in the following natural manner:

$$(\xi\cdot\varphi)(x_1,\dots,x_k) = \xi\cdot\varphi(x_1,\dots,x_k) - \sum_{i=1}^{k}\varphi(x_1,\dots,\xi\cdot x_i, x_{i+1},\dots,x_n)$$

with $\xi \in \mathfrak{sl}(2,\mathbb{C})$ and $\varphi \in \mathrm{Hom}(\otimes_{i=1}^k W_i, W_0)$. In particular, if $k=2$ and $W_0 = W_1 = W_2 = V_0$, then:

$$(\xi\cdot\varphi)\big(x_1,x_2\big) = \xi\cdot\varphi\big(x_1,x_2\big) - \varphi\big(\xi\cdot x_1,x_2\big) - \varphi\big(x_1,\xi\cdot x_2\big).$$

An element $\varphi \in \mathrm{Hom}(V_0\otimes V_0, V_0)$ is said to be invariant if $X_+\cdot\varphi = 0$, that is:

$$X_+\cdot\varphi\big(x_1,x_2\big) - \varphi\big(X_+\cdot x_1,x_2\big) - \varphi\big(x_1,X_+\cdot x_2\big) = 0, \quad \forall x_1,x_2 \in V. \tag{4.3}$$

Note that $\varphi \in \mathrm{Hom}(V_0\otimes V_0, V_0)$ is invariant if and only if φ is a maximal vector.

We are going to consider the structure of irreducible $\mathfrak{sl}(2,\mathbb{C})$-module in $V_0 = \langle X_1,\dots,X_n\rangle = L_0/\mathbb{C}X_0$, thus in particular:

$$\begin{cases}X_+\cdot X_i = X_{i+1}, & 1 \le i \le n-1,\\ X_+\cdot X_n = 0.\end{cases}$$

Next, we identify the multiplication of X_+ and X_i in the $\mathfrak{sl}(2,\mathbb{C})$-module $V_0 = \langle X_1,\dots,X_n\rangle$, with the bracket $[X_0,X_i]$ in L_0 and thanks to these identifications, the expressions (4.2) and (4.3) are equivalent. Thus, we have the following result:

Proposition 16. *Any skew-symmetric bilinear map φ, $\varphi : V_0\wedge V_0 \to V_0$ will be an element of the space of cocycles A if and only if φ is a maximal vector of the $\mathfrak{sl}(2,\mathbb{C})$-module* $\mathrm{Hom}(V_0\wedge V_0, V_0)$*, with $V_0 = \langle X_1,\dots,X_n\rangle$.*

Corollary 17. *As each irreducible $\mathfrak{sl}(2,\mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of the space of cocycles A is equal to the number of summands of any decomposition of* $\mathrm{Hom}(V_0 \wedge V_0, V_0)$ *into the direct sum of irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules.*

We use the fact that each irreducible module contains either a unique (up to scalar multiples) vector of weight 0 (in case the dimension of the irreducible module is odd) or a unique (up to scalar multiples) vector of weight 1 (in case the dimension of the irreducible module is even). We therefore have:

Corollary 18. *The dimension of the space of cocycles A is equal to the dimension of the subspace of* $\mathrm{Hom}(V_0 \wedge V_0, V_0)$ *spanned by the vectors of weight* 0 *or* 1.

At this point, we are going to apply the $\mathfrak{sl}(2,\mathbb{C})$-module method aforementioned in order to obtain the dimension of the space of cocycles A.

We consider a natural basis $\mathcal{B}$ of $\mathrm{Hom}(V_0 \wedge V_0, V_0)$ consisting of the following maps:

$$\varphi_{i,j}^s(X_k, X_l) = \begin{cases} X_s & \text{if } (i,j) = (k,l) \\ 0 & \text{in all other cases} \end{cases}$$

where $1 \le i,j,k,l,s \le n$, with $i \neq j$ and $\varphi_{i,j}^s = -\varphi_{j,i}^s$.

Thanks to Corollary 18, it will be enough to find the basis vectors $\varphi_{i,j}^s$ with weight 0 or 1. The weight of an element $\varphi_{i,j}^s$ (with respect to H) is:

$$\lambda\big(\varphi_{i,j}^s\big) = \lambda\big(X_s\big) - \lambda\big(X_i\big) - \lambda\big(X_j\big) = n + 2(s - i - j) + 1.$$

In fact,

$$\begin{aligned} \big(H \cdot \varphi_{i,j}^s\big)\big(X_i, X_j\big) &= H \cdot \varphi_{i,j}^s\big(X_i, X_j\big) - \varphi_{i,j}^s\big(H \cdot X_i, X_j\big) - \varphi_{i,j}^s\big(X_i, H \cdot X_j\big) \\ &= H \cdot X_s - \varphi_{i,j}^s\big((-n-1+2i)X_i, X_j\big) - \varphi_{i,j}^s\big(X_i, (-n-1+2j)X_j\big) \\ &= (-n-1+2s)X_s - (-n-1+2i)X_s - (-n-1+2j)X_s \\ &= \big[n + 2(s-i-j) + 1\big]X_s. \end{aligned}$$

We observe that if n is even, then $\lambda(\varphi)$ is odd; and if n is odd, then $\lambda(\varphi)$ is even. So, if n is even, it will be sufficient to find the elements $\varphi_{i,j}^s$ with weight 1 and if n is odd it will be sufficient to find those of them with weight 0.

We can consider the three sequences that correspond with the weights of $V = \langle X_1, X_2, \dots, X_{n-1}, X_n\rangle$ in order to find the elements with weight 0 or 1:

$$-n+1, -n+3, \dots, n-3, n-1; \quad -n+1, -n+3, \dots, n-3, n-1; \quad -n+1, -n+3, \dots, n-3, n-1.$$

and we have to count the number of all possibilities to obtain 1 (if n is even) or 0 (if n is odd). Remember that $\lambda(\varphi_{i,j}^s) = \lambda(X_s) - \lambda(X_i) - \lambda(X_j)$, where $\lambda(X_s)$ belongs to the last sequence, and $\lambda(X_i)$, $\lambda(X_j)$ belong to the first and second sequences respectively. For example, if n is odd, we have to obtain 0, so we can fix an element (a weight) of the last sequence and then count the possibilities to sum the same quantity between the two first sequences. Taking into account the skew-symmetry of $\varphi_{i,j}^s$, that is $\varphi_{i,j}^s = -\varphi_{j,i}^s$ and $i \neq j$, and repeating the above reasoning for all the elements of the last sequence, we obtain the following theorem:

Theorem 19. *Let $Z^2(L;L)$ be the 2-cocycles $Z_0^2(L^{n,m,p};L^{n,m,p})$ that vanish on the characteristic vector X_0. Then, if $A = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_0, L_0)$, we have that*

$$\dim A = \begin{cases} \dfrac{n(3n-2)}{8} & \textit{if } n \textit{ is even}, \\[2ex] \dfrac{3n^2-4n+1}{8} + \left\lfloor \dfrac{n+1}{4} \right\rfloor & \textit{if } n \textit{ is odd}. \end{cases}$$

Proof. It is convenient to distinguish the following four cases where the reasoning for each case is not hard:

(1) $n \equiv 0 \pmod 4$.

(2) $n \equiv 1 \pmod 4$.

(3) $n \equiv 2 \pmod 4$.

(4) $n \equiv 3 \pmod 4$. □

5 Dimension of $B = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_1, L_1)$

In general, any cocycle $b \in Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_1, L_1)$ will be any skew-symmetric bilinear map from $L_0 \wedge L_1$ to L_1 such that:

$$\begin{aligned} &[X_i, b(X_j, Y_k)] - [X_j, b(X_i, Y_k)] - b([X_i, X_j], Y_k) + b([X_i, Y_k], X_j) \\ &\quad + b(X_i, [X_j, Y_k]) = 0 \quad \forall X_i, X_j \in L_0,\ Y_k \in L_1 \end{aligned} \tag{5.1}$$

with $b(X_0, X) = 0\ \forall X \in L$. This condition reduces to

$$\big[X_0, b\big(X_j, Y_k\big)\big] - b\big(\big[X_0, X_j\big], Y_k\big) - b\big(X_j, \big[X_0, Y_k\big]\big) = 0, \quad 1 \le j \le n,\ 1 \le k \le m. \tag{5.2}$$

In order to obtain the dimension of the space of cocycles B, we apply an adaptation of the $\mathfrak{sl}(2,\mathbb{C})$-module method that we have already used in the precedent section.

Recall that if $W_0, W_1, \ldots, W_k$ are $\mathfrak{sl}(2,\mathbb{C})$-modules, then the space $\mathrm{Hom}(\otimes_{i=1}^k W_i, W_0)$ will be a $\mathfrak{sl}(2,\mathbb{C})$-module in the following natural manner:

$$(\xi \cdot \varphi)\big(x_1, \ldots, x_k\big) = \xi \cdot \varphi\big(x_1, \ldots, x_k\big) - \sum_{i=1}^{k} \varphi\big(x_1, \ldots, \xi \cdot x_i, x_{i+1}, \ldots, x_n\big)$$

with $\xi \in \mathfrak{sl}(2,\mathbb{C})$ and $\varphi \in \mathrm{Hom}(\otimes_{i=1}^k W_i, W_0)$. In particular, if $k = 2$ and $V_0 = W_1$, $V_1 = W_2 = W_0$, then:

$$(\xi \cdot \varphi)\big(x_1, x_2\big) = \xi \cdot \varphi\big(x_1, x_2\big) - \varphi\big(\xi \cdot x_1, x_2\big) - \varphi\big(x_1, \xi \cdot x_2\big).$$

An element $\varphi \in \mathrm{Hom}(V_0 \otimes V_1, V_1)$ is said to be invariant if that is:

$$X_+ \cdot \varphi\big(x_1, x_2\big) - \varphi\big(X_+ \cdot x_1, x_2\big) - \varphi\big(x_1, X_+ \cdot x_2\big) = 0, \quad \forall x_1 \in V_0,\ \forall x_2 \in V_1. \tag{5.3}$$

Note that $\varphi \in \mathrm{Hom}(V_0 \otimes V_1, V_1)$ is invariant if and only if φ is a maximal vector.

In this case, we are going to consider the structure of irreducible $\mathfrak{sl}(2,\mathbb{C})$-module in $V_0 = \langle X_1, \ldots, X_n \rangle = L_0/\mathbb{C}X_0$ and in $V_1 = \langle Y_1, \ldots, Y_n \rangle = L_1$. Thus, in particular:

$$\begin{cases} X_+ \cdot X_i = X_{i+1}, & 1 \le i \le n-1 \\ X_+ \cdot X_n = 0 \\ X_+ \cdot Y_j = Y_{j+1}, & 1 \le j \le m-1 \\ X_+ \cdot Y_m = 0. \end{cases}$$

We identify the multiplication of X_+ and X_i in the $\mathfrak{sl}(2,\mathbb{C})$-module $V_0 = \langle X_1, \ldots, X_n \rangle$, with the bracket product $[X_0, X_i]$ in L_0. Analogously with $X_+ \cdot Y_j$ and $[X_0, Y_j]$. Thanks to these identifications, the expressions (5.2) and (5.3) are equivalent, so we have the following result:

Proposition 20. *Any skew-symmetric bilinear map φ, $\varphi : V_0 \wedge V_1 \to V_1$ will be an element of B if and only if φ is a maximal vector of the $\mathfrak{sl}(2,\mathbb{C})$-module* $\mathrm{Hom}(V_0 \wedge V_1, V_1)$*, with $V_0 = \langle X_1, \ldots, X_n \rangle$ and $V_1 = L_1$.*

Corollary 21. *As each $\mathfrak{sl}(2,\mathbb{C})$-module has (up to nonzero scalar multiples) a unique maximal vector, then the dimension of B is equal to the number of summands of any decomposition of* $\mathrm{Hom}(V_0 \wedge V_1, V_1)$ *into direct sum of irreducible $\mathfrak{sl}(2,\mathbb{C})$-modules.*

As each irreducible module contains either a unique (up to scalar multiples) vector of weight 0 or a unique vector of weight 1, then we have the following corollary.

Corollary 22. *The dimension of B is equal to the dimension of the subspace of* $\mathrm{Hom}(V_0 \wedge V_1, V_1)$ *spanned by the vectors of weight* 0 *or* 1*.*

Next, we consider a natural basis of $\mathrm{Hom}(V_0 \wedge V_1, V_1)$ consisting of the following maps where $1 \le s, j, l \le m$ and $1 \le i, k \le n$:

$$\varphi_{i,j}^{s}(X_k, Y_l) = \begin{cases} Y_s & \text{if } (i,j) = (k,l), \\ 0 & \text{in all other cases.} \end{cases}$$

Thanks to Corollary 22, it will be enough to find the basis vectors $\varphi^s_{i,j}$ with weight 0 or 1. It is not difficult to see that the weight of an element $\varphi^s_{i,j}$ (with respect to H) is:

$$\lambda(\varphi^s_{i,j}) = \lambda(Y_s) - \lambda(X_i) - \lambda(Y_j) = n + 2(s-i-j) + 1.$$

Thus, if n is even, then $\lambda(\varphi)$ is odd; and if n is odd, then $\lambda(\varphi)$ is even. So, if n is even, it will be sufficient to find the elements $\varphi^s_{i,j}$ with weight 1; and if n is odd, it will be sufficient to find those with weight 0. To do that we consider the three sequences that correspond with the weights of $V_0 = \langle X_1,\ldots,X_n\rangle$, $V_1 = \langle Y_1, Y_2,\ldots,Y_m\rangle$ and $V_1 = \langle Y_1, Y_2,\ldots,Y_m\rangle$:

$$-n+1, -n+3, \ldots, n-3, n-1; \quad -m+1, -m+3, \ldots, m-3, m-1; \quad -m+1, -m+3, \ldots, m-3, m-1.$$

We shall have to count the number of all possibilities to obtain 1 (if n is even) or 0 (if n is odd). Remember that $\lambda(\varphi^s_{i,j}) = \lambda(Y_s) - \lambda(X_i) - \lambda(Y_j)$, where $\lambda(Y_s)$ belongs to the last sequence, and $\lambda(X_i)$, $\lambda(Y_j)$ belong to the first and second sequences respectively. Thus, we obtain the following theorem.

Theorem 23. *Let $Z^2(L;L)$ be the 2-cocycles $Z^2_0(L^{n,m,p};L^{n,m,p})$ that vanish on the characteristic vector X_0. Then, if $B = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_1, L_1)$, we have that:*

$$\dim B = \begin{cases} \dfrac{4nm - n^2 + 1}{4} & \text{if } n \text{ is odd}, \ n < 2m+1, \\ \dfrac{4nm - n^2}{4} & \text{if } n \text{ is even}, \ n < 2m+1, \\ m^2 & \text{if } n \geq 2m+1. \end{cases}$$

Proof. It is convenient to distinguish the following four cases where the reasoning for each case is not hard:

(1) $n \equiv 0 \pmod 4$.
(2) $n \equiv 1 \pmod 4$.
(3) $n \equiv 2 \pmod 4$.
(4) $n \equiv 3 \pmod 4$. □

6 Dimension of $C = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_2, L_2)$

Similarly to the previous section, we can obtain the equivalent result for C.

Theorem 24. *Let $Z^2(L;L)$ be the 2-cocycles $Z^2_0(L^{n,m,p};L^{n,m,p})$ that vanish on the characteristic vector X_0. Then, if $C = Z^2(L;L) \cap \mathrm{Hom}(L_0 \wedge L_2, L_2)$, we have that:*

$$\dim C = \begin{cases} \dfrac{4np - n^2 + 1}{4} & \text{if } n \text{ is odd}, \ n < 2p+1, \\ \dfrac{4np - n^2}{4} & \text{if } n \text{ is even}, \ n < 2p+1, \\ p^2 & \text{if } n \geq 2p+1. \end{cases}$$

7 Conclusions

Theorems 1, 2, and 3, together with those obtained by Khakimdjanov and Navarro [11,12], lead to obtain the total dimension of the infinitesimal deformations of the model $\mathbb{Z}_3$-filiform Lie algebra $L^{n,m,p}$. Thus, we have the following theorem.

Main theorem. *The dimension of the space of infinitesimal deformations of the model $\mathbb{Z}_3$-filiform Lie algebra $L^{n,m,p}$ that vanish on the characteristic vector X_0, is exactly $A+B+C+D+E+F$ where*

$$A = \begin{cases} \dfrac{n(3n-2)}{8} & \text{if } n \text{ is even} \\ \dfrac{3n^2 - 4n + 1}{8} + \left\lfloor \dfrac{n+1}{4} \right\rfloor & \text{if } n \text{ is odd} \end{cases}$$

$$B = \begin{cases} \dfrac{4nm - n^2 + 1}{4} & \text{if } n \text{ is odd}, \ n < 2m+1 \\ \dfrac{4nm - n^2}{4} & \text{if } n \text{ is even}, \ n < 2m+1 \\ m^2 & \text{if } n \geq 2m+1 \end{cases}$$

$$C=\begin{cases}\dfrac{4np-n^2+1}{4} & \textit{if } n \textit{ is odd}, \ n<2p+1\\ \dfrac{4np-n^2}{4} & \textit{if } n \textit{ is even}, \ n<2p+1\\ p^2 & \textit{if } n\geq 2p+1\end{cases}$$

$$D=\begin{cases}\dfrac{m(m-1)}{2} & \textit{if } p\geq 2m-1\\ \dfrac{1}{8}\big(4mp-p^2-2p-1\big) & \textit{if } p<2m-1, \ p\equiv 1 \ (\mathrm{mod}\,4) \textit{ and } m \textit{ odd, or } p\equiv 3 \ (\mathrm{mod}\,4) \textit{ and } m \textit{ even}\\ \dfrac{1}{8}\big(4mp-p^2-2p+3\big) & \textit{if } p<2m-1, \ p\equiv 3 \ (\mathrm{mod}\,4) \textit{ and } m \textit{ odd, or } p\equiv 1 \ (\mathrm{mod}\,4) \textit{ and } m \textit{ even}\\ \dfrac{1}{8}\big(4mp-p^2-2p\big) & \textit{if } p<2m-1 \textit{ and } p \textit{ even}\end{cases}$$

$$F=\begin{cases}\dfrac{p(p-1)}{2} & \textit{if } m\geq 2p-1\\ \dfrac{1}{8}\big(4pm-m^2-2m-1\big) & \textit{if } m<2p-1, \ m\equiv 1 \ (\mathrm{mod}\,4) \textit{ and } p \textit{ odd, or } m\equiv 3 \ (\mathrm{mod}\,4) \textit{ and } p \textit{ even}\\ \dfrac{1}{8}\big(4pm-m^2-2m+3\big) & \textit{if } m<2p-1, \ m\equiv 3 \ (\mathrm{mod}\,4) \textit{ and } p \textit{ odd, or } m\equiv 1 \ (\mathrm{mod}\,4) \textit{ and } p \textit{ even}\\ \dfrac{1}{8}\big(4pm-m^2-2m\big) & \textit{if } m<2p-1 \textit{ and } m \textit{ even.}\end{cases}$$

(1) *If* $m+p-n$ *is even, then*

$$E=\begin{cases}mn & \textit{if } p\geq m+n\\ np-1 & \textit{if } p<m+n, \ p=m-n+2\\ np & \textit{if } p<m+n, \ p<m-n+2\\ \dfrac{1}{4}\big(-m^2-n^2-p^2+2np+2mn+2mp\big) & \textit{if } p<m+n, p>m-n+2, \ p\geq n-m+2\\ mp & \textit{if } p<m+n, \ p>m-n+2, \ p<n-m+2\end{cases}$$

(2) *If* $m+p-n$ *is odd, then*

$$E=\begin{cases}mn & \textit{if } p\geq m+n-1\\ np & \textit{if } p<m+n-1, \ p\leq m-n+1\\ \dfrac{1}{4}\big(-m^2-n^2-p^2+2np+2mn+2mp+1\big) & \textit{if } p<m+n-1, \ p>m-n+1, \ p\geq n-m+1\\ mp & \textit{if } p<m+n-1, \ p>m-n+1, \ p<n-m+1.\end{cases}$$

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