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Interpolation-Collocation Method of Solution for Solving Poisson Equation

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Abstract

In this paper, we consider the system of algebraic equations arising from the discretization of elliptic partial differential equation with respect to x and y axes. To compute the solution of the resulting equations we use the new method to solve various elliptic equations. We study the numerical accuracy of the method. The numerical results have shown that the method provided exact result depending on the particular equation on which the scheme is applied.

Keywords: Continuous method; Lines; Multistep collocation; Elliptic; Taylor's polynomial

Introduction

A finite difference scheme with continuous coefficients for the approximate solution of elliptic partial differential equation of the form

$$\nabla^2 U(x,y) = \frac{\partial^2 U}{\partial x^2}(x,y) + \frac{\partial^2 U(x,y)}{\partial y^2} = f(x,y) \qquad for\ (x,y)\in R \quad (1.0)$$

And $U(x,y)=G(x,y)$ for $(x,y)\in S$ where $R=\{(x,y){:}a<x<b,\ c<y<d$ and S enotes the boundary of R is proposed. For this discussion we assume that both f and g are continuous on their domains so that a unique solution to equation (1.0) is ensured. The method to be used is the adaptation of the canonical polynomials $Q_r(x,y)$ [1-17]. Many problems in engineering and sciences cannot be formulated in terms of partial differential equations. The vast majority of equations encountered in practice cannot, however, be solved analytically, and recourse must necessarily be made to numerical methods.

Our Present Method

The basic method seeks an approximation of the form:

$$U(x,y) = \sum_{r=0}^{p-2} a_r Q_r(x,y), \quad x\in\left[x_{m,}\, x_{m+l}\right] \qquad r=0,1\ldots,p-2 \qquad (2.0)$$

Such that $0=x_0<\ldots<x_i\ldots<x_N=X$ The basis function, $Q_r(x,y)=x^r y^r, r=0,1,\ldots, p\text{-}2$ are assumed known, a_r are constants to be determined and $p \le l+s$, where s is the number of collocation points. The equality holds if the number of interpolation points used is equal to l. There will be flexibility in the choice of the basis function $Q_r(x,y)$ as may be desired for specific application. For this work, we consider the Taylor's polynomial $Q_r(x,y)=x^r y^r$. The interpolation values $U_{m,n},\ldots, U_{m+l-1},_n$ are assumed to have been determined from previous steps, while the method seeks to obtain $U_{m+l-1},_n$ [8-27].

We apply the above interpolation conditions on eqn. (2.0) to obtain:

$$a_0Q_0\left(x_{m+g},y_n\right)+\ldots+a_{p-2}Q_{p-2}\left(x_{m+g},y_n\right)=U(x_{m+g},y_n) \qquad g=-1(1)l-2 \quad (2.1)$$

We can write eqn. (2.1) as a simple matrix equation as,

$$\begin{bmatrix} Q_0(x_{m-1},y_n) & \ldots & Q_{p-2}(x_{m-1},y_n) \\ Q_0(x_m,y_n) & \ldots & Q_{p\ 2}(x_m,y_n) \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots \\ Q_0(x_{m+l-2},y_n) & \ldots & Q_{p-2}(x_{m+l-2},y_n) \end{bmatrix} \begin{bmatrix} \\ \\ \ldots \\ \ldots \\ \ldots \\ \\ \end{bmatrix} = \begin{bmatrix} U(x_m\ ,y_n) \\ U(x_m\ y_n) \\ \ldots \\ \ldots \\ \ldots \\ U(x_{m+l-}\ ,y_n) \end{bmatrix} \qquad (2.2)$$

Using three interpolation points and one collocation point, eqn. (2.1) becomes,

$$a_0Q_0\left(x_{m+g},y_n\right)+a_1Q_1\left(x_{m+g},y_n\right)+a_2Q_2\left(x_{m+g},y_n\right)=U_{m+g},_n \quad (2.3)$$

Putting the values of g in eqn. (2.3) and writing it as a matrix we obtain,

$$\begin{bmatrix} Q_0(x_{m-1},y_n) & Q_1(x_{m-1},y_n) & Q_2(x_{m-1},y_n) \\ Q_0(x_m,y_n) & Q_1(x_m,y_n) & Q_2(x_m,y_n) \\ Q_0(x_{m+1},y_n) & Q_1(x_{m+1},y_n) & Q_2(x_{m+1},y_n) \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_3 \end{bmatrix} = \begin{bmatrix} U_{m-1,n} \\ U_{m.n} \\ U_{m+1,n} \end{bmatrix} \qquad (2.4)$$

From eqn. (2.4) we obtain

$$\begin{bmatrix} 1 & x_{m-1}y_n & x^2{}_{m-1}y^2{}_n \\ 1 & x_m y_n & x^2{}_m y^2{}_n \\ 1 & x_{m+1}y_n & x^2{}_{m+1}y^2{}_n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} U_{m-1,n} \\ U_{m.n} \\ U_{m+1,n} \end{bmatrix} \qquad (2.5)$$

We solve eqn. (2.5) to obtain the value of a_2 as:

$$a_2 = \frac{U_{m+1,n}+U_{m-1,n}-2U_{m,n}}{2h^2y^2{}_n},$$

Using 3 interpolation points and 1 collocation point, implies that r. Putting the values of r in eqn. (2.0) we obtain

$$U(x,y) = a_0Q_0 + a_1Q_1 + a_2Q_2 \qquad (2.6)$$

By substitution of Q_0, Q_1 and Q_2 in eqn. (2.6) we obtain

$$U(x,y) = a_0 + a_1xy + a_2x^2y^2 \qquad (2.7)$$

Substituting the value of a_2 in eqn. (2.7) we have

$$U(x,y) = a_0 + a_1xy + x^2y^2\left(\frac{U_{m+1,n}+U_{m-1,n}-2U_{m,n}}{2h^2y^2{}_n}\right) \qquad (2.8)$$

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Taken the first and second derivatives of eqn. (2.8) with respect to x we have

$$U''(x,t) = y^2\left(\frac{U_{m+1,n}+U_{m-1,n}-2U_{m,n}}{h^2 y^2{}_n}\right) \quad (2.9)$$

We collocate eqn. (2.9) at $y=y_n$, we obtain:

$$U''(x,y) = \frac{U_{m+1,n}+U_{m-1,n}-2U_{m,n}}{h^2} \quad (2.10)$$

We interchange the roles of x and y in eqn. (2.0) and applying the same interpolation conditions we obtain,

$$a_0Q_0\left(x_m,y_{n+g}\right)+...+a_{p-2}Q_{p-2}\left(x_m,y_{n+g}\right)=U(x_m,y_{n+g}) \quad (2.11)$$

We can write eqn. (2.11) as a simple matrix equation as,

$$\begin{bmatrix} Q_0(x_m,y_{n-1}) & \cdots & Q_{p-2}(x_m,y_{n-1}) \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ Q_0(x_m,y_{n+l-2}) & \cdots & Q_{p-2}(x_m,y_{n+l-2}) \end{bmatrix}\begin{bmatrix} a_0 \\ \cdots \\ \cdots \\ \cdots \\ a_{p-2}\end{bmatrix}=\begin{bmatrix} U(x_m,y_{n-1}) \\ \cdots \\ \cdots \\ \cdots \\ U(x_m,y_{n+l-2})\end{bmatrix} \quad (2.12)$$

Using 3 interpolation and 1 collocation points, eqn. (2.11) becomes,

$$a_0Q_0\left(x_m,y_{n+g}\right)+a_1Q_1\left(x_m,y_{n+g}\right)+a_2Q_2\left(x_m,y_{n+g}\right)=U_{m,n+g} \qquad g=-1,0,1 \quad (2.13)$$

Putting the values of g in eqn. (2.13) and writing it as a matrix we obtain,

$$\begin{bmatrix} Q_0(x_m,y_{n-1}) & Q_1(x_m,y_{n-1}) & Q_2(x_m,y_{n-1}) \\ Q_0(x_m,y_n) & Q_1(x_m,y_n) & Q_2(x_m,y_n) \\ Q_0(x_m,y_{n+1}) & Q_1(x_m,y_{n+1}) & Q_2(x_m,y_{n+1}) \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_3\end{bmatrix}=\begin{bmatrix} U_{m,n-1} \\ U_{m.n} \\ U_{m,n+1}\end{bmatrix} \quad (2.14)$$

From eqn. (2.14) we obtain

$$\begin{bmatrix} 1 & x_my_{n-1} & x^2{}_my^2{}_{n-1} \\ 1 & x_my_n & x^2{}_my^2{}_n \\ 1 & x_my_{n+1} & x^2{}_my^2{}_{n+1} \end{bmatrix}\begin{bmatrix} a_0 \\ a_1 \\ a_2\end{bmatrix}=\begin{bmatrix} U_{m,n-1} \\ U_{m.n} \\ U_{m,n+1}\end{bmatrix} \quad (2.15)$$

We solve eqn. (2.15) to obtain the value of a_2 as,

$$a_2 = \frac{U_{m,n+1}+U_{m,n-1}-2U_{m,n}}{2h^2y^2{}_n},$$

Using 3 interpolation points and 1 collocation point implies that r. Putting the values of r in eqn. (2.0) we obtain,

$$U\left(x,y\right)=a_0Q_0+a_1Q_1+a_2Q_2 \quad (2.16)$$

By substitution of Q_0, Q_1 and Q_2 in eqn. (2.16) we obtain

$$U(x,y)=a_0+a_1xy+a_2x^2y^2 \quad (2.17)$$

Substituting the value of a_2 in eqn. (2.17) we have

$$U(x,y)=a_0+a_1xy+x^2y^2\left(\frac{U_{m,n+1}+U_{m,n-1}-2U_{m,n}}{2h^2y^2{}_n}\right) \quad (2.18)$$

Taken the first and second derivatives of eqn. (2.18) with respect to y we have

$$U''(x,y) = x^2\left(\frac{U_{m,n+1}+U_{m,n-1}-2U_{m,n}}{h^2x^2{}_n}\right) \quad (2.19)$$

We collocate eqn. (2.19) at $x=x_n$ to arrive at

$$U''(x,y) = \frac{U_{m,n+1}+U_{m,n-1}-2U_{m,n}}{h^2} \quad (2.20)$$

Substituting eqns. (2.10) and (2.20) in eqn. (1.0) we obtain a scheme that solves elliptic equation. To illustrate the method we use it to solve two test problems (3.1) and (3.2) respectively.

Specific Problem

Example 3.1

Use the scheme to approximate the solution of a problem of determining the steady-state heat in a thin metal plate in the shape of a square with dimensions 0.5 meters by 0.5 meters, which is held at 0° Celsius on two adjacent boundaries while the heat on the other boundaries increase linearly from 0° Celsius at one corner to 100° Celsius where these sides meet. If we replace the sides with zero boundary conditions along the x- and y –axes, the problem is expressed mathematically as:

$$\frac{\partial^2U}{\partial x^2}-\frac{\partial^2U}{\partial y^2}=0,$$

for (x,y) in theR={(x,y):0<x<.5, 0<y<.5 with the boundary conditions $U(0,y)=U(x,0)=0$ and $U(x,.5)=200x$, $U(.5,y)=200y$

The exact solution of the problem is $U(x,.y)=400xy$

Using mesh size of 0.125 on each axis, the method gives us the result as shown in Table 1.

Example 3.2

Use the scheme to approximate the solution to the Poisson's equation

$$\frac{\partial^2U}{\partial x^2}+\frac{\partial^2U}{yx^2}=xe^y \qquad 0<x<2, \quad 0<y<1$$

With the boundary conditions

$$U\left(0,y\right)=0, \quad U\left(2,y\right)=2e^y, \qquad 0\le y\le 1$$

$$U\left(x,0\right)=x, \quad U\left(x,1\right)=ex, \qquad 0\le x\le 2$$

The exact solution of the problem is $U(x,y)=xe^y$. Using a mesh size of 0.3333 on the axis x and 0.2000 on the y- axis we obtain the following result (Table 2).

Conclusion

A continuous interpolation collocation method is proposed for solving elliptic partial differential equations. To check the numerical method, it is applied to solve two (2) different test problems with known

i	j	x_i	Y_j	Our Method	Exact result
1	3	0.125	0.375	18.75	18.75
2	3	0.250	0.375	37.50	37.50
3	3	0.375	0.375	56.25	56.25
1	2	0.125	0.250	12.50	12.50
2	2	0.250	0.250	25.00	25.00
3	2	0.375	0.250	37.50	37.50
1	1	0.125	0.125	6.25	6.25
2	1	0.250	0.125	12.50	12.50
3	1	0.375	0.125	18.75	18.75

Table 1: Result of action of eqn. (2.21) on problem 3.1.

i	*j*	x_i	y_j	Our method	Exact result	Error
1	1	0.3333	0.2000	0.40726	0.40713	1.30×10^{-4}
1	2	0.3333	0.4000	0.49748	0.49727	2.08×10^{-4}
1	3	0.3333	0.6000	0.60760	0.60737	2.23×10^{-4}
1	4	0.3333	0.8000	0.74201	0.74185	1.60×10^{-4}
2	1	0.6667	0.2000	0.81452	0.81472	2.55×10^{-4}
2	2	0.6667	0.4000	0.99496	0.99455	4.08×10^{-4}
2	3	0.6667	0.6000	1.21520	1.21470	4.37×10^{-4}
2	4	0.6667	0.8000	1.48400	1.48370	3.15×10^{-4}
3	1	1.0000	0.2000	1.22180	1.22140	3.64×10^{-4}
3	2	1.0000	0.4000	1.49240	1.49180	5.80×10^{-4}
3	3	1.0000	0.6000	1.82270	1.82210	6.24×10^{-4}
3	4	1.0000	0.8000	2.22600	2.22550	4.51×10^{-4}
4	1	1.3333	0.2000	1.62900	1.62850	4.27×10^{-4}
4	2	1.3333	0.4000	1.98980	1.98910	6.79×10^{-4}
4	3	1.3333	0.6000	2.43020	2.42950	7.35×10^{-4}
4	4	1.3333	0.8000	2.96790	2.96740	5.40×10^{-4}
5	1	1.6670	0.2000	2.03600	2.03570	3.71×10^{-4}
5	2	1.6670	0.4000	2.48700	2.48640	5.84×10^{-4}
5	3	1.6670	0.6000	3.03750	3.03690	6.41×10^{-4}
5	4	1.6670	0.8000	3.70970	3.70920	4.89×10^{-4}

Table 2: Result of action of eqn. (2.21) on problem 3.2.

exact solutions. The scheme produced real values in test problem 1, while there is small deviation from the exact solutions in the result of the second test problem. The numerical results confirm the validity of the new numerical scheme and suggest that it is a viable numerical method which involves the reduction of PDE to a system of ODEs.

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