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Initial and Final Characterized Fuzzy $T_{3\frac{1}{2}}$ and Finer Characterized Fuzzy $R_{2\frac{1}{2}}$-Spaces

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Abstract

Basic notions related to the characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces are introduced and studied. The metrizable characterized fuzzy spaces are classified by the characterized fuzzy $R_{2\frac{1}{2}}$ and the characterized fuzzy T_4-spaces in our sense. The induced characterized fuzzy space is characterized by the characterized fuzzy $T_{3\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space if and only if the related ordinary topological space is $\varphi_{1,2}R_{2\frac{1}{2}}$-space and $\varphi_{1,2}T_{3\frac{1}{2}}$-space, respectively. Moreover, the α-level and the initial characterized spaces are characterized $R_{2\frac{1}{2}}$ and characterized $T_{3\frac{1}{2}}$-spaces if the related characterized fuzzy space is characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$, respectively. The categories of all characterized fuzzy $R_{2\frac{1}{2}}$ and of all characterized fuzzy $T_{3\frac{1}{2}}$-spaces will be denoted by CFR-Space and CRF-Tych and they are concrete categories. These categories are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych. That is, all the initial and final characterized fuzzy $R_{2\frac{1}{2}}$ spaces and all the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces exist in CFR-Space and in CRF-Tych. The initial and final characterized fuzzy spaces of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space are characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. As special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space are also characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. Finally, three finer characterized fuzzy $R_{2\frac{1}{2}}$-spaces and three finer characterized fuzzy $T_{3\frac{1}{2}}$-spaces are introduced and studied.

Keywords: Fuzzy filter; Fuzzy topological space; Operation; Characterized fuzzy space; Metriz-able characterized fuzzy space; Induced characterized fuzzy space; α-Level characterized space; $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous; Initial and final characterized fuzzy spaces; Characterized fuzzy $T_{3\frac{1}{2}}$-space; Characterized fuzzy $T_{3\frac{1}{2}}$-space; AMS classification; Primary 54E35, 54E52; Secondary 54A4003E72

Introduction

Eklund and Gahler [1] introduced the notion of fuzzy filter and by means of this notion the point-based approach to the fuzzy topology related to usual points has been developed. The more general concept for the fuzzy filter introduced by Gahler [2] and fuzzy filters are classified by types. Because of the specific type of the L-filter however the approach of Eklund and Gahler [1] is related only to the L-topologies which are stratified, that is, all constant L-sets are open. The more specific fuzzy filters considered in the former papers are now called homogeneous. The notion of fuzzy real numbers is introduced by Gahler and Gahler [3], as a convex, normal, compactly supported and upper semi-continuous fuzzy subsets of the set of all real numbers R. The set of all fuzzy real numbers is called the fuzzy real line and will be denoted by R_L, where L is complete chain.

The operation on the ordinary topological space *(X,T)* has been defined by Kasahara [4] as a mapping φ from T into 2^X such that $A \subseteq A^\varphi$, for all $A \in T$. Abd El-Monsef et al. [5], extend Kasahara [4] operation to the power set P (X) of the set X Kandil et al. [6] extended Kasahars's and Abd El-Monsef's operations by introducing operation on the class of all fuzzy sets endowed with an fuzzy topology τ as a mapping φ: $L^X \to L^X$ such that int $\mu \le \mu^\varphi$ for all $\mu \in L^X$, where μ^φ denotes the value of φ at μ. The notions of fuzzy filters and the operations on the class of all fuzzy sets on X endowed with an fuzzy topology τ are applied in ref. [7] to introduce a more general theory including all the weaker and stronger forms of the fuzzy topology. By means of these notions the notion of $\varphi_{1,2}$-interior of the fuzzy set, $\varphi_{1,2}$-fuzzy convergence and $\varphi_{1,2}$-fuzzy

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neighborhood filters are defined. The notion of $\varphi_{1,2}$-interior operator for the fuzzy sets is also defined as a mapping $\varphi_{1,2}$.int: $L^X \to L^X$ which fulfill (I1) to (I5). Since there is a one-to-one correspondence between the class of all $\varphi_{1,2}$-open fuzzy subsets of X and these operators, then the class $\varphi_{1,2}$OF (X) of all $\varphi_{1,2}$-open fuzzy subsets of X is characterized by these operators. Hence, the triple (X, $\varphi_{1,2}$.int) as will as the triple (X, $\varphi_{1,2}$ OF (X)) will be called the characterized fuzzy space of $\varphi_{1,2}$-open fuzzy subsets. For each characterized fuzzy space (X, $\varphi_{1,2}$.int) the mapping which assigns to each point x of X the $\varphi_{1,2}$-fuzzy neighborhood filter at x is said to be $\varphi_{1,2}$-fuzzy filter pre topology [7]. It can be identified itself with the characterized fuzzy space (X, $\varphi_{1,2}$.int). The characterized fuzzy spaces are characterized by many of characterizing notions, for example by: $\varphi_{1,2}$-fuzzy neighborhood filters, $\varphi_{1,2}$-fuzzy interior of the fuzzy filters and by the set of all $\varphi_{1,2}$-inner points of the fuzzy filters. Moreover, the notions of closeness and compactness in characterized fuzzy spaces are introduced and studied in ref. [8]. For an fuzzy topological space (X, τ), the operations on (X, τ) and on the fuzzy topological space (I_L, I), where I=[0, 1] is the closed unit interval and I is the fuzzy topology defined on the left unit interval I_L are applied to introduced and studied the notions of characterized fuzzy $R_{2\frac{1}{2}}$-spaces and characterized fuzzy $T_{3\frac{1}{2}}$-spaces or (characterized Tychonoff spaces) [9]. In this paper, Basic notions related to the characterized fuzzy $R_{2\frac{1}{2}}$ and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces are introduced and studied. Some of this the metrizable characterized fuzzy spaces, initial and final characterized fuzzy spaces and three finer characterized fuzzy $R_{2\frac{1}{2}}$-spaces are introduced and classified by the characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces. The metrizable characterized fuzzy space is introduce as a generalization of the weaker and stronger forms of the fuzzy metric space introduced by Gahler and Gahler [3]. For every stratified fuzzy topological space (X, τ_d) generated canonically by an fuzzy metric d on X, the metrizable characterized fuzzy space (X, $\varphi_{1,2}$.int$_{\tau d}$) is characterized fuzzy T_4-space in sense of Abd-Allah [10] and therefore it is characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$ L-space. The induced characterized fuzzy space (X, $\varphi_{1,2}$.int$_\omega$) is characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space if and only if the related ordinary topological space (X, T) is $\varphi_{1,2}$ $T_{3\frac{1}{2}}$-space and $\varphi_{1,}$ $T_{3\frac{1}{2}}$-space, respectively, that is, the notions of characterized fuzzy $R_{2\frac{1}{2}}$-spaces and characterized fuzzy $T_{3\frac{1}{2}}$-spaces are good extension as in sense of Lowen [11]. Moreover, the α-level characterized space (X, $\varphi_{1,2}$.int$_\alpha$) and the initial characterized space (X, $\varphi_{1,2}$.int$_i$) are characterized $R_{2\frac{1}{2}}$-space and characterized $T_{3\frac{1}{2}}$-space if the related characterized fuzzy space (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$-space and characterized fuzzy $T_{3\frac{1}{2}}$-space, respectively. We show that the finer characterized fuzzy space of the characterized fuzzy $R_{2\frac{1}{2}}$-space and of the characterized fuzzy $T_{3\frac{1}{2}}$-space is also characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space, respectively. The categories of all characterized fuzzy $R_{2\frac{1}{2}}$ and of all characterized fuzzy $T_{3\frac{1}{2}}$-spaces will be denoted by CFR-Space and CRF-Tych, respectively. We show that these categories are concrete categories and they are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych, respectively. That is, all the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces and all the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces are exist in the categories CFR-Space and CRF-Tych. Moreover, we show that the initial and final characterized fuzzy spaces of the characterized fuzzy $R_{2\frac{1}{2}}$-space and of the characterized fuzzy $T_{3\frac{1}{2}}$-space are characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. As an special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of the characterized fuzzy $R_{2\frac{1}{2}}$-space and of the characterized fuzzy $T_{3\frac{1}{2}}$-space are also characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. Finally, in section 5, we introduce and study three finer characterized fuzzy $R_{2\frac{1}{2}}$ and three finer characterized fuzzy $T_{3\frac{1}{2}}$-spaces as a generalization of the weaker and stronger forms of the completely regular and fuzzy $T_{3\frac{1}{2}}$-spaces introduced [1,12,13]. The relations between such new characterized fuzzy $R_{2\frac{1}{2}}$-spaces and our characterized fuzzy $T_{3\frac{1}{2}}$-spaces are introduced. More general the relations between such new characterized fuzzy $T_{3\frac{1}{2}}$-spaces and our characterized fuzzy $T_{3\frac{1}{2}}$-spaces are also introduced. Meany special cases from these finer characterized fuzzy $R_{2\frac{1}{2}}$-spaces and from finer characterized fuzzy $T_{3\frac{1}{2}}$-spaces are listed in Table 1.

Preliminaries

We begin by recalling some facts on fuzzy sets and fuzzy filters. Let L be a completely distributive complete lattice with different least and last elements 0 and 1, respectively. Consider $L_0=L\backslash\{0\}$ and $L_1=L\backslash\{1\}$. Recall that the complete distributivity of L means that the distributive law $\vee_{i\in I}(\alpha_i \wedge \alpha)=\left(\vee_{i\in I}\alpha_i\right)\wedge\alpha$. Sometimes we will assume more specially that L is a complete chain, that is, L is a complete lattice whose partial ordering is a linear one. The standard example of L is the real closed unit interval I=[0, 1]. For a set X, let L^X be the set of all fuzzy subsets of X, that is, of all mappings μ: $X \to L$. Assume that an order-reversing involution α 7→α′ is fixed. For each fuzzy set μ, let co μ denote the complement of μ defined by: (co μ) (x)=co $\mu(x)$ for all $x \in$ X. For all $x \in$ X and $\alpha \in L_0$. Supμ means the supremum of the set of values of μ. The fuzzy sets on X will be denoted by Greek letters as μ, η, ρ,. . . etc. Denote by $\overline{\alpha}$ the constant fuzzy subset of X with value α ∈ L. The fuzzy singleton x_α is an fuzzy set in X defined by $x_\alpha(x)$=α and $x_\alpha(y)$=0 for all $y \neq x$, $\alpha \in L_0$. The class of all fuzzy singletons in X will be denoted by $S(X)$. For every $x_\alpha \in$ S(X) and $\mu \in L^X$, we write $x_\alpha \leq \mu$ if and only if $\alpha \leq \mu(x)$. The fuzzy set μ is said to be quasi-coincident with the fuzzy set ρ and written μ q ρ if and only if there exists $x \in X$ such that $\mu(x)+\rho(x)>1$. If μ not quasi-coincident with the fuzzy set ρ, then we write $\mu\overline{q}\rho$. The fuzzy filter on X [14] is the mapping M: $L^X \to L$ such that the following conditions are fulfilled:

(F1) $M\left(\overline{\alpha}\right) \leq \alpha$ for all α ∈ L and $\mathcal{M}(1)$=1.

(F2) $\mathcal{M}(\mu \wedge \eta)=\mathcal{M}(\mu) \wedge \mathcal{M}(\eta)$ for all $\mu, \eta \in L^X$.

The fuzzy filter $\mathcal{M}$ is said to be homogeneous [14] if $M(\overline{\alpha})=\alpha$ for all $\alpha \in L$. For each $x \in X$, the mapping $\dot{x}: L^X \to L$ defined by $\dot{x}(\mu)=\mu(x)$ for all $\mu \in L^X$ is a homogeneous fuzzy filter on X. The homogenous fuzzy filter at the fuzzy set is defined by the same way as follows, for each $\mu \in L^X$, the mapping $\dot{\mu}$: $L^X \to L$ defined by $\dot{\mu}(\sigma)=\underset{0<\sigma(x)}{\wedge}\sigma(x)$ for all $\sigma \in L^X$ is also homogenous fuzzy filter on X, called homogenous fuzzy filter at $\mu \in L^X$. Obviously, the relation between homogenous fuzzy filter $\dot{\mu}$ at $\mu \in L^X$ and the homogenous fuzzy filter $\dot{x}$ at $x \in X$ is given by:

$$\dot{\mu}(\eta)=\underset{\mu(x)\geq 0}{\wedge}\sigma(x) \tag{2.1}$$

for all $\eta \in L^X$. As shown in ref. [15], $\mu \leq \eta$ if and only if $\dot{\mu} \leq \dot{\eta}$ holds for all $\mu, \eta \in L^X$. Let $\mathcal{F}_L X$ and $\mathcal{F}_L X$ denote to the sets of all fuzzy filters and of all homogeneous fuzzy filters on X, respectively. If $\mathcal{M}$ and $\mathcal{N}$ are fuzzy filters on the set X, then $\mathcal{M}$ is said to be finer than $\mathcal{N}$, denoted by $\mathcal{M} \leq \mathcal{N}$, provided $\mathcal{M}(\mu) \geq \mathcal{N}(\mu)$ holds for all $\mu \in L^X$. Noting that if L is a complete chain then M is not finer than N, denoted by $\mathcal{M}/\leq \mathcal{N}$, provided there exists $\mu \in L^X$ such that $\mathcal{M}(\mu) < \mathcal{N}(\mu)$ holds. As shown in ref. [4], if $\mathcal{M}, \mathcal{N}$ and L are three fuzzy filters on a set X, then we have:

$M \neq L \geq N$ implies $M \neq N$ and $M \geq L \neq N$ implies $M \neq N$.

The coarsest fuzzy filter $\mathcal{M}$ on X is the fuzzy filter has the value 1 at 1 and 0 otherwise. Suprema and infimum of sets of fuzzy filters are meant with respect to the finer relation. An fuzzy filter $\mathcal{M}$ on X is said to be ultra [2] fuzzy filter if it does not have a properly finer fuzzy filter. For each fuzzy filter $\mathcal{M} \in \mathcal{F}_L X$ there exists a finer ultra fuzzy filter $U \in \mathcal{F}_L X$ such that $U/\leq \mathcal{M}$. Consider $\mathcal{A}$ is a non-empty set of fuzzy filters on X, then the supremum $\underset{M\in A}{\vee}M$ exists [2] and given by $(\underset{M\in A}{\vee}M)(\mu)=\underset{M\in A}{\wedge}M(\mu)$ for all $\mu = L^X$ but the infimum $\underset{M\in A}{\wedge}M$ does not exists, in general. As shown in ref. [16], the infimum $\underset{M\in A}{\wedge}M$ of $\mathcal{A}$ with respect to the finer relation for fuzzy filters exists if and only if $M_1(\mu_1)\wedge\ldots\wedge M_n(\mu_n)\leq \sup(\mu_1\wedge\ldots\wedge\mu_n)$ holds for all finite subset $\{M_1,\ldots M_n\}$ of $\mathcal{A}$ and $\mu_1,\ldots,\mu_n \in L^X$. In this case the infimum is given by:

$$(\underset{M\in A}{\wedge}M)(\mu)=\underset{\substack{\mu_1\wedge\ldots\wedge\mu_n\leq\mu\\ M_1,\ldots M_n\in A}}{\vee}(M_1(\mu_1)\wedge\ldots\wedge M_n(\mu_n)),$$

for all $\mu \in L^X$.

Fuzzy filter bases. A family $(B_\alpha)_{\alpha\in L_0}$ of non-empty subsets of L^X is called a valued fuzzy filter base [2] if the following conditions are fulfilled:

(V1) $\mu \in \mathcal{B}_\alpha$ implies $\alpha \leq \sup \mu$.

(V2) For all $\alpha, \beta \in L_0$ with $\alpha\wedge\beta \in L_0$ and all $\mu \in \mathcal{B}_\alpha$ and $\eta \in \mathcal{B}_\beta$ there are $\gamma \geq \alpha\wedge\beta$ and $\sigma \leq \mu \wedge \eta$ such that $\sigma \in \mathcal{B}_\gamma$.

As shown in ref. [2], each valued fuzzy filter base $(\mathcal{B}^\alpha)^{\alpha\in L}$ defines an fuzzy filter $\mathcal{M}$ on X by $M(\mu)=\underset{\eta\in B\alpha,\eta\leq\mu}{\vee}\alpha$ for all $\mu \in L^X$. Conversely, each fuzzy filter $\mathcal{M}$ can be generated by a valued fuzzy filter base, e.g., by (α-pr $\mathcal{M})_{\alpha\in L0}$ with α-pr $M=\{\mu\in L^X \mid \alpha\leq\mathcal{M}(\mu)\}$. ($\alpha$-pr $\mathcal{M})_{\alpha\in L0}$ is a family of pre filters on X and it is called the large valued filter base of $\mathcal{M}$. Recall that a pre filter on X [17] is a non-empty proper subset of $\mathcal{F}$ of L^X such that (1) $\mu, \eta \in \mathcal{F}_X$ implies $\mu \wedge \eta \in \mathcal{F}$ and (2) from $\mu \in \mathcal{F}$ and $\mu \leq \eta$ it follows $\eta \in \mathcal{F}$. $\mathcal{A}$ subset $\mathcal{B}$ of L^X is said to be superior fuzzy filter base [2] if the following conditions are fulfilled:

(S1) $\overline{\alpha} \in B$ for every $\alpha \in L$.

(S2) For all $\mu, \eta \in \mathcal{B}$ there is a fuzzy set $\sigma \in \mathcal{B}$ such that $\sigma \leq \mu$, $\sigma \leq \eta$ and sup σ=sup $\mu \wedge$ sup η.

Each superior fuzzy filter base $\mathcal{B}$ generated a homogeneous fuzzy filter $\mathcal{M}$ on X by $M(\mu)=\underset{\eta\in B,\eta\leq\mu}{\vee}$ sup η for all $\mu \in L^X$ and each fuzzy filter $\mathcal{M}$ can be generated by a superior fuzzy filter base, e.g., by base $M=\{\mu\in L^X \mid M(\mu)=\sup\mu\}=\mu\wedge\overline{M\mu} \mid \mu\in L^X\}$, where base M will be called the large superior fuzzy filter base of $\mathcal{M}$. If X is a non-empty set and μ is an fuzzy subset of X, then $B=\{\mu\wedge\overline{\alpha} \mid \alpha\in L\}\cup\{\overline{\alpha} \mid \alpha\in L\}$ is a superior fuzzy filter base of a homogeneous fuzzy filter on X, called superior principal fuzzy filter generated by μ and will be denoted by $[\mu]$. In case L is a complete chain and μ is not constant we have $[\mu](\eta)$=sup μ, when $\mu \leq \eta$ and $[\mu](\eta)=\underset{\eta(x)<\mu(x)}{\wedge}\eta(x)$ otherwise for all $\eta \in L^X$. For each ordinary subset M of X we have that $[\chi_M]=\underset{x\in M}{\vee}\dot{x}$, where χ_M is the characteristic function of M.

Fuzzy topology

By the fuzzy topology on a set X, we mean a subset of L^X which is closed with respect to all supreme and all finite infimum and contains the constant fuzzy sets $\overline{0}$ and $\overline{1}$ [16,18]. A set X equipped with an fuzzy topology τ on X is called an fuzzy topological space. For each fuzzy topological space (X, τ), the elements of τ are called the open fuzzy subsets of this space. If τ_1 and τ_2 are fuzzy topologies on a set X, then τ_1 is said to be finer than τ_2 and τ_2 is said to be coarser than τ_1, provided $\tau_2 \subseteq \tau_1$ holds. For each fuzzy set $\mu \in L^X$, the strong α-cut and the weak α-cut of μ are the ordinary subsets $S_\alpha(\mu)=\{x\in X \mid \mu(x)>\alpha\}$ and $W_\alpha(\mu)=\{x\in X \mid \mu(x)\geq\alpha\}$ of X respectively. For each complete chain L, the α-level topology and the initial topology [19] of an fuzzy topology τ on the set X are defined as follows:

$$\tau_\alpha=\{S_\alpha(\mu)\in P(X) : \mu\in\tau\} \text{ and } i(\tau)=\inf\{\tau_\alpha : \alpha\in L_1\},$$

respectively, where *inf* is the infimum with respect to the finer relation for topologies. On other hand if (X, T) is an ordinary topological space, then the induced fuzzy topology on X is given by Lowen [17] as the following:

$$\omega(T)=\{\mu\in L^X : S_\alpha(\mu)\in T \text{ for all } \alpha\in L_1\}.$$

The fuzzy topological space(X, τ) and also τ are said to be stratified provided $\alpha \in \tau$ holds for all $\alpha \in L$, that is, all constant fuzzy sets are open [19].

The fuzzy unit interval

The fuzzy unit interval will be denoted by I_L an it is defined in [3] as the fuzzy subset:

$$I_L=\{x\in R^*_L \mid x \leq 1^\sim\},$$

where I=[0, 1] is the real unit interval and $\mathbf{R}^*_L=\{x\in R_L \mid x(0)=1 \text{ and } 0^\sim\leq x\}$ is the set of all positive fuzzy real numbers. Note that, the binary relation $\leq$ is defined on R_L as follows:

$$x \leq y \Leftrightarrow x_{\alpha 1}\leq y_{\alpha 1} \text{ and } x_{\alpha 2}\leq y_{\alpha 2},$$

for all $x, y \in R_L$, where $x_{\alpha 1}=\inf\{z\in R \mid x(z)\geq\alpha\}$ and $x_{\alpha 2}=\sup\{z\in R \mid x(z)\geq\alpha\}$ for all $\alpha \in L_0$. Note that the family Ω which is defined by:

$$\Im=\{R_\delta|I_L \mid \delta\in I\}\cup\{R^\delta|I_L \mid \delta\in I\}\cup\{0^\sim \mid I_L\}$$

is a base for an fuzzy topology I on I^L, where R_δ and R^δ are the fuzzy subsets of R^L defined by $R_\delta(x)=\underset{\alpha>\delta}{\vee}x(\alpha)$ and $R^\delta=(\underset{\alpha\geq\delta}{\vee}x(\alpha))'$ for all x

$\in R_L$ and $\delta \in R$. The restrictions of R_δ and R^δ on I_L are the fuzzy subsets $R_\delta I_L$ and $R^\delta I_L$, respectively. Recall that:

$$R^{\delta}(x) \wedge R^{\gamma}(y) \le R^{\delta+\gamma}(x+y), \tag{2.2}$$

where, $x+y$ is the fuzzy real number defined by $(x+y)(\xi) = \bigvee_{\gamma,\zeta\in R,\gamma+\zeta=\xi}(x(\gamma)\wedge y(\zeta))$ for all $\xi \in R$.

Operation on fuzzy sets

In the sequel, let a fuzzy topological space (X, τ) be fixed. By the operation [6] on the set X we mean the mapping $\varphi: L^X \to L^X$ such that int $(\mu) \le \mu^\varphi$ holds for all $\mu \in L^X$, where, μ^φ denotes the value of φ at μ. The class of all operations on X will be denoted by $O_{(L^X,\tau)}$. By the identity operation on $O_{(L^X,\tau)}$, we mean the operation $1_L^X: L^X \to L^X$ such that $1_L^X(\mu)=\mu$ for all $\mu \in L^X$. The constant operation on $O_{(L^X,\tau)}$ is the operation $c_L^x: L^X \to L^X$ defined by $c_L^x(\mu)=1$ for all $\mu \in L^X$. If $\le$ is a partially order relation on $O_{(L^X,\tau)}$ defined as follows: $\varphi_1 \le \varphi_2 \Leftrightarrow \varphi_1(\mu) \le \varphi_2(\mu)$ for all $\mu \in L^X$, then $(O_{(L^X,\tau)}, \le)$ is a completely distributive lattice. The operation $\varphi: L^X \to L^X$ is called:

(i) Isotone if $\mu \le \eta$ implies $\varphi\mu \le \varphi\eta$, for all $\mu, \eta \in L^X$.

(ii) Weakly finite intersection preerving (wfip, for short) with respect to $A \subseteq L^X$ if $\eta \wedge \varphi(\mu) \le \varphi(\eta \wedge \mu)$ holds, for all $\eta \in \mathcal{A}$ and $\mu \in L^X$.

(iii) Idempotent if $\varphi(\mu) = \varphi(\varphi(\mu))$, for all $\mu \in L^X$.

The operations $\varphi, \psi \in O_{(L^X,\tau)}$ are said to be dual if $\psi(\mu)=co(\varphi(co\mu))$ or equivalently $\varphi(\mu)=co(\psi(co\mu))$ for all $\mu \in L^X$, where coμ denotes the complement of μ. The dual operation of φ is denoted by $\varphi^{\sim}$. In the classical case of L={0, 1}, by the operation on a set X we mean the mapping φ: P (X) → P (X) such that int $A \subseteq A^\varphi$ for all $A \in P(X)$ and the identity operation on the class of all ordinary operations $O_{(P(X),T)}$ on X will be denoted by $i_{P(X)}$ and it defined by: $i_{P(X)}(A)=A$ for all $A \in P(X)$.

The φ-open fuzzy sets

Let a fuzzy topological space (X, τ) be fixed and $\varphi \in O_{(L^X,\tau)}$. The fuzzy set μ: X → L is said to be φ-open fuzzy set if $\mu \le \mu^\varphi$ holds. We will denote the class of all φ-open fuzzy sets on X by φ of (X). The fuzzy set μ is called φ-closed if its complement coμ is φ-open. The operations φ, $\psi \in O_{(L^X,\tau)}$ are equivalent and written $\varphi \sim \psi$ if φ of $(X)=\psi$ of (X).

The $\varphi_{1,2}$-interior fuzzy sets

Let a fuzzy topological space (X, τ) be fixed and

$\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the $\varphi_{1,2}$-interior of the fuzzy set μ: X → L is a mapping φ_{12}.intμ: X → L defined by:

$$\varphi_{1,2}.int\ \mu = \bigvee_{\eta\in\varphi_1 OF(X)\varphi_2\eta\le\mu} \eta. \tag{2.3}$$

That is, the $\varphi_{1,2}$.intμ is the greatest φ_1-open fuzzy set η such that η^{φ_2} less than or equal to μ [19]. The fuzzy set μ is said to be $\varphi_{1,2}$-open if and only if $\mu \le \varphi_{1,2}$.int μ. The class of all $\varphi_{1,2}$-open fuzzy sets on X will be denoted by $\varphi_{1,2}$OF (X). The complement co μ of the $\varphi_{1,2}$-open fuzzy subset μ will be called $\varphi_{1,2}$-closed, the class of all $\varphi_{1,2}$-closed fuzzy subsets of X will be denoted by $\varphi_{1,2}CF(X)$. In the classical case of L={0, 1}, the fuzzy topological space (X, τ) is up to an identification by the ordinary topological space (X, T) and $\varphi_{1,2}$.int μ is the classical one. Hence in this case the ordinary subset A of X is $\varphi_{1,2}$-open if $A \subseteq \varphi_{1,2}$.int A. The complement of a $\varphi_{1,2}$-open subset A of X will be called $\varphi_{1,2}$-closed. The class of all $\varphi_{1,2}$-open and the class of all $\varphi_{1,2}$-closed subsets of X will be denoted by $\varphi_{1,2}$O(X) and $\varphi_{1,2}$C(X), respectively. Clearly, F is $\varphi_{1,2}$-closed if and only if $\varphi_{1,2}.cl_T$ F=F.

Proposition

For each two operations $\varphi_1, \varphi_2 \in O(L^X,\tau)$ and for each μ, η∈φ1, φ2 ∈ L^X, the mapping $\varphi_{1,2}$.int: X → L fulfills the following axioms [7]:

(i) If $\varphi_2 \ge 1_L^X$, then $\varphi_{1,2}$.intμ ≤ μ.

(ii) $\varphi_{1,2}$.int is isotone, i.e if μ ≤ η, then $\varphi_{1,2}$.intμ ≤ $\varphi_{1,2}$.intη.

$\varphi_{1,2}.int\bar{1} = \bar{1}$.

If $\varphi_2 \ge 1_L X$ is isotone and φ_1 is with respect to $\varphi_1 O\mathcal{F}$ (X), then $\varphi_{1,2}.int(\mu\wedge\eta) = \varphi_{1,2}.int\mu \wedge \varphi_{1,2}.int\eta$.

If φ_2 is isotone and idempotent operation, then $\varphi_{1,2}.int\mu \le \varphi_{1,2}.int(\varphi_{1,2}.int\mu)$.

$\varphi_{1,2}.int(\bigvee_{i\in I}\mu_i) = \bigvee_{i\in I}\varphi_{1,2}.int\mu_i$ for all $\mu_i \in \varphi_{1,2}OF\ (X)$.

Proposition

Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the following are fulfilled:

(i) If $\varphi_2 \ge 1_L X$, then the class $\varphi_{1,2}$OF (X) of all $\varphi_{1,2}$-open fuzzy sets on X forms an extended fuzzy topology on X [7,21].

If $\varphi_2 \ge 1_{L^X}$ and $\varphi_{1,2}.int\bar{1} = \bar{1}$, then the class $\varphi_{1,2}$OF (X) of all $\varphi_{1,2}$-open fuzzy sets on X forms a supra fuzzy topology on X [21].

If $\varphi_2 \ge 1_L X$ is isotone and φ_1 is with respect to φ_1OF (X), then $\varphi_{1,2}$OF (X) is an fuzzy pre topology on X [21].

If $\varphi_2 \ge 1_L X$ is isotone and idempotent operation and φ_1 is with respect to φ_1OF (X), then $\varphi_{1,2}$OF (X) is fuzzy topology on X [16,18].

Because of Propositions 2.1 and 2.2, if the fuzzy topological space(X, τ) be fixed and

$\varphi_1, \varphi_2\ O_{(L^X,\tau)}$. Then the relation between the class $\varphi_{1,2}$OF (X) of all $\varphi_{1,2}$-open fuzzy sets on X and the mapping $\varphi_{1,2}$.int is given by:

$$\varphi_{1,2}OF\ (X) = \{\mu \in L^X \mid \mu \le \varphi_{1,2}.int\mu\} \tag{2.4}$$

and the following axioms are fulfilled:

(I1) If $\varphi_2 \ge 1_L^X$, then $\varphi_{1,2}.int\mu \le \mu$ holds, for all $\mu \in L^X$.

(I2) If $\mu \le \eta$, then $\varphi_{1,2}$.intμ ≤ $\varphi_{1,2}$.intη for all $\mu, \eta \in L^X$.

(I3) $\varphi_{1,2}.int\ \bar{1} = \bar{1}$

(I4) If $\varphi_2 \ge 1_{L^X}$ is isotone and φ_1 is with respect to φ_1OF (X), then $\varphi_{1,2}$.intμ ∧∧ $\varphi_{1,2}$.intη=$\varphi_{1,2}$.int (μ ∧ η) for all μ, η ∈ L^X. s

(I5) If φ_2 is isotone and idempotent, then $\varphi_{1,2}$.int ($\varphi_{1,2}$.intμ)=$\varphi_{1,2}$.intμ for all μ ∈ L^X.

Characterized Fuzzy Spaces

Independently on the fuzzy topologies, the notion of $\varphi_{1,2}$-interior operator for the fuzzy sets can be defined as a mapping $\varphi_{1,2}$.int: $L^X \to L^X$ which fulfill (I1) to (I5). It is well-known that (2.3) and (2.4) give a one-to-one correspondence between the class of all $\varphi_{1,2}$-open fuzzy sets and these operators, that is, $\varphi_{1,2}$OF (X) can be characterized by the $\varphi_{1,2}$-interior operators. In this case the triple (X, $\varphi_{1,2}$.int) as well as the triple (X, $\varphi_{1,2}$OF (X)) will be called characterized fuzzy space [7] of the $\varphi_{1,2}$-open fuzzy subsets of X. The characterized fuzzy space (X, $\varphi_{1,2}$.int) is said to be stratified if and only if $\varphi_{1,2}.int\bar{\alpha} = \bar{\alpha}$ for all α ∈ L. As shown in ref. [7], the characterized fuzzy space $(X, \varphi_{1,2}.int)$ is stratified if the related fuzzy topology is stratified. Moreover, the characterized fuzzy space $(X, \varphi_{1,2}.int)$ is said to have the weak infimum property [21], provided $\varphi_{1,2}.int(\mu \wedge \bar{\alpha}) = \varphi_{1,2}.int\mu \wedge \varphi_{1,2}.int\bar{\alpha}$ for all $\mu \in L^X$ and $\alpha \in L$. The characterized fuzzy space (X, $\varphi_{1,2}$.int) is said to be strongly stratified [21], provided $\varphi_{1,2}$.int is stratified and have the weak infimum property. If (X, $\varphi_{1,2}$.int) and (X, $\psi_{1,2}$.int) are two characterized fuzzy spaces, then (X, $\varphi_{1,2}$.int) is said to be finer than (X, $\psi_{1,2}$.int) and denoted by $\varphi_{1,2}$.int ≤ $\psi_{1,2}$.int, provided $\varphi_{1,2}$.intμ ≥ $\psi_{1,2}$.intμ holds for all μ ∈ L^X. If τ is a fuzzy topology on the set X and $\varphi_1, \varphi_2 \in O_{(L}X,_{\tau)}$, then by the initial characterized space of (X, τ) we mean the characterized spaces (X, $(\varphi_{1,2}O(X))_\alpha$) and (X, i($\varphi_{1,2}$O(X))), respectively where $(\varphi_{1,2}O(X))_\alpha$ and i($\varphi_{1,2}$O(X)) are defined as follows:

$$(\varphi_{1,2}O(X))_\alpha = \{ S_\alpha\ \mu \in P\ (X) \mid \mu \in \varphi_{1,2}OF\ (X)\ \} \text{ and } i(\varphi_{1,2}O(X)) = \cap\{(\varphi_{1,2}OF\ (X))_\alpha \mid \alpha \in L_1\}.$$

Sometimes we denoted to the α-level characterized space and the initial characterized space of (X, τ) by $(X,\ \varphi_{1,2}.int_\alpha)$ and $(X,\ \varphi_{1,2}.int_i)$, respectively. If T is an ordinary topology on a set X and $\varphi_1,\ \varphi_2 \in O_{(P(X),T)}$, then by the induced characterized fuzzy space on X we mean the characterized fuzzy space $(X,\ \omega(\varphi_{1,2}OF\ (X)))$ which is defined by:

$$\omega(\varphi_{1,2}OF(X)) = \{\mu \in L^X \mid S_\alpha\mu \in \varphi_{1,2}O(X) \text{ for all } \alpha \in L_1\}.$$

Sometimes we denoted to the induced characterized fuzzy space for the ordinary topological space (X, *T*) by $(X,\ \varphi_{1,2}.int_\omega)$.

If $\varphi_1 = int_\tau$ and $\varphi_2 = 1_L X$, then the class $(\varphi_{1,2}OF\ (X))$ of all $\varphi_{1,2}$-open fuzzy of X coincide with τ which is defined in [22,23] and hence the characterized fuzzy space $(X,\ \varphi_{1,2}.int)$ coincide with the fuzzy topological space (X, τ).

$\varphi_{1,2}$-fuzzy neighborhood filters

An important notion in the characterized fuzzy space (X, $\varphi_{1,2}$.int) is that of the $\varphi_{1,2}$-fuzzy neighborhood filter at the points and at the ordinary subsets of this space. Let (X, τ) be a fuzzy topological space and $\varphi_1,\ \varphi_2 \in O_{(L^X,\tau)}$. As follows by (I1) to (I5) for each x ∈ X, the mapping $N_{\varphi 1,2}(x) : L^X \to L$ which is defined by:

$$N_{\varphi 1,2}(x)(\mu) = (\varphi_{1,2}.int\ \mu)(x) \tag{2.5}$$

for all $\mu \in L^X$, is a fuzzy filter on X, called $\varphi_{1,2}$-fuzzy neighborhood filter at x [7]. If the related $\varphi_{1,2}$-interior operator fulfill the axioms (I1) and (I2) only, then the mapping $N_{\varphi 1,2}(x) : L^X \to L$, defined by (2.5) is fuzzy stack [21], called $\varphi_{1,2}$-fuzzy neighborhood stack at x. Moreover, if the $\varphi_{1,2}$-interior operator fulfill the axioms (I1), (I2) and (I4) such that in (I4) instead of η ∈ L^X we take α^-, then the mapping $N_{\varphi 1,2}(x) : L^X \to L$, defined by (2.5) is a fuzzy stack with the cutting property, called $\varphi_{1,2}$-fuzzy neighborhood stack with the cutting property at *x*. The $\varphi_{1,2}$-fuzzy neighborhood filters fulfill the following conditions:

$$(N1)\ \dot{x} \le N_{\varphi 1,2}(x) \text{ holds for all } x \in X$$

$$(N2)\ N_{\varphi 1,2}(x)(\mu) \le N_{\varphi 1,2}(x)(\eta) \text{ holds for all } \mu,\ \eta \in L^X \text{ and } \mu \le \eta.$$

$$N_{\varphi 1,2}(x)(y \mapsto N_{\varphi 1,2}(y)(\mu)) = N_{\varphi 1,2}(y)(\mu), \text{ for all } x \in X \text{ and } \mu \in L^X.$$

Clearly $y \mapsto N_{\varphi 1,2}(y)(\mu)$ is the fuzzy set $\varphi_{1,2}.\text{int}\,\mu$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is characterized as the fuzzy filter pre topology [7], that is, as a mapping $N_{\varphi 1,2} : X \to F_L X$ such that (N1) to (N3) are fulfilled.

$\varphi_{1,2}\psi_{1,2}$-Fuzzy continuity

Let now the fuzzy topological spaces (X, τ_1) and (Y, τ_2) are fixed, φ_1, $\varphi_2 \in O_{(L}X,_{\tau 1)}$ and ψ_1, $\psi_2 \in O_{(L}Y,_{\tau 2)}$. The mapping f: (X, $\varphi_{1,2}$.int) →(Y, $\psi_{1,2}$.int) is said to be $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if

$$(\psi_{1,2}.int\ \eta) \circ \mu \le \varphi_{1,2}.int\ (\eta \circ \mu) \tag{2.6}$$

holds for all η ∈ L^Y [7]. If an order reversing involution ′ of L is given, we have that f is a $\varphi_{1,2}\ \psi_{1,2}$-fuzzy continuous if and only if $\varphi_{1,2}.cl\ (\eta \circ \mu) \le (\psi_{1,2}.cl\ \eta) \circ \mu$ holds for all $\eta \in L^Y$. Here $\varphi_{1,2}$.cl and $\psi_{1,2}$.cl, mean the closure operators related to $\varphi_{1,2}$.int and $\psi_{1,2}$.int, respectively which are defined by $\varphi_{1,2}$.cl μ=co ($\varphi_{1,2}$.int coμ) for all μ ∈ L^X. Obviously if f is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous, then the inverse f^{-1}: (Y, $\psi_{1,2}$.int) → (X, $\varphi_{1,2}$.int) is $\psi_{1,2}\varphi_{1,2}$-fuzzy continuous, that is $(\varphi_{1,2}.int\ h) \circ \mu^{-1} \le \psi_{1,2}.int\ (h \circ f^{-1})$ holds for all h ∈L^X

By means of characterizing $\varphi_{1,2}$-fuzzy neighborhoods $N_{\varphi 1,2}(x)$ of $\varphi_{1,2}.int$ and $N_{\psi 1,2}(x)$ of $\psi_{1,2}.int$, the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of f can also be characterized. The mapping f: (X, $\varphi_{1,2}$.int) → (Y, $\psi_{1,2}$.int) is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous if $N_{\psi 1,2}$ (f(x)) ≥ F_Lf($N_{\varphi 1,2}$ (x)) holds for all x ∈ X. Obviously, in case of L={ 0, 1 }, $\varphi_1=\psi_1$=int, $\varphi_2=1_L$X and $\psi_2=1_L$Y the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity coincides with the usual fuzzy continuity.

Initial characterized fuzzy spaces

In the following let X be a set, let *I* be a class and for each i ∈ I, let (X_i, $\delta_{1,2}$.int_i) be a characterized fuzzy space of $\delta_{1,2}$-open fuzzy subsets of X_i and f_i: X → X_i is the mapping from X into X_i. By the initial $\varphi_{1,2}$-fuzzy interior operator of $(\delta_{1,2}.int_i)_i \in_I$ with respect to $(f_i)_i \in_I$, we mean the coarsest $\varphi_{1,2}$-fuzzy interior operator $\varphi_{1,2}$.int on X for which all mappings f_i: (X, $\varphi_{1,2}$.int) → (X_i, $\delta_{1,2}$.int_i) are $\varphi_{1,2}\delta_{1,2}$-fuzzy continuous. The triple (X, $\varphi_{1,2}$.int) is said to be initial characterized fuzzy space [7] of $((X_i, \delta_{1,2}.int_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$. The initial $\varphi_{1,2}$-fuzzy interior operator $\varphi_{1,2}$.int: L^X → L^X of $(\delta_{1,2}.int_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ always exists and is given by:

$$\varphi_{1,2}.int\ \mu = \bigvee_{\mu_i f_i \le \mu,\ i \in I} (\delta_{1,2}.int_i\mu_i) \circ f_i \tag{2.7}$$

for all μ ∈ L^X. For each i ∈ I, let $N^i_{\delta 1,2} : X_i \to F_L X_i$ is the representation of $\delta_{1,2}$.int_i as an fuzzy filter pre topology. Then because of (2.5) and (2.7), the mapping $N_{\varphi 1,2}$: X → F_LX which is defined by:

$$N_{\varphi 1,2}(x)(\mu) = \bigvee_{\mu_i \circ f_i \le \mu,\ i \in I} N_{\delta}{}^{i}{}_{1,2}(f_i(x))(\mu_i)$$

for all x ∈ X and μ ∈ L^X, is the representation of the initial $\varphi_{1,2}$-fuzzy interior operator of $(\psi_{1,2}.int_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ as the fuzzy filter pre topology.

Characterized Fuzzy Subspaces

Let A be a subset of a characterized fuzzy space (X, $\varphi_{1,2}$.int) and i: A,→ X is the inclusion mapping of A into X. Then the mapping $\varphi_{1,2}$.int_A: L^A → L^A defined by:

$$\varphi_{1,2}.int_A\eta = \bigvee_{\mu \circ i \le \eta} (\varphi_{1,2}.int\ \mu) \circ i$$

for all η ∈ L^A is initial $\varphi_{1,2}$-fuzzy interior operator for $\varphi_{1,2}$.int with respect to the inclusion mapping i: A,→ X. $\varphi_{1,2}$.int_A will be called induced $\varphi_{1,2}$-interior operator of $\varphi_{1,2}$.int on the subset A of X. The triple (A, $\varphi_{1,2}$.int_A) is said to be characterized fuzzy subspace of (X, $\varphi_{1,2}$.int) [7].

Characterized Fuzzy Product Spaces

Assume that (X_i, $\delta_{1,2}$.int_i) is a characterized fuzzy space for each i I, where I is any class. Let X be the cartesian product πX_i of the family $(X_i)_{i\in I}$ and π_i: X → X_i the related projections. The i∈I, mapping $\varphi_{1,2}$.int: L^X → L^X, defined by:

$$\varphi_{1,2}.int\ \mu = \bigvee_{\mu_i \circ \pi_i \le \mu,\ i \in I} (\delta_{1,2}.int_i\mu_i) \circ \pi_i$$

for all μ ∈ L^X, will be called $\varphi_{1,2}$-fuzzy product of the $\delta_{1,2}$L-interior operators $\delta_{1,2}$.int_i. The triple (X, $\varphi_{1,2}$.int) is said to be characterized fuzzy product space [7] of the characterized fuzzy spaces (X_i, $\delta_{1,2}$.int$_i$). The $\varphi_{1,2}$.int will be denoted by $\underset{i\in I}{\pi}\,\delta_{1,2}.int_i$ and it is initial $\varphi_{1,2}$-fuzzy interior operator of $(\delta_{1,2}.\text{int}_i)_{i\in I}$ with respect to the family $(\pi_i)_{i\in I}$ of projections. The characterized fuzzy product space (X, $\varphi_{1,2}$.int) also will be denoted by $\underset{i\in I}{\pi}(X_i, \delta_{1,2}.int_i)$

Final characterized fuzzy spaces

It is well-known (cf. e.g., [11,24]) that in the topological category all final lifts uniquely exist and hence also all final structures exist. They are dually defined. In case of the category CF-Space of all characterized fuzzy spaces the final structures can easily be given, as is shown in the following:

Let I be a class and for each $i \in I$, let (X_i, $\delta_{1,2}$.int$_i$) be an characterized fuzzy space and f_i: $X_i \to X$ is the mapping of X_i into a set X. The final $\varphi_{1,2}$-fuzzy interior operator of $(\delta_{1,2}.\text{int}_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ is the finest $\varphi_{1,2}$.int on X for which all mappings f_i: (X_i, $\delta_{1,2}$.int$_i$) → (X, $\varphi_{1,2}$.int) are $\delta_{1,2}\varphi_{1,2}$-fuzzy continuous [7]. Hence, the triple (X, $\varphi_{1,2}$.int) is the final characterized fuzzy space of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$. The final $\varphi_{1,2}L$-interior operator $\varphi_{1,2}$.int: $L^X \to L^X$ of $(\delta_{1,2}.\text{int}_i)_{i\in I}$ with respect to $(f_i)_{i\in I}$ exists and is given by

$$\left(\varphi_{1,2}.int\ \mu\right)(x) \ = \bigwedge_{x_i \in f_i^{-1}\{x\}\ ,\ i\in I} \delta_{1,2}.int_i(\mu \circ f_i)(x_i) \wedge \mu(x)$$

for all x ∈ X and μ ∈ L^X.

Characterized Fuzzy Quotient Spaces

Let (X, $\varphi_{1,2}$.int) be a characterized fuzzy space and f: $X \to A$ is an surjective mapping. Then the mapping $\varphi_{1,2}$.int$_f$: $L^A \to L^A$, defined by:

$$\left(\varphi_{1,2}.int\ \mu\right)(a) \ = \bigwedge_{x_i \in f_i^{-1}\{a\}} \varphi_{1,2}.int(\mu \circ f_i)(x)$$

for all $a \in A$ and $\mu \in L^A$, is final $\varphi_{1,2}$-fuzzy interior operator of $\varphi_{1,2}$.int with respect to f which is not idempotent. Then the $\varphi_{1,2}$.int$_f$ will be called quotient $\varphi_{1,2}$-fuzzy interior operator and the triple (A, $\varphi_{1,2}$.int$_f$) is said to be characterized fuzzy quotient space [7].

Note that in this case $\varphi_{1,2}$.int is idempotent, $\varphi_{1,2}$.int$_f$ need not be. Even in the classical case of L={0, 1}, φ_1=int and φ_2=$1_L X$ we have the following: If $\varphi_{1,2}$.int is up to an identification the usual topology, then $\varphi_{1,2}$.int$_f$ is a pre topology which need not be idempotent. An example is given [25] (p. 234).

Characterized Fuzzy Sum Spaces

Assume that (X_i, $\delta_{1,2}$.int$_i$) is a characterized fuzzy space for each i ∈, where I is any class. Let X be the disjoint union $\bigcup_{i\in I}(X_i \times \{i\})$ of the family $(X_i)_{i\in I}$ and for each i ∈ I, let $\varphi_{1,2}$.int: $L^X \to L^X$, defined by:e_i: $X_i \to X$ be the canonical injection from X_i into X given by $e_i(x_i)=(x_i, i)$. Then the mapping $\varphi_{1,2}$.int: $L^X \to L^X$, defined by:

$$\left(\varphi_{1,2}.int\ \mu\right)(a,\ i) \ = \ \delta_{1,2}.int_i\left(\mu \circ e_i\right)(a)$$

for all i ∈ I, of a ∈ X_i and μ ∈ L^X, is said to be final $\varphi_{1,2}$-fuzzy interior operator with respect to $(e_i)_{i\in I}$.

$(\delta_{1,2}.\text{int}_i)_{i\in I}$ $\varphi_{1,2}$.int will be called sum $\varphi_{1,2}$-fuzzy interior operator will be denoted by $^{\Sigma}$ $\delta_{1,2}$.int$_i$. The pair (X, $\varphi_{1,2}$.int) is said to be characterized fuzzy sum space [7] and it will be denoted also by $\sum_{i\in I}(X_i, \delta_{1,2}.\text{int}_i)$.

Characterized Fuzzy T_1 And Fuzzy $\Phi_{1,2}T_1$-Spaces

The notions of characterized fuzzy T_s and of characterized fuzzy R_k-spaces are investigated and studied [9,10,26,27] for all $s \in \{0,\ 1,\ 2,\ 2\frac{1}{2},\ 3,\ 3\frac{1}{2},\ 4\}$ and $k \in \{0,\ 1,\ 2,\ 2\frac{1}{2}\}$. These characterized spaces depend only on the usual points and the operation defined on the class of all fuzzy subsets of X endowed with an fuzzy topology τ. Let the fuzzy topological space(X, τ) be fixed and φ_1, $\varphi_2 \in O_{(L^X, \tau)}$ then the characterized fuzzy space all fuzzy subsets of X endowed with an fuzzy topology τ. Let the fuzzy topological space (X, τ) be fixed and φ_1, $\varphi_2 \in O_{(L^X, \tau)}$ then the characterized fuzzy space all fuzzy subsets of X endowed with an fuzzy topology τ. Let the fuzzy topological space (X, τ) be fixed and φ_1, $\varphi_2 \in O_{(L^X, \tau)}$ then the characterized fuzzy space (X, $\varphi_{1,2}$.int) is said to be characterized fuzzy T_1-space if for all x, y ∈ X such that (X, $\varphi_{1,2}$.int) is said to be characterized fuzzy T_1-space if for all x, y ∈ X such that $x \neq y$ there exist $\mu, \eta \in LX$ and $\alpha, \beta \in L0$ such that $\mu(x) < \alpha \le (\varphi_{1,2}.\text{int}\mu)(y)$ and $\eta(y) < \beta \le (\varphi_{1,2}.\text{int}\eta)(x)$ are hold. The related fuzzy topological space(X, τ) is said to be fuzzy $\varphi_{1,2}$-T_1 if for all $x, y \in X$ such that $x \neq y$, we have $x^{\cdot} /\le N\varphi_{1,2}(y)$ and $y^{\cdot} /\le N\varphi_{1,2}(x)$.

Proposition

Let (X, T) be an ordinary topological space and φ1, φ2 ∈ ∈ $O_{(P(X),T)}$ such that $\varphi_2 \ge i_{P(X)}$ is isotone and idempotent. Then (X, T) is $\varphi_{1,2}T_1$-space if and only if the induced characterized fuzzy space (X, $\varphi_{1,2}$.intω) is characterized fuzzy T_1 [27].

Proposition

Let (X, τ) be an fuzzy $\varphi_{1,2}$-T_1 space and φ_1, $\varphi_2 \in O_{(L^X, \tau)}$ such that φ_2 is isotone and idempotent. Then the α-level characterized space (X, $\varphi_{1,2}$.int$_\alpha$) and the initial characterized space (X, $\varphi_{1,2}$.int$_i$) are T_1-spaces [27].

Proposition

Let X be a set, let I be a class and for each i ∈ I, let the characterized fuzzy space (X_i, $\delta_{1,2}$.int$_i$) is characterized fuzzy T_1 and f_i: X → X_i be an injective mapping for some i ∈ I. Then the initial characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is also characterized fuzzy T_1-space [10].

Proposition

Let X be a set, let I be a class and for each $i \in I$, let the characterized fuzzy space (X_i, $\delta_{1,2}$.int$_i$) is characterized fuzzy T_1 and f_i: $X_i \to X$ be an surjective mapping for some i ∈ I. Then the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ (X, $\varphi_{1,2}$.int) is characterized fuzzy T_1-space [27].

Proposition

Let the characterized fuzzy space (X, φ1,2.int) is characterized fuzzy T1 and δ1,2.int is finer than φ1,2.int. Then the characterized fuzzy space (X, δ1,2.int) is also fuzzy T1 [27].

Characterized Fuzzy $R_{2\frac{1}{2}}$ and Characterized Fuzzy R_3-Spaces

Let a fuzzy topological space(X, τ) be fixed and φ_1, $\varphi_2 \in O_{(L^X, \tau)}$. Then the characterized fuzzy space (X, $\varphi_{1,2}$.int) is said to be characterized fuzzy $R_{2\frac{1}{2}}$ [9] (resp. fuzzy R_3-space [10] if for all x ∈ X, F ∈ $\varphi_{1,2}$C(X) such that x/ F (resp. F_1, $F_2 \in \varphi_{1,2}$C(X) such that $F_1 \cap F_2$=∅), there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f: \left(X,\ \varphi_{1,2}.int\right) \to \left(I_L,\ \psi_{1,2}.int_3\right)$ such that

$$\mu \ = \ \{\overline{\alpha} \wedge R^{\delta}|_{R_L} : \ \delta > 0 \ and \ \alpha \in L\} \cup \{\overline{\alpha} : \alpha \in L\},$$

$$F_1,\ F_2 \in \varphi_{1,2}C\left(X\right) F_1 \cap F_2 = \varnothing.$$

for all y ∈ F (resp. the infimum) $\mathcal{N}_{\varphi 1,2}(F_1) \wedge \mathcal{N}_{\varphi 1,2}(F_2)$ does not exist).

Proposition 2.8 [9] Let (X, τ) be a fuzzy topological space, φ_1, φ_2 ∈

$O_{(X,\tau)}$ and Ω is a subbase for the characterized fuzzy space $(X, \varphi_{1,2}.int_\tau)$. Then, $(X, \varphi_{1,2}.int_\tau)$ is characterized fuzzy R_2 $\mathbf{1}_2$-space if and only if for all $F \in \Omega'$ and $x \in X$ such that $x \in/F$, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X, \varphi_{1,2}.int) \to (I_L, \psi_{1,2}.int_3)$ fuzzy $T_{3\frac{1}{2}}$ and characterized fuzzy T_4-spaces such that $f(x)=\bar{1}$ and $f(y)=\bar{0}$ for all $y \in F$.

Characterized

Let a fuzzy topological space(X, τ) be fixed and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.int)$ is said to be characterized fuzzy $R_{2\frac{1}{2}}$ or characterized Tychonoff fuzzy space [9] (resp. fuzzy T_4-space [10] if and only if it is characterized fuzzy $R_{2\frac{1}{2}}$ (resp. characterized fuzzy R_3) and characterized fuzzy T_1-space. The related fuzzy topological space(X, τ) is said to be fuzzy $\varphi_{1,2}$- $T_{3\frac{1}{2}}$ (resp. fuzzy $\varphi_{1,2}$-T_4) if and only if it is fuzzy $\varphi_{1,2}$- $R_{2\frac{1}{2}}$ (resp. fuzzy $\varphi_{1,2}$-R_3) and fuzzy $\varphi_{1,2}$-T_1 space.

Proposition

Every characterized fuzzy T_4-space is characterized fuzzy $T_{3\frac{1}{2}}$-space [9].

Metrizable Characterized Fuzzy Spaces and Characterized $T_{3\frac{1}{2}}$-Spaces

By the fuzzy metric on the set X [6], we mean that the mapping d: X × X:→ R^*_L such that the following conditions are fulfilled:

(1) $d(x, y)=0^\sim$ if and only if x=y.

(2) $d(x, y)=d(y, x)$ for all $x, y \in X$.

(3) $d(x, y) \le d(x, z)+ d(z, y)$ holds for all $x, y, z \in X$.

Where $0^\sim$ denotes the fuzzy number which has value 1 at 0 and 0 otherwise. The set X equipped with an fuzzy metric on X will be called fuzzy metric space. Each fuzzy metric on a set X generated canonically a stratified fuzzy topology τ_d which has the set $B=\{\xi \circ d_x: \xi \in \mu$ and $x \in X\}$ as a base, where d_x: X → R^*_L is the mapping defied by: $d_x(y)=d(x, y)$ and

$$\mu = \{\bar{\alpha} \wedge R^\delta|_{R_L^*}: \delta > 0 \text{ and } \alpha \in L\} \cup \{\bar{\alpha} : \alpha \in L\},$$

Where $\bar{\alpha}$ has the domain is R_L^* and $R^\delta|_{R_L^*}$ is the restriction of R^δ on R_L^*. Now, consider $\varphi_1, \varphi_2 \in O_{(L^X,\tau_d)}$, then as shown in ref. [20], the characterized fuzzy space $(X, \varphi_{1,2}.int_{\tau d})$ is stratified. The stratified characterized fuzzy space $(X, \varphi_{1,2}.int_{\tau d})$ is said to be metrizable characterized fuzzy space.

In the following proposition we shall prove that every metrizable characterized fuzzy space is characterized fuzzy T_4-space in sense of Abd-Allah [10].

Proposition

Let (X, τ_d) be an stratified fuzzy topological space generated canonically by an fuzzy metric d on X and $\varphi_1, \varphi_2 \in O_{(L^X,\tau d)}$, then the metrizable characterized fuzzy space $(X, \varphi_{1,2}.int_{\tau d})$ is characterized fuzzy T_4-space.

Proof: Let $F_1, F_2 \in \varphi_{1,2}C(X)$ such that $F_1 \cap F_2 = \emptyset$. Then for all $x \in F_1$ and $y \in F2$, we get $d(x, y) \ne 0\sim$, that is, there exists $\delta>0$ such that $d(x, y)(2\delta)>0$ and therefore

$$R^{2\delta}|_{R_L^*}\left(d(x, y)\right) = \left(\vee_{\alpha\ge 2\delta} d(x,y)(\alpha)\right)' < 1,$$

holds. Consider $\mu = R^\circ|_{R_L^*} \circ d_x$ and $\eta = R^\circ|_{R_L^*} \circ d_y$, then

$$\mu(x) = R^\delta|_{RL^*}\left(d_x(x)\right) = R^\delta|_{RL^*}(0^\sim) = \left(\vee_{\alpha\ge\delta}(0^\sim)(\alpha)\right) = 1 \text{ for all}$$

$$x \in F_1 \text{ and } \eta(y) = R^\delta|_{RL^*}\left(d_y(y)\right) = R^\delta|_{RL^*}(0^\sim) = \left(\vee_{\alpha\ge\delta}(0^\sim)(\alpha)\right) = 1$$

for all $y \in F_2$. Hence, μ and η are φ1,2-fuzzy neighborhoods in (X, φ1,2.intτd) at all x ∈ F1 and all y ∈ F2, respectively, this means $\wedge_{x\in F_1} N_{\varphi 1,2}(x)(\mu) \wedge \wedge_{x\in F_2} N_{\varphi 1,2}(y)(\eta) = 1$. Because of the symmetry and triangle inequality of d and (2.2), we get $R^\delta|_{R_L^*}\left(d(x, z)\right) \wedge R^\delta|_{R_L^*}\left(d(y, z)\right) \le R^{2\delta}|_{R_L^*}\left(d(x, y)\right) < 1$ and therefore $(\mu \wedge \eta)(z) = \left(R^\delta|_{R_L^*} \circ d_x\right)(z) \wedge \left(R^\delta|_{R_L^*} d_y\right)(z) < 1$ holds for all $z \in X$, that is, sup $(\mu \wedge \eta)<1$. Hence, the infimum $N\varphi1,2$ (F1) ∧ $N\varphi1,2$ (F2) does exists and therefore $(X, \varphi_{1,2}.int_{\tau d})$ is characterized fuzzy R_3-space. Because of Theorem 3.1 [27], it is clear that $(X, \varphi_{1,2}.int_{\tau d})$ is characterized fuzzy T_1-space. Consequently, $(X, \varphi_{1,2}.int_{\tau d})$ is characterized fuzzy T_4-space.

Example 3.1

From Propositions 2.9 and 3.1, we get that the metrizable fuzzy space in sense of Gahler and Gahler [3] is an example of a metrizable characterized fuzzy T_4-space and that is also example of a metrizable characterized fuzzy T_k-space for $k \in \{0, 1, 2, 2\frac{1}{2}, 3, 3\frac{1}{2}\}$

Characterized $R_{2\frac{1}{2}}$ and characterized $T_{3\frac{1}{2}}$-spaces

In the following we introduce and study the concepts of characterized $R_{2\frac{1}{2}}$-space and of characterized $T_{3\frac{1}{2}}$-spaces in the classical case. Let (X, T) be an ordinary topological space and $\varphi_1, \varphi_2 \in O_{(P(X),T)}$. Then the characterized space $(X, \varphi_{1,2}.int_T)$ is said to be characterized $R_{2\frac{1}{2}}$-space if for all $x \in X$, $F \in \varphi_{1,2}C(X)$ such that x/F, there exists an $\varphi_{1,2}\psi_{1,2}$ continuous mapping f: $(X, \varphi_{1,2}.int_T) \to (I, \psi_{1,2}.int_{TI})$ such that f(x)=1 and f(y)=0 for all $y \in F$, where $\psi_{1,2}.int_I$ is the usual $\psi_{1,2}$-interior operator on the closed unit interval I and $\psi_1, \psi_2 \in O_{(P(I),TI)}$. Moreover, the ordinary characterized space $(X, \varphi_{1,2}.int_T)$ is said to be characterized $T_{3\frac{1}{2}}$-space or classical characterized-Tychonoff space if and only if it is characterized T_1-space and characterized $R_{2\frac{1}{2}}$-space.

Proposition

Let (X, T) be an ordinary topological space and $\varphi_1, \varphi_2 \in O_{(P(X),T)}$ such that $\varphi_2 \ge i_{P(X)}$ is isotone and idempotent. Then, $(X, \varphi_{1,2}.int_T)$ is characterized $R_{2\frac{1}{2}}$-space if and only if the induced characterized fuzzy space $(X, \varphi_{1,2}.int_\omega)$ is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Proof: Let $(X, \varphi_{1,2}.int_T)$ is characterized $R_{2\frac{1}{2}}$-space, $x \in X$ and $F \in \left(\omega\left(\varphi_{1,2}O(X)\right)\right)'$ such that x /F. Then, there exists $\varphi_{1,2}\delta_{1,2}$-continuous mapping g: $(X, \varphi_{1,2}.int_T) \to (I, \delta_{1,2}.int_{T_I})$ such that $g(x)=1$ and $g(y)=0$ for all $y \in S_\alpha S = F$ and for all $\alpha \in L_1$, where $\delta_1, \delta_2 \in O_{(P(I),T_I)}$. Hence, the mapping g: $(X, \varphi_{1,2}.int_\omega) \to (I, \delta_{1,2}.int_{\omega(T_I)})$ is $\varphi_{1,2}\delta_{1,2}$-fuzzy continuous. Consider h: $\left(I, \delta_{1,2}.int_{\omega(T_I)}\right) \to (I_L, \psi_{1,2}.int_3)$ is the map-ping defied by $h(z) = \bar{z}$ for all $z \in I$, then h is $\delta_{1,2}\psi_{1,2}$-fuzzy continuous and there-fore there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f = h \circ g:(X, \varphi_{1,2}.int_\omega) \to (I_L, \psi_{1,2}.int_3)$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all y ∈F. Consequently, (X, $\varphi_{1,2}$.int$_\omega$) is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Conversely, let (X, $\varphi_{1,2}$.int$_\omega$) is characterized fuzzy $R_{2\frac{1}{2}}$-space, $x \in X$ and F ∈ $\varphi_{1,2}$C(X) such that $x \notin F$. Then, $x \notin \chi_F$. $\chi_F \in \varphi_{1,2}C(X)$ and $\chi_F \in \left(\omega\left(\varphi_{1,2}O(X)\right)\right)'$. Therefore, there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping $f:(X, \varphi_{1,2}.int_\omega) \to (I_L, \psi_{1,2}.int_\Im)$ such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all $y \in \chi_F$. Since $\varphi_{1,2}int_T = \left(\varphi_{1,2}.int_\omega\right)_\alpha$ and $\psi_{1,2}.int_\Im = \psi_{1,2}.int_{T_I}$, then there could be found the mapping $f_\alpha = \left(X, \varphi_{1,2}.int\right)_T \to \left(I, \psi_{1,2}.int_{T_I}\right)$ which is $\psi_{1,2}\psi_{1,2}$-continuous with $f_\alpha(x) = 1$ and $f_\alpha(y) = 0$ for all $y \in F$. Hence, $\left(X, \varphi_{1,2}.int_T\right)$ is characterized $R_{2\frac{1}{2}}$-space.

Corollary 3.1

Let (X, T) be an ordinary topological space and φ_1, φ_2 ∈ $O_{(P(X),T)}$ such that $\varphi_2 \geq i_{P(X)}$ is isotone and idempotent. Then, (X, $\varphi_{1,2}$.int$_T$) is characterized $T_{3\frac{1}{2}}$-space if and only if the induced characterized fuzzy space (X, $\varphi_{1,2}$.int$_\omega$) is characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Immediate from Propositions 2.3 and 3.2.

Proposition 3.2 and Corollary 3.1, show that the notions of characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces are good extension as in sense of Lowen [11].

In the following proposition for each fuzzy topological space (X, τ), we show that the α-level characterized space (X, $\varphi_{1,2}$.int$_\alpha$) and the initial characterized space (X, $\varphi_{1,2}$.int$_i$) are characterized $R_{2\frac{1}{2}}$-spaces if the characterized fuzzy space (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$.

Proposition 3.3

Let (X, τ) be a fuzzy topological space and φ_1, φ_2 ∈ $O_{(L^X,\tau)}$ such that $\varphi_2 \geq 1_L{}^X$ is isotone and idempotent. Then the α-level characterized space (X, $\varphi_{1,2}$.int$_\alpha$) and the initial characterized space (X, $\varphi_{1,2}$.int$_i$) are characterized $R_{2\frac{1}{2}}$-spaces if (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$-space, there exists

Proof: Consider (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$-space, x ∈ X and $F \in \left(\left(\varphi_{1,2}O(X)\right)_\alpha\right)'$ such that $x \notin F$. Then $x \notin \chi_F$. and $\chi_F \in \varphi_{1,2}C(X)$. Because of (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$ Space,-space, there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: (X, $\varphi_{1,2}$.int$_\tau$) → (I_L, $\psi_{1,2}$.int$_I$) and f(y)=0 such that $f(x) = \bar{1}$ and $f(y) = \bar{0}$ for all $y \in X_F$. Since $\varphi_{1,2}$.in t$_\tau$= $\varphi_{1,2}$.int$_\alpha$ and $\psi_{1,2}$.int$_I$=$\psi_{1,2}$.int$_{TI}$, then there could be found the mapping f_α: (X, $\varphi_{1,2}$.int$_\alpha$) → (I, $\psi_{1,2}$.int$_{TI}$) which is $\varphi_{1,2}\psi_{1,2}$-continuous with f_α(x)=1 and f_α(y)=0 for all y ∈ F. Consequently, (X, $\varphi_{1,2}$.int$_\alpha$) is characterized $R_{2\frac{1}{2}}$ space. The second case is similarly, that is, if (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Corollary 3.2

Let (X, τ) be a fuzzy topological space and φ_1, φ_2 ∈ $O_{(L}X,_{\tau)}$ such that $\varphi_2 \geq 1_L X$ is isotone and idempotent. Then the α-level characterized space (X, $\varphi_{1,2}$.int$_\alpha$) and the initial characterized space (X, $\varphi_{1,2}$.int$_i$) are characterized $T_{3\frac{1}{2}}$-spaces if the characterized fuzzy space (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $T_{3\frac{1}{2}}$.

Proof: Immediate from Propositions 2.4 and 3.3.

In the following it will be shown that the finer characterized fuzzy space of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space is also characterized completely fuzzy $R_{2\frac{1}{2}}$-space and characterized fuzzy $T_{3\frac{1}{2}}$-space, respectively.

Proposition

Let (X, τ) is a fuzzy topological space and φ_1, φ_2∈ O(L^X, τ). If the characterized fuzzy space (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$ and $\delta_{1,2}$.int$_\tau$ is finer than $\varphi_{1,2}$.int$_\tau$, then (X, $\delta_{1,2}$.int$_\tau$) is also characterized fuzzy and $\delta_{1,2}$.int$_\tau$ $R_{2\frac{1}{2}}$-space.

Proof: Let Ω is a sub base for the characterized fuzzy space (X, $\varphi_{1,2}$.int$_\tau$), x ∈ X and F ∈ Ω′ such that $x \notin F$. Such that $x \notin F$ Then, there is V_1,. . ., V_n ∈ Ω such that x ∈ (V_1 ∩.. . ∩V_n) ⊆ F′ and therefore $x \notin V_i'$, V_i' ∈ Ω′ for all i ∈ {1,. . ., n}. Because of Proposition 2.8, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mappings f_i: (X, $\varphi_{1,2}$.int$_\tau$) → (I_L, $\psi_{1,2}$.int$_I$) such that $f_i(x) = \bar{1}$ and $f_i(y)' = \bar{0}$ is also fulfilled for all $y \in (V_1' \cup \ldots \cup V_n)$. In particular this means that $f_i(x) = \bar{1}$ and $f_i(y) = \bar{0}$ for all y ∈ F and i ∈ {1,. . ., n}. Since $\delta_{1,2}$.int$_\tau$ is finer than $\varphi_{1,2}$.int$_\tau$, then any one of these mappings f_i: X → I_L gives us the required $\delta_{1,2}\psi_{1,2}$-fuzzy continuous mappings g: (X, $\delta_{1,2}$.int$_\tau$) → (I_L, $\psi_{1,2}$.int$_I$) such that $g(x) = \bar{1}$ and $g(y) = \bar{0}$ and f_i(y)=0 for all y ∈ F and i ∈ {1,. . ., n}. Since $\delta_{1,2}$.int$_\tau$ is finer than $\varphi_{1,2}$.int$_\tau$, then any one of these mappings f_i: X → I_L gives us the required $\delta_{1,2}\psi_{1,2}$-fuzzy for all y ∈ F. Consequently, (X, $\delta_{1,2}$.int$_\tau$) is characterized fuzzy $R_{2\frac{1}{2}}$ Space.

Corollary 3.3 Let (X, τ) be a fuzzy topological space and φ_1, φ_2 ∈ $O_{(L}X,_{\tau)}$. If (X, $\varphi_{1,2}$.int$_\tau$) is characterized fuzzy $T_{3\frac{1}{2}}$-space and $\delta_{1,2}$.int$_\tau$ is finer than $\varphi_{1,2}$.int$_\tau$, then (X, $\delta_{1,2}$.int$_\tau$) is also characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Immediate from Propositions 2.7 and 3.4.

Initial and Final Characterized Fuzzy $R_{2\frac{1}{2}}$ and Fuzzy $T_{3\frac{1}{2}}$-Spaces

In this section we are going to introduce and study the notion of initial and final characterized fuzzy $R_{2\frac{1}{2}}$-spaces and the notions of initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces. The characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space are studied as special case from the initial and final characterized fuzzy $R_{2\frac{1}{2}}$ and fuzzy $T_{3\frac{1}{2}}$-spaces. New additional properties for the initial and final characterized fuzzy $R_{2\frac{1}{2}}$-spaces and for the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces are given. The categories of all characterized fuzzy $R_{2\frac{1}{2}}$ and of all characterized fuzzy $R_{2\frac{1}{2}}$-spaces will be denoted by CFR-Space and CRF-Tych, respectively. Note that the categories CFR-Space and CRF-Tych are concrete categories. The concrete categories CFR-Space and CRF-Tych are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets. Hence, all the initial and final lifts exist uniquely in the categories CFR-Space and CRF-Tych, respectively.

This means that they also topological over the category SET. That is, all the initial and final characterized fuzzy $R_{2\frac{1}{2}}$-spaces and all the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces exist in CFR-Space and CRF-Tych, respectively.

In the following let X be a set, let I be a class and for each $i \in I$, let the characterized fuzzy space $(X_i, \delta_{1,2}.\mathrm{int}_i)$ of all $\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $R_{2\frac{1}{2}}$-space. For some $i \in I$, let $f_i: X \to X_i$ is $\varphi_{1,2}\delta_{1,2}$-closed injective mapping from X into X_i. Then we show in the following that the initial characterized fuzzy space $(X,\varphi_{1,2}.\mathrm{int})$ of $((X_i, \delta_{1,2}.\mathrm{int}_i))_i\in_I$ with respect to $(f_i)_{i\in I}$ is also characterized fuzzy $R_{2\frac{1}{2}}$-space. More general, we show under the same conditions, that the initial characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ of $((X_i, \delta_{1,2}.\mathrm{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is characterized fuzzy $T_{3\frac{1}{2}}$-space if all the characterized fuzzy spaces $(X_i, \delta_{1,2}.\mathrm{int}_i)$ are characterized fuzzy $T_{3\frac{1}{2}}$-spaces for all $i \in I$. Moreover, as special cases we show that the characterized fuzzy subspace, characterized fuzzy product space and characterized fuzzy filter pre topology of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space are characterized fuzzy $R_{2\frac{1}{2}}$-spaces and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively.

Proposition

Let X be a set and I be a class. For each $i \in I$, let the characterized fuzzy space $(X_i, \delta_{1,2}.\mathrm{int}_i)$ of all $\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $R_{2\frac{1}{2}}$-space. If $f_i: X \to X_i$ is an $\varphi_{1,2}\delta_{1,2}$-closed injective mapping from X into X_i for some $i \in I$, then the initial characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ of $((X_i, \delta_{1,2}.\mathrm{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is also characterized fuzzy $R_{2\frac{1}{2}}$-space.

Proof: Let $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that x F. Since $f_i: X \to X_i$ is $\varphi_{1,2}\delta_{1,2}$-closed injective for some $i \in I$, then $f_i(F) \in \delta_{1,2}C(X_i)$ and $f_i(x) \notin f_i(F)$. Because of $(X_i, \delta_{1,2}.\mathrm{int}_i)$ is characterized fuzzy $R_{2\frac{1}{2}}$-space for all $i \in I$, then there

exists an $\delta_{1,2}\psi_{1,2}$-fuzzy continuous mapping g: $(X_i, \delta_{1,2}.\mathrm{int}_i) \to (I_L, \psi_{1,2}.\mathrm{int}_1)$ such that $g(fi(x)) = \overline{1}$ and $g(fi(x)) = \overline{0}$ for all $y \in F$. Therefor the composition $h=g\, fi$: $(X, \varphi_{1,2}.\mathrm{int}) \to (I_L, \psi_{1,2}.\mathrm{int}_1)$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping such that $h(x) = (g \circ fi)(x)) = \overline{1}$ and $h(y) = (g \circ fi)(y)) = \overline{0}$ for all y F. Consequently, $(X, \varphi_{1,2}.\mathrm{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Corollary 4.1 Let X be a set and I be a class. For each $i \in I$, let the characterized fuzzy space $(X_i, \delta_{1,2}.\mathrm{int}_i)$ of all $\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $T_{3\frac{1}{2}}$-space. If $f_i: X \to X_i$ is an $\varphi_{1,2}\delta_{1,2}$-closed injective mapping from X into X_i for some $i \in I$, then the initial characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ $((X_i, \delta_{1,2}.\mathrm{int}_i))_{i\in I}$ of with respect to $(f_i)_{i\in I}$ is also characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Immediate from Propositions 2.5 and 4.1.

Corollary 4.2

The characterized fuzzy subspace $(A, \varphi_{1,2}.\mathrm{int}_A)$ and the characterized fuzzy product space $\prod_{i\in I}(X_i, \psi_{1,2}.\mathrm{int}_i)$ of a characterized fuzzy $R_{2\frac{1}{2}}$-space (resp. characterized fuzzy $T_{3\frac{1}{2}}$-space) are also characterized fuzzy $R_{2\frac{1}{2}}$-space (resp. characterized $T_{3\frac{1}{2}}$-space)

Proof: Follows immediately from Proposition 4.1 and Corollary 4.1. 2

As shown in ref. [7], the characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ is characterized as a fuzzy filter pre topology, then we have the following result:

Corollary 4.3

For each $i \in I$, let $\mathcal{N}^i_{\delta_{1,2}} X_i \to F_L X_i$ is $\delta_{1,2}.\mathrm{int}_i$ as the fuzzy filter pre topology is characterized fuzzy R_2 fuzzy $T_{3\frac{1}{2}}$). Then, the representation of the initial $\varphi_{1,2}$-interior operator $\mathcal{N}_{\varphi_{1,2}} X \to F_L X$ of the initial characterized fuzzy space $(X, \varphi_{1,2}.\mathrm{int})$ of $((X_i, \delta_{1,2}.\mathrm{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ as a fuzzy filter pre topology which is defined by:

$$\mathcal{N}_{\varphi 1,2}(x)(\mu) = \bigvee_{\mu_i \circ f_i \le \mu, i\in I} \mathcal{N}^{\,i}_{\delta\,1,2}(f_i(x))(\mu_i)$$

for all $x \in X$ and $\mu \in L^X$ is also characterized fuzzy $R_{2\frac{1}{2}}$ (resp. characterized fuzzy $T_{3\frac{1}{2}}$).

Now, if we consider the case of I being a singleton, then we have the following results as special cases from Proposition 4.1 and Corollary 4.1.

Proposition

Let (X, τ_1) and (Y, τ_2) are two fuzzy topological spaces, $\delta_1, \delta_2 \in O_{(L^Y,\tau_2)}$ and $\delta_1, \delta_2 \in O_{(L^Y,\tau_2)}$. If the mapping f: $X \to Y$ is an $\varphi_{1,2}\delta_{1,2}$-closed injective from X into Y and $(Y, \delta_{1,2}.\mathrm{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$ terized fuzzy $T_{3\frac{1}{2}}$) L-space, then the initial characterized fuzzy space $(X(Y, \delta_{1,2}.\mathrm{int})$ with respect to f is also characterized fuzzy $R_{2\frac{1}{2}}$ (resp. fuzzy $T_{3\frac{1}{2}}$) L-space.

Proof: Straight forward.

Corollary 4.4

Let (Y, τ_2) be an fuzzy topological spaces and $\delta_1, \delta_2\, f$: $X \to Y$ is an $\varphi_{1,2}\delta_{1,2}$-closed injective mapping from X into Y fuzzy $\delta_{1,2}T_{3\frac{1}{2}}$-space), then the initial fuzzy topological space $(X, f^{-1}(\tau_2))$ of (Y, τ_2) with respect to f is fuzzy $\varphi_{1,2}$ $R_{2\frac{1}{2}}$ – space (resp. fuzzy $\varphi_{1,2}T_{3\frac{1}{2}}$-space) for all $\varphi_1, \varphi_2 \in O_{(L^X, f^{-1}(\tau_2))}$.

Proof: Follows immediately from Proposition 4.2. 2

In the following let X be a set and I be a class. For each $i\in I$, let the characterized fuzzy space $(X_i, \delta_{1,2}.\mathrm{int}_i)$ of all$\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $R_{2\frac{1}{2}}$-space. For some $i \in I$, let f_i: $X_i \to X$

is surjective mapping from X_i into X and f_i^{-1} is $\varphi_{1,2}\delta_{1,2}$-closed in the classical sense. Then as in case of the initial characterized fuzzy spaces, we show in the following that the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to$(f_i)_{i\in I}$ is also characterized fuzzy $R_{2\frac{1}{2}}$-space. More general, we show under the same conditions that, the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is characterized fuzzy $T_{3\frac{1}{2}}$ space if each of the characterized fuzzy spaces (X_i, $\delta_{1,2}$.int$_i$) is characterized fuzzy $T_{3\frac{1}{2}}$-spaces for all $i \in I$. Moreover, as special cases we show that the characterized fuzzy quotient space and the characterized fuzzy sum space of the characterized fuzzy $R_{2\frac{1}{2}}$-space and of the characterized fuzzy $T_{3\frac{1}{2}}$-space are characterized fuzzy $R_{2\frac{1}{2}}$-spaces and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. Proposition 4.3 Let X be a set and I be a class. For each $i \in I$, let the characterized fuzzy space (X_i, $\delta_{1,2}$.int$_i$) of all $\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $R_{2\frac{1}{2}}$-space. If f_i: $X_i \to X$ is an subjective $\delta_{1,2}\varphi_{1,2}$-fuzzy open mapping from X_i into X and f_i^{-1} is $\varphi_{1,2}\delta_{1,2}$-closed for some $i \in I$, then the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is also characterized fuzzy $R_{2\frac{1}{2}}$-space.

Proof: Let $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that x F. Since f_i: $X_i \to X$ is surjective and f_i^{-1} is $\varphi_{1,2}\delta_{1,2}$-closed for some $i \in I$, then there exists $K \in \delta_{1,2}C(X_i)$ and $x_i \in X_i$ for which $x_i=f_i^{-1}(x)$ and $K=f_i^{-1}(F)$ such that $x_i \notin K$. Because of (X_i, $\delta_{1,2}$.int$_i$) is characterized fuzzy $R_{2\frac{1}{2}}$-space for all $i \in I$, then there exists an $\delta_{1,2}$ $\psi_{1,2}$ fuzzy continuous mapping g: (X_i, $\delta_{1,2}$. int$_i$)→ (I_L,$\psi_{1,2}$.int$_\mathfrak{I}$) such that $g(x_i)=\bar{1}$ and $g(z)=\bar{0}$ for all $z \in K$, that is, $g(f_i^{-1}(x))=\bar{1}$ and $g(f_i^{-1}(s))=\bar{0}$ for all $s\in F$. Therefore, there exists a mapping $h=g\circ f_i^{-1}:(X,\varphi_{1,2}.\text{int})\to(I_L,\psi_{1,2}.\text{int}_{\mathfrak{I}})$ such that $h(x)=\bar{1}$ and $h(s)=\bar{0}$ for all $s\in F$. Since f_i is $\delta_{1,2}\phi_{1,2}$ fuzzy open, then $\varphi_{1,2}\,\text{int}\,\mu\circ f_i^{-1}=f_i(\varphi_{1,2}.\text{int}\,\mu)\le\delta_{1,2}.\text{int}_i\,f_i(\mu)=\delta_{1,2}.\text{int}_i(\mu\circ f_i^{-1})$ holds for all $\in L^X$ and $i\in I$, which means that $f_i^{-1}:(X,\varphi_{1,2}.\text{int})\to(X_i,\varphi_{1,2}.\text{int}_i)$ is $\varphi_{1,2}$ $\delta_{1,2}$-fuzzy continuous. Hence, the composition $h=g\circ f_i^{-1}:(X,\varphi_{1,2}.\text{int})\to(I_L,\varphi_{1,2}.\text{int}_{\mathfrak{I}})$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping and therefore the final characterized fuzzy space (X, $\varphi_{1,2}$.int) is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Corollary 4.5

Let X be a set and I be a class. For each $i \in I$, let the characterized fuzzy space (X_i, $\delta_{1,2}$.int$_i$) of all $\delta_{1,2}$-open fuzzy subsets of X_i is characterized fuzzy $T_{3\frac{1}{2}}$-space. If f_i: $X_i \to X$ is an surjective $\delta_{1,2}\varphi_{1,2}$-fuzzy open mapping from X_i into X and f_i^{-1} is $\varphi_{1,2}\delta_{1,2}$-closed for some $i \in I$, then the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of $((X_i, \delta_{1,2}.\text{int}_i))_{i\in I}$ with respect to $(f_i)_{i\in I}$ is also characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Immediate from Propositions 2.6 and 4.3. 2

Corollary 4.6

The characterized fuzzy quotient space (A, $\varphi_{1,2}$.int$_f$) and the char characterized fuzzy $T_{3\frac{1}{2}}$-space) are also characterized fuzzy $R_{2\frac{1}{2}}$ (resp. characterized fuzzy $T_{3\frac{1}{2}}$) L-spaces.

Proof: Follows immediately from Proposition 4.3 and Corollary 4.5. 2

Now, if we consider the case of I being a singleton, then we have the following results as special cases from Proposition 4.3 and Corollary 4.5.

Proposition 4.4 Let (X, τ_1) and (Y, τ_2) are two fuzzy topological spaces, $\varphi_1, \varphi_2 \in O_{(L^X, f(\tau_2))}$ and $\delta_1, \delta_2 \in O_{(L^Y, \tau_2)}$. If f: $Y \to X$ is an subjective $\delta_{1,2}\varphi_{1,2}$-fuzzy open mapping from X into Y and f^{-1} is $\varphi_{1,2}\delta_{1,2}$-closed, then the final characterized fuzzy space (X, $\varphi_{1,2}$.int) of (Y, $\delta_{1,2}$.int) with respect to f is characterized fuzzy $R_{2\frac{1}{2}}$ (resp. characterized fuzzy $T_{3\frac{1}{2}}$)L-space if (Y, $\delta_{1,2}$.int) is characterized fuzzy $R_{2\frac{1}{2}}$ (resp. characterized fuzzy $T_{3\frac{1}{2}}$) L-spaces.

Proof: Straight forward.

Corollary 4.7

Let (Y, τ_2) be an fuzzy topological spaces and $\delta_1, \delta_2 \in O_{(L^Y, \tau_2)}$, f: $Y \to X$ is an $\delta_{1,2}\varphi_{1,2}$-fuzzy open surjective mapping from Y into X and f^{-1} $\varphi_{1,2}\delta_{1,2}$-closed, then the final fuzzy topological space(X, $f(\tau_2)$) of (Y, τ_2) with respect to f is fuzzy $\varphi_{1,2}$ $R_{2\frac{1}{2}}$-space (resp. fuzzy $\varphi_{1,2}$ $T_{3\frac{1}{2}}$)-space if (Y, τ_2) is fuzzy $\delta_{1,2}$ $R_{2\frac{1}{2}}$-space (resp. fuzzy $\delta_{1,2}$ $T_{3\frac{1}{2}}$)-space for all $\varphi_1, \varphi_2 \in O_{(L^X, f(\tau_2))}$.

Proof: Follows immediately from Proposition 4.4. 2.

Finer Characterized Fuzzy $R_{2\frac{1}{2}}$ and Finer Characterized Fuzzy $T_{3\frac{1}{2}}$-Spaces

In this section we are going to introduce and study some finer characterized fuzzy $R_{2\frac{1}{2}}$ and finer characterized fuzzy $T_{3\frac{1}{2}}$-paces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy $T_{3\frac{1}{2}}$-spaces introduced [28,12,13]. The relations between such characterized fuzzy $R_{2\frac{1}{2}}$-spaces and our characterized fuzzy $R_{2\frac{1}{2}}$-spaces which presented [9] are introduced. More generally, the relations between such characterized fuzzy $T_{3\frac{1}{2}}$-spaces and our characterized fuzzy $T_{3\frac{1}{2}}$-spaces are also introduced.

Characterized fuzzy $R_{2\frac{1}{2}}$ H and characterized fuzzy $T_{3\frac{1}{2}}$ H-spaces. In the following we introduce and study the concept of characterized completely fuzzy regular Hutton and characterized fuzzy $T_{3\frac{1}{2}}$ Hutton-spaces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy $T_{3\frac{1}{2}}$-spaces in sense of Hutton [28], respectively. The relation between characterized completely fuzzy regular Hutton-spaces and the characterized fuzzy $R_{2\frac{1}{2}}$-spaces in our sense is introduced. More generally, the relations between characterized fuzzy $T_{3\frac{1}{2}}$ Hutton-spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces in our sense is also introduced. Let (X, τ) be a fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X, \tau)}$. Then the characterized fuzzy space (X, $\varphi_{1,2}$.int) is said to be characterized completely fuzzy regular Hutton-space or (characterized fuzzy $R_{2\frac{1}{2}}$ H-space, for short) if for an

$\mu \in \varphi_{1,2}OF(X)$, there exists a collection $(\eta_\alpha)_{\alpha\in L}$ in L^X and an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping g: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_I)$ such that $\mu = \bigvee_{\alpha\in L}\eta_\alpha$ and $\eta_\alpha(y) \le g(y)(1-) = \bigwedge_{t<1} g(y)(t) \le g(y)(0_+) = \bigvee_{s>0} g(y)(s) \le \mu(y)$ holds for all $y \in X$. Then characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized fuzzy $T_{3\frac{1}{2}}$ Hutton-space or (characterized fuzzy $T_{3\frac{1}{2}}$ H-space, for short) if and only if it is characterized fuzzy $R_{2\frac{1}{2}}$ H and characterized fuzzy $T_{3\frac{1}{2}}$-spaces.

In the classical case of $L=\{0, 1\}$, $\varphi_1=\text{int}_\tau$, $\psi_1=\text{int}_I$, 1 and $\psi_2 = 1_{L^I}$, the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of f is up to an identification the usual fuzzy continuity of f. Then in this case the notions of characterized fuzzy $R_{2\frac{1}{2}}$ H-spaces and of characterized fuzzy $T_{3\frac{1}{2}}$ H-spaces are coincide with the notion of fuzzy completely regular spaces and the notion fuzzy $T_{3\frac{1}{2}}$-spaces defined by Hutton [28], respectively. Another special choices for the operations φ_1, φ_2, ψ_1 and ψ_2 are obtained (Table 1).

In the following proposition, we show that the characterized fuzzy $R_{2\frac{1}{2}}$-spaces which are presented [9] are more general than the characterized fuzzy $R_{2\frac{1}{2}}$ H-spaces.

Proposition 5.1

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$.

Then every characterized fuzzy $R_{2\frac{1}{2}}$ H-space $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Proof: Let $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$ H-space, $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Then, $\chi_{F'} \in \varphi_{1,2} \in OF(X)$ and $\chi_{F'}(x)=1$, therefore $\chi_{F'}(x) \ge \alpha$ holds for all $\alpha \in L$. Hence, $\chi_{F'} = \bigvee_{\alpha\in F', \alpha\in L} x_\alpha$ and therefore for all $x \in F'$, there exists a family $(x_\alpha)_{\alpha\in L}$ in L^X such that $\chi_{F'} = \bigvee_{\alpha\in L} x_\alpha$ and $x_\alpha(y) < g(y)(1-) < g(y)(0+) < \chi_{F'}(y)$ holds for all $y \in X$. In case of $y \in F$, we get $0 \le g(y)(1-) \le g(y)(0+) \le 0$ holds for all $y \in F$ and therefore $g(y) = \overline{0}$ for all $y \in F$. In case of $y=x$, we get $x_\alpha(x)=\alpha$ $g(x)(1-) \le g(x)(0+) \le 1$ holds for all $\alpha \in L$ and this means that $g(x)(s)=1$ for all $s < 1$ and therefore $g(y) = \overline{1}$ Consequently, $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9].

Corollary 5.1 Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized fuzzy $T_{3\frac{1}{2}}$ H-space is characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Follows immediately from Proposition 5.1.

The following example shows that the inverse of Proposition 5.1 and of Corollary 5.1 is not true in general.

Example 5.1.

Let $X=\{x, y\}$ with $x \neq y$ and $\tau = \{\overline{0}, \overline{1}, x_1, x_{\frac{1}{2}}, x_1 \vee y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_1, x_{\frac{1}{2}} \vee y_{\frac{1}{2}}\}$ is an fuzzy topology on X. Choose $\varphi_1=\text{int}_\tau$, $\varphi_2=\text{cl}_\tau$, $\psi_1=\text{int}_I$ and $\psi_2=\text{cl}_I$. Hence, $\varphi_{1,2}CF(X) = \{\overline{0}, \overline{1}, y_1, x_{\frac{1}{2}}, y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_{\frac{1}{2}}, x_{\frac{1}{2}} \vee y_1\}$ and there is the only case of $x \in X$, $F=\{y\} \in \varphi_{1,2}C(X)$ such that $x \notin F$. Since the mapping f: $(X, \varphi_{1,2}.\text{int}_\tau) \to (I_L, \psi_{1,2}.\text{int}_I)$ which is defined by $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \neq x$ is $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous, then $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9]. Obviously, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy T1-space, therefore $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $T_{3\frac{1}{2}}$-space.

On other hand, let $(X, \varphi_{1,2}.\text{int}\tau)$ is characterized fuzzy $T_{3\frac{1}{2}}$ H-space, then $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $R_{2\frac{1}{2}}$ H and characterized fuzzy T_1-space. Since $x_{\frac{1}{2}} \in \tau = \varphi_{1,2}OF(X)$ and $x_{\frac{1}{2}} = \bigvee_{\alpha\in L}\left(\frac{1}{2} \wedge x\alpha\right)$ then there exists a collection $(\eta_\alpha)_{\alpha\in L} = \bigvee_{\alpha\in L}\left(\overline{\frac{1}{2}} \wedge x_\alpha\right)_{\alpha\in L}$ such that $x_{\frac{1}{2}} = \bigvee_{\alpha\in L}\eta_\alpha$. Moreover, for an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}\tau) \to (I_L, \psi_{1,2}.\text{int}_I)$ such that $f(x) = \overline{1}$ and $f(y) = \overline{0}$ for all $y \neq x$, we get the inequality

$$\eta_\alpha(z) \le f(z)(1-) \le f(z)(0+) \le x_{\frac{1}{2}}(z)$$

holds only when $z=y$, but it is not holds when $z=x$, because $(\frac{1}{2} \wedge \alpha) \le 1 \le \frac{1}{2}$ and this is a contradiction. Hence, $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized fuzzy $R_{2\frac{1}{2}}$ H-space and therefore it is not characterized fuzzy $T_{3\frac{1}{2}}$ H-space.

Characterized fuzzy $R_{2\frac{1}{2}}$ K and characterized fuzzy $T_{3\frac{1}{2}}$ K-spaces. In the following we introduce and study the concept of characterized completely fuzzy regular Katasars spaces and characterized fuzzy $T_{3\frac{1}{2}}$ Katasars spaces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy $T_{3\frac{1}{2}}$-spaces introduced by Katasars [13], respectively. The relation between characterized fuzzy completely regular Katasars spaces and the characterized fuzzy $R_{2\frac{1}{2}}$-spaces in sense Abd-Allah and Khedhairi [9] is introduced. More generally, the relations between characterized fuzzy $T_{3\frac{1}{2}}$ Katasars spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces in sense of [9] is also introduced.

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized completely fuzzy regular Katasars-space or (characterized fuzzy $R_{2\frac{1}{2}}$ K-space, for short) if for every $x \in X$ and $\mu \in L^X$ such that $\mu(x) > \alpha$, $\alpha \in L_0$, there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping g: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_I)$ such that $g(y)(0+) \le \mu(y)$ and $g(y)(1-) > \alpha$ are holds for all $y \in X$ and $\alpha \in L_0$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized fuzzy $T_{3\frac{1}{2}}$ Katasars-space or (characterized fuzzy $T_{3\frac{1}{2}}$ K-space, for short) if and only if it is characterized fuzzy $R_{2\frac{1}{2}}$ K-space and characterized fuzzy T_1-space.

In the classical case of $L=\{0, 1\}$, $\varphi_1=\text{int}_\tau$, $\psi_1=\text{int}_I$, $\psi_2 = 1_{L^I}$ and $\psi_2 = 1_{L^I}$, the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of f is up to an identification the usual fuzzy continuity of f. Then in this case the notions of characterized fuzzy $R_{2\frac{1}{2}}$ K-space and of characterized fuzzy $T_{3\frac{1}{2}}$ K-spaces are coincide with the notion of completely fuzzy regular spaces and the notion of fuzzy $T_{3\frac{1}{2}}$-spaces presented by Katasars [13], respectively. Another special choices for the operations φ_1, φ_2, ψ_1 and ψ_2 are obtained in Table 1. In the following proposition we show that the notion of characterized fuzzy $R_{2\frac{1}{2}}$-spaces which are presented [9] are more general than the characterized fuzzy $R_{2\frac{1}{2}}$ K-spaces.

Proposition

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized fuzzy $R_{2\frac{1}{2}}$ K-space $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$ space.

Proof: Let $(X, \varphi_{1,2}.\text{int})$ is a characterized fuzzy $R_{2\frac{1}{2}}$ K-space, x X and $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Then, $\chi_{F'}(x)=1$ and $\chi_{F'}(x)=1$, therefore $\chi_{F'}(x) \geq \alpha$ holds for all $\alpha \in L$. Because of $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$ K-space, then there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping g: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_1)$ such that $\underset{t>0}{\vee} g(y)(t) \leq \chi_{F'}(y)$ and $\underset{s>1}{\vee} g(y)(s) > \alpha$ are hold for all $y \in X$ and $\alpha \in L$. In case of $y \in F$, we have $\underset{t>0}{\vee} g(y)(t) \leq 0$, that is, $g(y)(t)=0$ for all $t>0$, $y \in F$ and therefore $g(y)$ $\overline{0}$ for all $y \in F$. In case of y=x, we have $\underset{s>1}{\vee} g(x)(s) |> \alpha$ holds for all $\alpha \in L$, and therefore $g(x)=\overline{1}$. Hence, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping g: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_1)$ such that $g(y)=\overline{0}$ and $g(y)=\overline{0}$ for all $y \in F$. Consequently, $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9].

Corollary 5.2 Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized fuzzy $T_{3\frac{1}{2}}$ K-space is characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Follows immediately from Proposition 5.2.

The following example shows that the inverse of Proposition 5.2 and of Corollary 5.2 is not true in general.

Example 5.2.

Consider the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ which is defined in Example 5.1, then as shown in Example 5.1, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9] and characterized fuzzy T_1-space, therefore $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $T_{3\frac{1}{2}}$-space in sense [9].

On other hand, for any $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}_\tau) \to (I_L, \psi_{1,2}.\text{int}_1)$ such that $f(y)=\overline{1}$ and $f(y)=\overline{0}$ for all $y \neq x$, we shall consider $x_{\frac{1}{2}} \in \varphi_{1,2}.OF(X)$ with $x_{\frac{1}{2}}(x)=\frac{1}{2}>0$, that is, there exists some $\alpha=\frac{1}{2} \in L$ such that $x_{\frac{1}{2}}(x)=\alpha$. Therefore, $f(z)(1-)=\underset{r<1}{\wedge} f(z)(t)>\frac{1}{2}$ holds only when $z=x$ and it is not fulfilled when $z=y$. Moreover, $f(z)(0+)=\underset{s>0}{\wedge} f(z)(s) \leq x_{\frac{1}{2}}(z)$ holds only when $z=y$ and it is not fulfilled when $z=x$. Hence, $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized fuzzy $R_{2\frac{1}{2}}$ K-space and therefore it is not characterized fuzzy $T_{3\frac{1}{2}}$ K-space.

Characterized Fuzzy $R_{2\frac{1}{2}}$ KE and Characterized Fuzzy $T_{3\frac{1}{2}}$ KE-Spaces

In the following we introduce and study the concepts of characterized completely fuzzy regular Kandil and Shafee spaces and of characterized fuzzy $T_{3\frac{1}{2}}$ Kandil and Shafee spaces as a generalization of the weaker and stronger forms of the completely fuzzy regular and fuzzy $R_{2\frac{1}{2}}$-spaces presented by Kandil and Shafee [12], respectively. The relation between characterized completely fuzzy regular Kandil and Shafee spaces and the characterized fuzzy $R_{2\frac{1}{2}}$-spaces which are presented [6]. More generally, the relations between characterized fuzzy $T_{3\frac{1}{2}}$ Kandil El-Shafee-spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces in sense [9] is also introduced.

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then the characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized completely fuzzy regular Kandil and Shafee space or (characterized fuzzy $R_{2\frac{1}{2}}$ KE-space, for short) if for every $x_\alpha \in S(X)$ and $\mu \in \varphi_{1,2}CF(X)$ such that $x_\alpha \overline{q} \mu$, there exists an $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_1)$ such that $f(y)(0+) \leq \mu'(y)$ and $f(y)(1-) \geq x_\alpha(y)$ are hold for all $y \in X$ and $\alpha \in L$. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to characterized quasi fuzzy T_1-space or (characterized QFT_1-space, for short) if for all $x, y \in X$ such that $x \neq y$ we have $x_\alpha \overline{q} \varphi_{1,2}.\text{cl} y_\beta$ and $\varphi_{1,2}.\text{cl} x_\alpha \overline{q} y_\beta$ for all $\alpha, \beta \in L$. As easily seen that every characterized QFT_1-space is characterized fuzzy T_1-space. The characterized fuzzy space $(X, \varphi_{1,2}.\text{int})$ is said to be characterized fuzzy $T_{3\frac{1}{2}}$ Kandil El-Shafee-space or (characterized fuzzy $T_{3\frac{1}{2}}$ KE-space, for short) if and only if it is characterized fuzzy $T_{3\frac{1}{2}}$ KE and characterized QFT_1-spaces. Obviously, every characterized fuzzy $T_{3\frac{1}{2}}$ KE-space is characterized fuzzy $T_{3\frac{1}{2}}$ K-space. In the classical case of $L=\{0, 1\}$, $\varphi_1=\text{int}_\tau$, $\psi_1=\text{int}_I$, $\varphi_2 = 1_{L^X}$ and $\psi_2 = 1_{L^I}$, the $\varphi_{1,2}\psi_{1,2}$-fuzzy continuity of f is up to an identification the usual fuzzy continuity of f. Hence, the notions of characterized fuzzy $R_{2\frac{1}{2}}$ KE-spaces and of characterized fuzzy $T_{3\frac{1}{2}}$ KE-spaces are coincide with the notion of completely fuzzy regular spaces and the notion fuzzy fuzzy $T_{3\frac{1}{2}}$-spaces presented by Kandil and Shafee [12], respectively. Another special choices for the operations φ_1, φ_2, ψ_1 and ψ_2 are obtained in Table 1.

In the following proposition we show that the characterized fuzzy $R_{2\frac{1}{2}}$-spaces which are presented [9] are more general than the characterized fuzzy $R_{2\frac{1}{2}}$ KE-spaces.

Proposition 5.3

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$ Then every characterized fuzzy $R_{2\frac{1}{2}}$ KE-space $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space.

Proof: Let $(X, \varphi_{1,2}.\text{int})$ is a characterized fuzzy $R_{2\frac{1}{2}}$ KE-space, $x \in X$ and $F \in \varphi_{1,2}C(X)$ such that $x \notin F$. Then, $\chi_{F'} \in \varphi_{1,2}OF(X)$ and $\chi_{F'}(x)=1$, therefore $x_1 \overline{q} \chi_F$. Because of $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$ KE-space, then there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}) \to (IL, \psi_{1,2}.\text{int}_I)$ such that $f(y)(0+) \leq \chi_{F'}(y)$ and $f(y)(1-) \geq x_1(y)$ are hold for all $y \in X$. In case of $y \in F$, we have $0 \leq f(y)(1-) \leq f(y)(0+) \leq 0$, that is, $f(y)(s)=0$ for all $s>0$and therefore $f(y)=\overline{0}$ for all $y \in F$. In case of y=x, we have $1 \leq f(x)(1-) \leq f(x)(0+) \leq 1$ holds and then $f(x)(s)=1$ for all $s < 1$, therefore $f(x)=\overline{1}$. Hence, there exists a $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}) \to (I_L, \psi_{1,2}.\text{int}_I)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y \in F$. Consequently, $(X, \varphi_{1,2}.\text{int})$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9].

Corollary 5.3

Let (X, τ) be an fuzzy topological space and $\varphi_1, \varphi_2 \in O_{(L^X,\tau)}$. Then every characterized fuzzy $T_{3\frac{1}{2}}$ KE-space is characterized fuzzy $T_{3\frac{1}{2}}$-space.

Proof: Follows immediately from Proposition 5.3 and the fact that every characterizedQFT$_1$-space is characterized fuzzy T_1-space.

The following example shows that the inverse of Proposition 5.3 and Corollary 5.3 are not true in general.

Example 5.3.

Consider the characterized fuzzy space $(X, \varphi_{1,2}.\text{int}_\tau)$ which is defined in Example 5.1, then as shown in Example 5.1, $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $R_{2\frac{1}{2}}$-space in sense [9] and characterized fuzzy T_1-space, therefore $(X, \varphi_{1,2}.\text{int}_\tau)$ is characterized fuzzy $T_{3\frac{1}{2}}$-space in sense [9].

Now, choose $x_{\frac{1}{2}} \in S(X)$ and $\mu = x_{\frac{1}{2}} \in \varphi_{1,2}CF(X)$ then $\mu' = x_{\frac{1}{2}} \vee y_1 \in \varphi_{1,2}OF(X)$ such that $x_{\frac{1}{2}}\overline{q}\mu$ .Hence, for any $\varphi_{1,2}\psi_{1,2}$-fuzzy continuous mapping f: $(X, \varphi_{1,2}.\text{int}_\tau) \to (I_L, \psi_{1,2}.\text{int}_I)$ such that $f(x)=\overline{1}$ and $f(y)=\overline{0}$ for all $y \neq x$, we get $x_{\frac{1}{2}}(z) \leq f(z)(1-) = \underset{\tau<1}{\wedge} f(z)(t)$ holds for all $z \in X$. But $\mu'(z) = \left(x_{\frac{1}{2}} \vee y_1\right)(z) \geq f(z)(0+) \underset{\sigma>0}{\wedge} f(z)(s)$ holds only for $z=y$ and it is not fulfilled for $z=x$. Consequently, $(X, \varphi_{1,2}.\text{int}_\tau)$ is not characterized fuzzy $R_{2\frac{1}{2}}$ KE-space and therefore it is not characterized fuzzy $T_{3\frac{1}{2}}$ KE-space.

Conclusion

In this paper, basic notions related to the characterized fuzzy $R_{2\frac{1}{2}}$ and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces which are presented [9] are introduced and studied. These notions are named metrizable characterized fuzzy spaces, initial and final characterized fuzzy spaces, some finer characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces. The metrizable characterized fuzzy space is introduced as a generalization of the weaker and stronger forms of the fuzzy metric space introduced by Gahler and Gahler [3]. For every stratified fuzzy topological space generated canonically by an fuzzy metric we proved that, the metrizable characterized fuzzy space is characterized fuzzy T_4-space in sense of Abd-Allah [10] and therefore, it is characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space. The induced characterized fuzzy space is characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space if and only if the related ordinary topological space is $\varphi_{1,2}$ $R_{2\frac{1}{2}}$-space and $\varphi_{1,2}$ $T_{3\frac{1}{2}}$-space, respectively. Hence, the notions of characterized fuzzy $R_{2\frac{1}{2}}$ and of characterized fuzzy $T_{3\frac{1}{2}}$ are good extension in sense of Lowen [11]. Moreover, the α-level characterized space and the initial characterized space are characterized -space and characterized $T_{3\frac{1}{2}}$-space if the related characterized fuzzy space is characterized fuzzy $R_{2\frac{1}{2}}$-space and characterized fuzzy $T_{3\frac{1}{2}}$-space, respectively. We shown that the finer characterized fuzzy space of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space is also characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-space, respectively. The categories of all characterized fuzzy $R_{2\frac{1}{2}}$ and of all characterized fuzzy $T_{3\frac{1}{2}}$-spaces will be denoted by CFR-Space and CRF-Tych and they are concrete categories. These categories are full subcategories of the category CF-Space of all characterized fuzzy spaces, which are topological over the category SET of all subsets and hence all the initial and final lifts exist uniquely in CFR-Space and CRF-Tych, respectively. That is, all the initial and final characterized fuzzy $R_{2\frac{1}{2}}$-spaces exist in CFR-Space and also all the initial and final characterized fuzzy $T_{3\frac{1}{2}}$-spaces exist in CRF-Tych. We shown that the initial and final characterized fuzzy spaces of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of characterized fuzzy $T_{3\frac{1}{2}}$-space are characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. As special cases, the characterized fuzzy subspace, characterized fuzzy product space, characterized fuzzy quotient space and characterized fuzzy sum space of a characterized fuzzy $R_{2\frac{1}{2}}$-space and of a characterized fuzzy $T_{3\frac{1}{2}}$-space are also characterized fuzzy $R_{2\frac{1}{2}}$ and characterized fuzzy $T_{3\frac{1}{2}}$-spaces, respectively. Finally, we introduced and studied three finer characterized fuzzy $R_{2\frac{1}{2}}$ and three finer characterized fuzzy $T_{3\frac{1}{2}}$ L-spaces as a generalization of the weaker and stronger forms of the completely regular and the fuzzy $T_{3\frac{1}{2}}$-spaces introduced [28,12,13]. These fuzzy spaces are named characterized fuzzy $R_{2\frac{1}{2}}$ H, characterized fuzzy $R_{2\frac{1}{2}}$ K, characterized fuzzy $R_{2\frac{1}{2}}$ KE, characterized fuzzy $T_{3\frac{1}{2}}$ H, characterized fuzzy $T_{3\frac{1}{2}}$ K and characterized fuzzy $T_{3\frac{1}{2}}$ KE-spaces. The relations between characterized fuzzy $R_{2\frac{1}{2}}$ H, characterized fuzzy $R_{2\frac{1}{2}}$ K, characterized fuzzy $R_{2\frac{1}{2}}$ KE-spaces and the characterized fuzzy $R_{2\frac{1}{2}}$-space which are presented [9] are introduced. More generally, the relations between characterized fuzzy $T_{3\frac{1}{2}}$ H, characterized fuzzy $T_{3\frac{1}{2}}$ K, characterized fuzzy $T_{3\frac{1}{2}}$ KE-spaces and the characterized fuzzy $T_{3\frac{1}{2}}$-spaces are also introduced. Meany special cases from these finer characterized fuzzy $R_{2\frac{1}{2}}$ and finer characterized fuzzy $T_{3\frac{1}{2}}$-spaces are listed in Table 1.

	Operations	Char.fuzzy $R_{2\frac{1}{2}}$ H-space	Char.fuzzy $R_{2\frac{1}{2}}$ K-space	Char.fuzzy $R_{2\frac{1}{2}}$ KE-space	Char.fuzzy $T_{3\frac{1}{2}}$ H-space	Char.fuzzy $T_{3\frac{1}{2}}$ K-space	Char.fuzzy $T_{3\frac{1}{2}}$ KE-space
1	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz. $R_{2\frac{1}{2}}$ H space [10,21]	Fuz. $R_{2\frac{1}{2}}$ K space [10,25]	Fuz. $R_{2\frac{1}{2}}$ KE space [10,23]	Fuz. $T_{3\frac{1}{2}}$ H space [10,21]	Fuz. $T_{3\frac{1}{2}}$ K space [10,25]	Fuz. $T_{3\frac{1}{2}}$ KE space [10,23]
2	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{cl}_\tau$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz.θ $R_{2\frac{1}{2}}$ H-space	Fuz.θ $R_{2\frac{1}{2}}$ K-space	Fuz.θ $R_{2\frac{1}{2}}$ KE-space	Fuz.θ $T_{3\frac{1}{2}}$ H-space	Fuz.θ $T_{3\frac{1}{2}}$ K-space	Fuz.θ $T_{3\frac{1}{2}}$ KE-space
3	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{int}_\tau\circ\mathrm{cl}_\tau$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{int}_I\circ\mathrm{cl}_I$	Fuz.δ $R_{2\frac{1}{2}}$ H-space	Fuz.δ $R_{2\frac{1}{2}}$ K-space	Fuz.δ $R_{2\frac{1}{2}}$ KE-space	Fuz.δ $T_{3\frac{1}{2}}$ H-space	Fuz.δ $T_{3\frac{1}{2}}$ K-space	Fuz.δ $T_{3\frac{1}{2}}$ KE-space
4	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz.W $R_{2\frac{1}{2}}$ H-space	Fuz.W $R_{2\frac{1}{2}}$ K-space	Fuz.W $R_{2\frac{1}{2}}$ KE-space	Fuz.W $T_{3\frac{1}{2}}$ H-space	Fuz.W $T_{3\frac{1}{2}}$ K-space	Fuz.W $T_{3\frac{1}{2}}$ KE-space
5	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{cl}_\tau$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.S.θ $R_{2\frac{1}{2}}$ H-space	Fuz.S.θ K-space	Fuz.S.θ $R_{2\frac{1}{2}}$ KE-space	Fuz.S.θ $T_{3\frac{1}{2}}$ H-space	Fuz.S.θ $T_{3\frac{1}{2}}$ K-space	Fuz.S.θ $T_{3\frac{1}{2}}$ KE-space
6	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{int}_I\circ\mathrm{cl}_I$	Fuz.A $R_{2\frac{1}{2}}$ H-space	Fuz.A $R_{2\frac{1}{2}}$ K-space	Fuz.A $R_{2\frac{1}{2}}$ KE-space	Fuz.A $T_{3\frac{1}{2}}$ H-space	Fuz.A $T_{3\frac{1}{2}}$ K-space	Fuz.A $T_{3\frac{1}{2}}$ KE-space
7	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{cl}_\tau$ ψ_1 $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{int}_I\circ\mathrm{cl}_I$	Fuz.A.S.θ $R_{2\frac{1}{2}}$ H-space	Fuz.A.S.θ $R_{2\frac{1}{2}}$ K-space	Fuz.A.S.θ $R_{2\frac{1}{2}}$ KE-space	Fuz.A.S.θ $T_{3\frac{1}{2}}$ H-space	Fuz.A.S.θ $T_{3\frac{1}{2}}$ K-space	Fuz.A.S.θ $T_{3\frac{1}{2}}$ KE-space
8	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{int}_\tau\circ\mathrm{cl}_\tau$	Fuz. super $R_{2\frac{1}{2}}$ H-space	Fuz. super $R_{2\frac{1}{2}}$ K-space	Fuz. super $R_{2\frac{1}{2}}$ KE-space	Fuz. super $T_{3\frac{1}{2}}$ H-space	Fuz. super $T_{3\frac{1}{2}}$ K-space	Fuz. super $T_{3\frac{1}{2}}$ KE-space
	$\varphi_1=\mathrm{int}_\tau$, $\varphi_2=\mathrm{int}_\tau\circ\mathrm{cl}_\tau$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz.W.θ $R_{2\frac{1}{2}}$ H-space	Fuz.W.θ $R_{2\frac{1}{2}}$ K-space	Fuz.W.θ $R_{2\frac{1}{2}}$ KE-space	Fuz.W.θ $T_{3\frac{1}{2}}$ H-space	Fuz.W.θ $T_{3\frac{1}{2}}$ K-space	Fuz.W.θ $T_{3\frac{1}{2}}$ KE-space
10	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.semi $R_{2\frac{1}{2}}$ H-space	Fuz.semi $R_{2\frac{1}{2}}$ K-space	Fuz.semi $R_{2\frac{1}{2}}$ KE-space	Fuz.semi $T_{3\frac{1}{2}}$ H-space	Fuz.semi $T_{3\frac{1}{2}}$ K-space	Fuz.semi KE-space
11	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{cl}_I\circ\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.irr. $R_{2\frac{1}{2}}$ H-space	Fuz.irr. $R_{2\frac{1}{2}}$ K-space	Fuz.irr. $R_{2\frac{1}{2}}$ KE-space	Fuz.irr. $T_{3\frac{1}{2}}$ H-space	Fuz.irr. $T_{3\frac{1}{2}}$ K-space	Fuz.irr. $T_{3\frac{1}{2}}$ KE-space
12	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{cl}_I\circ\mathrm{int}_I$, $\psi_2=\mathrm{Scl}_I$	Fuz. semi -irr. $R_{2\frac{1}{2}}$ H-space	Fuz. semi –irr. $R_{2\frac{1}{2}}$ K-space	Fuz. semi -irr. $R_{2\frac{1}{2}}$ KE-space	Fuz. semi-irr. $T_{3\frac{1}{2}}$ H-space	Fuz. semi -irr. $T_{3\frac{1}{2}}$ K-space	Fuz. semi -irr. $T_{3\frac{1}{2}}$ KE-space
13	$\varphi_1=\mathrm{cl}_I\circ\mathrm{int}_I$, $\varphi_2=\mathrm{Scl}_I$ $\psi_1=\mathrm{cl}_I\circ\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.S-irr. $R_{2\frac{1}{2}}$ H-space	Fuz.S-irr. $R_{2\frac{1}{2}}$ K-space	Fuz.S-irr. $R_{2\frac{1}{2}}$ KE-space	Fuz.S-irr. $T_{3\frac{1}{2}}$ H-space	Fuz.S-irr. $T_{3\frac{1}{2}}$ K-space	Fuz.S-irr. $T_{3\frac{1}{2}}$ KE-space
14	$\varphi_1=\mathrm{int}_\tau\circ\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.λ $R_{2\frac{1}{2}}$ H-space	Fuz.λ $R_{2\frac{1}{2}}$ K-space	Fuz.λ $R_{2\frac{1}{2}}$ KE-space	Fuz.λ $T_{3\frac{1}{2}}$ H-space	Fuz.λ $T_{3\frac{1}{2}}$ K-space	Fuz.λ $T_{3\frac{1}{2}}$ KE-space
15	$\varphi_1=\mathrm{int}_\tau\circ\mathrm{cl}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.pre $R_{2\frac{1}{2}}$ H-space	Fuz.pre $R_{2\frac{1}{2}}$ K-space	Fuz.pre $R_{2\frac{1}{2}}$ KE-space	Fuz.pre $T_{3\frac{1}{2}}$ H-space	Fuz.pre $T_{3\frac{1}{2}}$ K-space	Fuz.pre $T_{3\frac{1}{2}}$ KE-space
16	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau\circ\mathrm{cl}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=1_L I$	Fuz.β $R_{2\frac{1}{2}}$ H-space	Fuz.β $R_{2\frac{1}{2}}$ K-space	Fuz.β $R_{2\frac{1}{2}}$ KE-space	Fuz.β $T_{3\frac{1}{2}}$ H-space	Fuz.β $T_{3\frac{1}{2}}$ K-space	Fuz.β $T_{3\frac{1}{2}}$ KE-space
17	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz. W semi $R_{2\frac{1}{2}}$ H-space	Fuz. W semi $R_{2\frac{1}{2}}$ K-space	Fuz. W semi $R_{2\frac{1}{2}}$ KE-space	Fuz. W semi H-space	Fuz. W semi $T_{3\frac{1}{2}}$ K-space	Fuz. W semi $T_{3\frac{1}{2}}$ KE-space
18	$\varphi_1=\mathrm{int}_\tau\circ\mathrm{cl}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz. W pre $R_{2\frac{1}{2}}$ H-space	Fuz. W pre $R_{2\frac{1}{2}}$ K-space	Fuz. W pre $R_{2\frac{1}{2}}$ KE-space	Fuz. W pre $T_{3\frac{1}{2}}$ KE-space	Fuz. W pre $T_{3\frac{1}{2}}$ K-space	Fuz. W pre $T_{3\frac{1}{2}}$ KE-space
19	$\varphi_1=\mathrm{int}_\tau\circ\mathrm{cl}_\tau\circ\mathrm{int}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz.W λ $R_{2\frac{1}{2}}$ H-space	Fuz.W λ $R_{2\frac{1}{2}}$ K-space	Fuz.W λ $R_{2\frac{1}{2}}$ KE-space	Fuz.W λ $T_{3\frac{1}{2}}$ H-space	Fuz.W λ $T_{3\frac{1}{2}}$ K-space	Fuz.W λ $T_{3\frac{1}{2}}$ KE-space
20	$\varphi_1=\mathrm{cl}_\tau\circ\mathrm{int}_\tau\circ\mathrm{cl}_\tau$, $\varphi_2=1_L X$ $\psi_1=\mathrm{int}_I$, $\psi_2=\mathrm{cl}_I$	Fuz. W β v H-space	Fuz. W β $R_{2\frac{1}{2}}$ K-space	Fuz. W β $R_{2\frac{1}{2}}$ KE-space	Fuz. W β $T_{3\frac{1}{2}}$ H-space	Fuz. W β $T_{3\frac{1}{2}}$ K-space	Fuz. W β $T_{3\frac{1}{2}}$ KE-space

21	$\varphi 1=cl\tau \circ int\tau$, $\varphi 2=1_L X$ $\psi_1=int_l, \psi_2=int_l \circ cl_l$	Fuz. A semi $R_{2\frac{1}{2}}$ H-space	Fuz. A semi $R_{2\frac{1}{2}}$ K-space	Fuz. A semi $R_{2\frac{1}{2}}$ KE-space	Fuz. A semi $T_{3\frac{1}{2}}$ H-space	Fuz. A semi $T_{3\frac{1}{2}}$ K-space	Fuz. A semi $T_{3\frac{1}{2}}$ KE-space
22	$\varphi 1=int\tau \circ cl\tau \circ int\tau$, $\varphi 2=1_L X$ $\psi_1=int_l$, $\psi_2=int_l \circ cl_l$	Fuz.A λ $R_{2\frac{1}{2}}$ H-space	Fuz.A λ $R_{2\frac{1}{2}}$ K-space	Fuz.A λ $R_{2\frac{1}{2}}$ KE-space	Fuz.A λ $T_{3\frac{1}{2}}$ H-space	Fuz.A λ $T_{3\frac{1}{2}}$ K-space	Fuz.A λ $T_{3\frac{1}{2}}$ KE-space
23	$\varphi 1=cl\tau \circ int\tau \circ cl\tau$, $\varphi 2=1_L X$ $\psi_1=int_l, \psi_2=int_l \circ cl_l$	Fuz.A β $R_{2\frac{1}{2}}$ H-space	Fuz.A β $R_{2\frac{1}{2}}$ K-space	Fuz.A β $R_{2\frac{1}{2}}$ KE-space	Fuz.A β $T_{3\frac{1}{2}}$ H-space	Fuz.A β $T_{3\frac{1}{2}}$ K-space	Fuz.A β $T_{3\frac{1}{2}}$ KE-space
24	$\varphi 1=cl\tau \circ int\tau \circ cl\tau$, $\varphi 2=1_L X$ $\psi_1=int_l, \psi_2=int_l \circ cl_l$	Fuz.θ semi $R_{2\frac{1}{2}}$ H-space	Fuz.θ semi $R_{2\frac{1}{2}}$ K-space	Fuz.θ semi $R_{2\frac{1}{2}}$ KE-space	Fuz.θ semi. T $T_{3\frac{1}{2}}$ H-space	Fuz.θ semi K-space	Fuz.θ semi $R_{2\frac{1}{2}}$ KE-space
25	$\varphi 1=cl\tau \circ int\tau$, $\varphi 2=1_L X$ $\psi_1=int_l, \psi_2=Scl_l$	Fuz. semi. W. $R_{2\frac{1}{2}}$ H-space	Fuz. semi. W. K-space	Fuz. semi. W. $R_{2\frac{1}{2}}$ KE-space	Fuz. semi. W. $T_{3\frac{1}{2}}$ H-space	Fuz. semi. W. $T_{3\frac{1}{2}}$ K-space	Fuz. semi. W. $T_{3\frac{1}{2}}$ KE-space
26	$\varphi 1=int\tau \circ cl\tau \circ int\tau$, $\varphi 2=1_L X$ $\psi 1=intI \circ clI \circ intI$, $\psi 2=1_L I$	Fuz.λ. irr. $R_{2\frac{1}{2}}$ H-space	Fuz.λ. irr. $R_{2\frac{1}{2}}$ K-space	Fuz.λ. irr. $R_{2\frac{1}{2}}$ KE-space	Fuz.λ. irr. $T_{3\frac{1}{2}}$ H-space	Fuz.λ. irr. $T_{3\frac{1}{2}}$ K-space	Fuz.λ. irr. $T_{3\frac{1}{2}}$ KE-space
27	$\varphi 1=int\tau \circ cl\tau$, $\varphi 2=1_L X$ $\psi 1=intI \circ clI$, $\psi 2=1_L I$	Fuz. pre-irr. R $R_{2\frac{1}{2}}$ H-space	Fuz. pre-irr. $R_{2\frac{1}{2}}$ K-space	Fuz. pre-irr. $R_{2\frac{1}{2}}$ KE-space	Fuz. pre-irr. $T_{3\frac{1}{2}}$ H-space	Fuz. pre-irr. $T_{3\frac{1}{2}}$ K-space	Fuz. pre-irr. $T_{3\frac{1}{2}}$ KE-space
28	$\varphi 1=cl\tau \circ int\tau \circ cl\tau$, $\varphi 2=1_L X$ $\psi 1=clI \circ intI\ clI$, $\psi 2=1_L I$	Fuz.β. irr. $R_{2\frac{1}{2}}$ H-space	Fuz.β. irr. $R_{2\frac{1}{2}}$ K-space	Fuz.β. irr. $R_{2\frac{1}{2}}$ KE-space	Fuz.β. irr. $T_{3\frac{1}{2}}$ H-space	Fuz.β. irr. $T_{3\frac{1}{2}}$ K-space	Fuz.β. irr. $T_{3\frac{1}{2}}$ KE-space
29	$\varphi 1=int\tau$, $\varphi 2=1_L X$ $\psi_1=int_l \circ cl_l$, $\psi_2=cl_l$	Fuz.(θ, S) $R_{2\frac{1}{2}}$ H-space	Fuz.(θ, S) $R_{2\frac{1}{2}}$ K-space	Fuz.(θ, S) $R_{2\frac{1}{2}}$ KE-space	Fuz.(θ, S) $T_{3\frac{1}{2}}$ H-space	Fuz.(θ, S) $T_{3\frac{1}{2}}$ K-space	Fuz.(θ, S) $T_{3\frac{1}{2}}$ KE-space

Table 1: Some special classes of Char.fuzzy $R_{2\frac{1}{2}}$ H-spaces, Char.fuzzy $R_{2\frac{1}{2}}$ K-spaces, Char.fuzzy $R_{2\frac{1}{2}}$ KE-spaces, Char.fuzzy $T_{3\frac{1}{2}}$ H-spaces, Char. fuzzy $T_{3\frac{1}{2}}$ K-spaces, Char.fuzzy $T_{3\frac{1}{2}}$ KE-spaces.

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