

## Invariant Tensor Product

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### Abstract

In this paper, we define invariant tensor product and study invariant tensor products associated with discrete series representations. Let  $G(V_1) \times G(V_2)$  be a pair of classical groups diagonally embedded in  $G(V_1 \oplus V_2)$ . Suppose that  $\dim V_1 < \dim V_2$ . Let  $\pi$  be a discrete series representation of  $G(V_1 \oplus V_2)$ . We prove that the functor  $\pi \otimes_{G(V_1)}^*$  maps unitary representations of  $G(V_1)$  to unitary representations of  $G(V_2)$ . Here we enlarge the definition of unitary representations by including the zero dimensional representation.

### Invariant Tensor Products

Various forms of invariant tensor products appeared in the literature implicitly, for example, in Schur's orthogonality for finite groups [1]. In many cases, they are employed to study the space  $\text{Hom}_G(\pi_1, \pi_2)$  where one of the representations  $\pi_1$  and  $\pi_2$  is irreducible. In this paper, we formulate the concept of invariant tensor product uniformly. We also study the invariant tensor functor associated with discrete series representations for classical groups. For motivations and applications [2-4].

#### Definition 1

Let  $G$  be a locally compact tomography group and  $dg$  be a left invariant Haar measure. Let  $(H_\pi)$  and  $(\pi_1, H_{\pi_1})$  be two unitary representations of  $G$ . Let  $V$  and  $V_1$  be two dense subspaces of  $H_\pi$  and  $H_{\pi_1}$ . Formally, define the averaging operator

$$\mathcal{L}: V \otimes V_1 \rightarrow (V \otimes V_1)_{\mathbb{R}}^*$$

as follows,  $\forall u, v \in V, u_1, v_1 \in V_1$ ,

$$\mathcal{L}(v \otimes v_1)(uu_1) = \int_G ((\pi \otimes \pi_1)(g)(v \otimes v_1), (uu_1)) dg \quad (1)$$

$$= \int_G (\pi(g)v, u)(\pi_1(g)v_1, u_1) dg. \quad (2)$$

Suppose that  $\mathcal{L}$  is well-defined. The image of  $\mathcal{L}$  will be called the invariant tensor product. It will be denoted by  $V \otimes_G V_1$ . Whenever we use the notation  $V \otimes_G V_1$ , we assume  $V \otimes_G V_1$  is well-defined, that is, the integral (1) converges for all  $u, v \in V, u_1, v_1 \in V_1$ . Denote  $\mathcal{L}(v \otimes v_1)$  by  $v \otimes_G v_1$ . Define

$$(v \otimes_G v_1, u \otimes_G u_1)_G = \int_G (\pi(g)v, u)(\pi_1(g)v_1, u_1) dg.$$

For any unitary representation  $(\pi, \mathcal{H})$  of  $G$ , let  $(\pi^c, \mathcal{H}^c)$  be the same unitary representation of  $G$  equipped with the conjugate linear multiplication. If  $V$  is a subspace of  $\mathcal{H}$ , let  $V^c$  be the corresponding subspace of  $\mathcal{H}^c$ .

#### Lemma 1.1

Let  $G$  be a unimodular group. Suppose that  $V \otimes_G V_1$  is well-defined. Then the form  $(\cdot, \cdot)_G$  is a well-defined Hermitian form on  $V \otimes_G V_1$ .

The main result proved in this paper is as follows.

**Theorem 1.1:** Let  $G(m+n)$  be a classical group of type I with  $m > n$ . Let  $(G(n), G(m))$  be diagonally embedded in  $G$  (see Def. 2). Suppose that  $(\pi, H_\pi)$  is a discrete series representation of  $G(m+n)$  and  $(\pi_1, H_{\pi_1})$  is a unitary representation of  $G(n)$ . Let  $\mathcal{H}_\pi^\infty$  be the space of smooth vectors in  $H_\pi$ . Then  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  is well-defined. Suppose that  $H_\pi^\infty \otimes_{G(n)} \mathcal{H}_1 \neq 0$ . Then  $(\cdot, \cdot)_{G(n)}$  is positive definite. Furthermore,  $(H_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\cdot, \cdot)_{G(n)})$  completes to a unitary representation of  $G(m)$ .

### Example: $\pi_1$ Irreducible

#### Example I

Let  $G$  be a compact group. Let  $(\pi, H_\pi)$  and  $(\pi_1, H_{\pi_1})$  be two unitary representations of  $G$ . Then  $H_\pi \otimes_G H_{\pi_1}$  is always well-defined. Suppose that  $\pi_1$  is irreducible. Then the dimension of  $H_\pi \otimes_G H_{\pi_1}$  is the multiplicity of  $\pi_1^*$  occurring in  $H_\pi$ .

#### Example II

Let  $G$  be a real reductive group. Let  $\pi$  and  $\pi_1$  be two discrete series representations. Then  $H_\pi \otimes_G H_{\pi_1}$  is always well-defined. It is one dimensional if and only if  $\pi_1 \cong \pi^*$ . Otherwise, it is zero dimensional.

**Theorem 2.1:** Let  $\pi_1$  be an irreducible unitary representation of  $G$ . Suppose that  $V_1$  and  $V$  are both closed under the action of  $G$ . Suppose that  $V_1 \otimes_G V$  is well-defined. Then  $\mathcal{L}$  induces an injection from  $V_1 \otimes_G V$  to  $\text{Hom}_G(V^c, V_1^h)$ , the space of  $G$ -equivariant homomorphisms from  $V^c$  to the Hermitian dual  $V_1^h$ .

**Proof:** For each  $v_1 \in V_1, v, u \in V$ , define  $\mathcal{L}(v_1 \otimes v)(u) \in V_1^h$  as follows:

$$\forall u_1 \in V_1, \mathcal{L}(v_1 \otimes v)(u)(u_1) = (v_1 \otimes v, u_1 \otimes u)_G.$$

We have for every  $\lambda \in \mathbb{C}$ ,

$$\mathcal{L}(v_1 v)(u)(u_1) = \lambda \mathcal{L}(v_1 \otimes v)(u)(u_1);$$

$$(v_1 \otimes \lambda v)(u)(u_1) = \mathcal{L}(v_1 \otimes v)(u)(u_1);$$

$$\mathcal{L}(v_1 \otimes v)(\lambda u)(u_1) = \bar{\lambda} \mathcal{L}(v_1 \otimes v)(u)(u_1);$$

$$\mathcal{L}(v_1 \otimes v)(u)(\lambda u_1) = \bar{\lambda} \mathcal{L}(v_1 \otimes v)(u)(u_1).$$

We see that  $\mathcal{L}(v_1 \otimes v)(u)$  is in the Hermitian dual of  $V_1$ . In addition,  $\mathcal{L}(v_1 \otimes v)$  is  $G$ -equivariant:

$$\mathcal{L}(v_1 \otimes v)(\pi(g)u)(u_1) = \int_{hG} (\pi(h)v_1, u_1)(\pi(h)v, \pi(g)u) dh \quad (3)$$

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$$= \int_{hG} (\langle h, v_1, u_1 \rangle (\pi(g^{-1}h)v, u) dh) \tag{4}$$

$$= \int_{hG} (\langle gh, v_1, u_1 \rangle (\pi(h)v, u) dh) \tag{5}$$

$$= \int_{hG} (\langle h, v_1, \pi(g^{-1})u_1 \rangle (\pi(h)v, u) dh) \tag{6}$$

$$= \mathcal{L}(v_1 \otimes v)(u)(\langle g^{-1} \rangle u_1) \tag{7}$$

$$= [\pi_1^h(g) \mathcal{L}(v_1 \otimes v)(u)](u_1). \tag{8}$$

Here  $dh$  is a left invariant measure if  $G$  is not unimodular. Now it is easy to see that  $\mathcal{L}(v_1 \otimes v)(u) = 0$  for every  $u$  if and only if  $v_1 \otimes v = 0$ . So

$$\mathcal{L} : V_1 \otimes_G V \rightarrow \text{Hom}_G(V^c, V_1^h)$$

is an injection.

**Corollary 2.1:** Under the same assumption as in Theorem 2.1, let  $G$  be a real reductive group and  $K$  a maximal compact Lie group of  $G$ . Suppose that  $V$  and  $V_1$  are both smooth and  $K$ -finite. Then  $\mathcal{L}$  induces an injection from  $V_1 \otimes_G V$  into  $\text{Hom}_{g,K}(V^c, V_1^h)$ .

**Proof:** When  $V$  is  $K$ -finite,  $\mathcal{L}(v_1 \otimes v)(u)$  will land in the  $K$ -finite subspaces of  $V_1^h$  which is isomorphic to  $V_1$ .

### A Geometric Realization

Let  $G$  be a Lie algebra group and  $dg$  a left invariant Haar measure. Let  $X$  be a manifold with a continuous free (right)  $G$  action. Suppose that  $X/G$  is a smooth manifold. Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$ . For any  $f \in C_c(X)$ ,  $v \in \mathcal{H}$ , define

$$\mathcal{L}^0(f \otimes v)(x) = \int_G f(xg) \pi(g) v dg.$$

Then  $\mathcal{L}^0(f \otimes v)$  is a  $\mathbb{C}$ -valued function on  $X$ . We shall see that it realizes  $f \otimes_G v$  in the following sense.

**Theorem 3.1:** Let  $G$  be a Lie group and  $dg$  a left invariant Haar measure. Let  $X$  be a manifold with a continuous free (right)  $G$  action such that the topological quotient  $X/G$  is a smooth manifold. Suppose there exist measures  $(X, \mu)$  and  $(X/G, d[x])$  such that

$$\int_X f(x) d\mu(x) = \int_{[x] \in X/G} \int_G f(xg) dg d[x].$$

Let  $C_c(X)$  be the set of continuous functions with compact support. Let  $(\pi, \mathcal{H})$  be a representation of  $G$ . Then  $\mathcal{L}^0(f \otimes v) \in C_c(X_G \mathcal{H}, X/G)$  where  $C_c(X_G \mathcal{H}, X/G)$  is the set of continuous compactly supported sections of the vector bundle

$$X \times_G \rightarrow X/G.$$

Furthermore,

$$C_c(X) \otimes_G \mathcal{H} \cong \mathcal{L}^0(C_c(X) \otimes \mathcal{H}),$$

and for every  $f \in C_c(X)$  and  $v \in \mathcal{H}$ ,

$$(f \otimes_G v, f \otimes_G v)_G = (\mathcal{L}^0(f \otimes v), \mathcal{L}^0(f \otimes v))_{X/G}.$$

**Proof:** Let  $f \in C_c(X)$  and  $v \in \mathcal{H}$ . It is easy to see that  $\mathcal{L}^0(f \otimes v)$  is compactly supported in  $X/G$ . In addition

$$\mathcal{L}^0(f \otimes v)(xg_1) = \int_G f(xg_1g) \pi(g) v dg = \int_G f(xg) \pi(g_1^{-1}g) v dg = \pi(g_1)^{-1} \mathcal{L}^0(f \otimes v)(x).$$

So  $\mathcal{L}^0(f \otimes v) \in C_c(X \times_G, X/G)$ . Observe that

$$(f \otimes_G v, f \otimes_G v)_G \tag{9}$$

$$= \int_G (R(g)f, f) (\pi(g)v, v) dg \tag{10}$$

$$= \int_G \int_X f(xg) \overline{f(x)} dx (\pi(g)v, v) dg \tag{11}$$

$$= \int_G \int_{X/G} \int_G f(xg_1g) \overline{f(xg_1)} (\pi(g)v, v) dg_1 d[x] dg \tag{12}$$

$$= \int_{X/G} \int_G \int_G f(xg) \overline{f(xg_1)} (\pi(g_1^{-1}g)v, v) dg_1 dg d[x] \tag{13}$$

$$= \int_{X/G} \int_G \int_G f(xg) \overline{f(xg_1)} (\pi(g)v, \pi(g_1)v) dg_1 dg d[x] \tag{14}$$

$$= \int_{X/G} (\int_G f(xg) v dg, \int_G \overline{f(xg_1)} \pi(g_1) v dg_1) d[x] \tag{15}$$

Absolute convergence are guaranteed since  $f(g)$  is compactly supported. Notice that

$$\mathcal{L}^0(f \otimes v)(x) = \int_G f(xg) \pi(g) v dg.$$

We have

$$(f \otimes_G v, f \otimes_G v)_G = (\mathcal{L}^0(f \otimes v), \mathcal{L}^0(f \otimes v))_{X/G}.$$

Clearly,  $C_c(X) \otimes_G \cong \mathcal{L}^0(C_c(X) \otimes \mathcal{H})$ .

### Invariant Tensor Product and Representation Theory

#### Definition 2

Let  $G$  be a classical group that preserves a nondegenerate sesquilinear form  $\Omega$ . Write  $G = G(V, \Omega)$  or simply  $G(V)$ , where  $V$  is a vector field over  $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$  equipped with the nondegenerate sesquilinear form  $\Omega$ . Let  $V = V_1 \oplus V_2$  such that  $\Omega(V_1, V_2) = 0$ . Let  $G_1 = G(V_1, \Omega|_{V_1})$  and  $G_2 = G(V_2, \Omega|_{V_2})$ . For each  $g_1, g_2 \in G_i$ , let  $(g_1, g_2)$  acts on  $V_1 \oplus V_2 = V$  diagonally. We say that  $G_1 \times G_2$  is diagonally embedded in  $G$ .

#### Definition 3

Let  $(G_1, G_2)$  be diagonally embedded in  $G$ . Let  $(\pi, \mathcal{H})$  be a unitary representation of  $G$  and  $(\pi_1, \mathcal{H}_{\pi_1})$  be a unitary representation of  $G_1$ . Let  $V$  be a subspace of  $\mathcal{H}_{\pi_1}$  that is invariant under  $G_2$ . Let  $V_1$  be a subspace of  $\mathcal{H}_{\pi_1}$  such that  $V \otimes_{G_1} V_1$  is well-defined. Define a linear  $G_2$ -representation  $(\pi \otimes_G \pi_1, V \otimes_{G_1} V_1)$  as follows:

$$(\pi \otimes_{G_1} \pi_1)(g_2)(u \otimes_{G_1} u_1) = \pi(g_2)u \otimes_{G_1} u_1 \quad (g_2 \in G_2, u \in V, u_1 \in V_1).$$

Since the Lie group action of  $G_2$  commutes with the integration over  $G_1$ , the action of  $G_2$  on  $V \otimes_{G_1} V_1$  is well-defined.

The linear representation  $(\pi \otimes_{G_1} \pi_1, V \otimes_{G_1} V_1)$  is not necessarily continuous because no topology has been defined on  $V \otimes_{G_1} V_1$ .

#### Lemma 4.1

The form  $(\cdot, \cdot)_{G_1}$  on  $V \otimes_{G_1} V_1$  is  $G_2$ -invariant.

**Proof:** Let  $u, v \in V; u_1, v_1 \in V_1$  and  $g_2 \in G_2$ . Write  $\sigma = \pi \otimes_{G_1} \pi_1$ . Then

$$(\sigma(g_2)(u \otimes_{G_1} u_1), v \otimes_{G_1} v_1)_{G_1} \tag{16}$$

$$= \int_{G_1} (\pi(g_1) \pi(g_2) u, v) (\pi_1(g_1) u_1, v_1) dg_1 \tag{17}$$

$$= \int_{G_1} (\pi(g_2) \pi(g_1) u, v) (\pi_1(g_1) u_1, v_1) dg_1 \tag{18}$$

$$= \int_{G_1} (\pi(g_1) u, \pi(g_2^{-1}) v) (\pi_1(g_1) u_1, v_1) dg_1 \tag{19}$$

$$= (u \otimes_{G_1} u_1, \pi(g_2^{-1}) v \otimes_{G_1} v_1)_{G_1} \tag{20}$$

$$= (u \otimes_{G_1} u_1, \sigma(g_2^{-1})(v \otimes_{G_1} v_1))_{G_1} \tag{21}$$

Hence  $(\cdot, \cdot)_{G_1}$  is  $G_2$ -invariant.

### ITP Associated with Discrete Series Representations

Let  $G(m+n)$  be a classical group preserving a nondegenerate sesquilinear form. Let  $(G(n), G(m))$  be diagonally embedded in  $G$ . For any irreducible unitary representation of  $G(m+n)$ , let  $\mathcal{H}_{\pi}$  be the

Frechet space of smooth vectors.

**Theorem 5.1:** Suppose that  $(\pi, \mathcal{H}_\pi)$  is a discrete series representation of  $G(m+n)$ . Suppose that  $m > n$  and  $(\pi_1, \mathcal{H}_1)$  is a unitary representation of  $G(n)$ . Then  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  is well-defined. Suppose that  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1 \neq 0$ . Then  $(\cdot)_{G(n)}$  is positive definite. Furthermore,  $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\cdot)_{G(n)})$  completes to a unitary representation of  $G(m)$ .

The key of the proof is to realize  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  as a subspace of the  $L^2$ -sections of the Hilbert bundle

$$\mathcal{H}_1 : G(n) \times G(m+n) \rightarrow G(n) \backslash G(m+n).$$

**Proof:** Write  $G = G(m+n)$ . Fix a maximal compact subgroup  $K$  of  $G$  such that

$$K(m) = K \cap G(m), \quad K(n) = K \cap G(n)$$

are maximal compact subgroups of  $G(m)$  and  $G(n)$  respectively. Let  $\mathfrak{a}$  be a maximal Abelian subalgebra in the orthogonal complement of  $\mathfrak{k}$  with respect to the Killing form  $(\cdot, \cdot)_\mathfrak{k}$  such that

$$\mathfrak{a} = (\mathfrak{a} \cap \mathfrak{g}(m)) \oplus (\mathfrak{a} \cap \mathfrak{g}(n)).$$

Let  $A$  be the analytic group generated by  $\mathfrak{a}$ . The function  $\log : A \rightarrow \mathbb{R}$  is well-defined. Let  $\|H\|^2 = (H, H)_\mathfrak{k}$  for each  $H \in \mathfrak{a}$ .

Since  $(\pi, \mathcal{H})$  is a discrete series representation, without loss of generality, realize  $\mathcal{H}$  on a right  $K$ -finite subspace of  $L^2(G)$ . So  $\mathcal{H} \subseteq L^2(G)_K$ .

Let  $\Xi_G(g)$  be Harish-Chandra's basic spherical function. Let  $\mathcal{HCS}(G)$  be the space of Harish-Chandra's Schwartz space. It is well-known that every  $f \in \mathcal{H}_\pi^\infty \subseteq \mathcal{HCS}(G)$  satisfies  $|f(g)| \leq C_f \Xi_G(g)$  for some  $C_f$  (see for example Ch. 12.4 [?]). For every  $h \in G$ ,  $|f(hg)| \leq \Xi_G(hg) \leq C_h C_f \Xi_G(g)$  for a constant  $C_h$ . Observe that  $\mathcal{H}_\pi^\infty$  is  $G(m)$ -invariant.

Fix a positive root system in  $\Sigma(\mathfrak{g}, \mathfrak{a})$ . Let  $A^+$  be the corresponding closed Weyl Chamber. Let  $\rho$  be the half sum of positive roots. Let  $u, v \in \mathcal{H}_\pi^\infty \subseteq L^2(G)$ . Then  $|L(g)u, v| \leq C_{u,v} \Xi_G(g)$  for a positive constant  $C_{u,v}$  [5,6]. Notice that for  $a \in A^+$ ,  $k_1, k_2 \in K$ ,

$$\Xi_G(k_1 a k_2) \leq C(1 + \|\log a\|^q \exp(\rho(\log a)))$$

for some  $q > 0$  and  $C > 0$ . Let  $\rho(n)$  be the half sum of positive roots of the restricted root system  $\Sigma(\mathfrak{g}(n), \mathfrak{a} \cap \mathfrak{g}(n))$ . Let  $(\mathfrak{a} \cap \mathfrak{g}(n))^+$  be the positive Weyl chamber of  $\mathfrak{a} \cap \mathfrak{g}(n)$  with respect to the root system  $\Sigma(\mathfrak{g}(n), \mathfrak{a} \cap \mathfrak{g}(n))$ . Since

$$\rho \Big|_{\mathfrak{a} \cap \mathfrak{g}(n)}(H) > 2\rho(n)(H) \quad (H \in (\mathfrak{a} \cap \mathfrak{g}(n))^+),$$

$\Xi_G(g) \Big|_{G(n)} \in L^1(G(n))$ . It follows that  $(L(g)u, v) \Big|_{G(n)} \in L^1(G(n))$  for every  $u, v \in \mathcal{H}_\pi^\infty$ . Notice that  $g_1 \in G(n) \implies (\pi_1(g_1)u_1, v_1)$  is always bounded for  $u_1, v_1 \in \mathcal{H}_1$ . We see that

$$\int_{G(n)} (\pi(g_1)u, v) (\pi_1(g_1)u_1, v_1) dg_1$$

always converges. So  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  is well-defined. Now suppose that  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1 \neq 0$ .

Notice that  $u \in \mathcal{H}_\pi^\infty \subseteq L^2(G)_K$  is bounded by a multiple of  $\Xi_G(g)$ . So  $u \Big|_{G(n)} \in L^1(G(n))$ . For each  $u \in \mathcal{H}_\pi^\infty$  and  $u_1 \in \mathcal{H}_1$ , define  $\mathcal{L}^0(u \otimes u_1)$  to be the  $\mathcal{H}_1$ -valued function on  $G$ :

$$g \in G \rightarrow \int_{g_1 \in G(n)} [L(g_1)u](g) \pi_1(g_1)u_1 dg_1 = \int_{g_1 \in G(n)} u(g_1^{-1}g) \pi_1(g_1)u_1 dg_1$$

in the strong sense. Notice that for  $g \in G, h_1 \in G(n)$ ,

$$\mathcal{L}^0(u \otimes u_1)(h_1 g) = \int_{g_1 \in G(n)} [L(g_1)u](h_1 g) \pi_1(g_1)u_1 dg_1 \tag{22}$$

$$= \int_{g_1 \in G(n)} u(g_1^{-1}h_1 g) \pi_1(g_1)u_1 dg_1 \tag{23}$$

$$= \int_{g_1 \in G(n)} u(g_1^{-1}g) \pi_1(h_1 g_1)u_1 dg_1 \tag{24}$$

$$= \pi(h_1) \left[ \int_{g_1 \in G(n)} L(g_1)u(g) \pi_1(g_1)u_1 dg_1 \right] \tag{25}$$

$$= \pi(h_1) \mathcal{L}^0(u \otimes u_1)(g) \tag{26}$$

So  $\mathcal{L}^0(u \otimes u_1)$  can be regarded as a section of the Hilbert bundle

$$\mathcal{H}_{1|G(n)} \times GG(n) \backslash G.$$

In addition, we have

$$(u \otimes_{G(n)} u_1, v \otimes_{G(n)} v_1)_{G(n)} \tag{27}$$

$$= \int_{G(n)} (L(g_1)u, v) (\pi_1(g_1)u_1, v_1) dg_1 \tag{28}$$

$$= \int_{G(n)} \int_G u(g_1^{-1}g) \overline{v(g)} dg (\pi_1(g_1)u_1, v_1) dg_1 \tag{29}$$

$$= \int_{G(n)} \int_{G(n) \backslash G} \int_{G(n)} u(g_1^{-1}h_1 g) \overline{v(h_1 g)} dh_1 d[g] (\pi_1(g_1)u_1, v_1) dg_1 \tag{30}$$

$$= \int_{G(n)} \int_{G(n) \backslash G} \int_{G(n)} u(g_1^{-1}h_1 g) \overline{v(h_1 g)} (\pi_1(g_1)u_1, v_1) dg_1 dh_1 d[g] \quad (g_1 = h_1 g_1) \tag{31}$$

$$= \int_{G(n) \backslash G} \int_{G(n) \times G(n)} u(g_1^{-1}g) \overline{v(h_1 g)} (\pi_1(h_1 g_1)u_1, v_1) dg_1 dh_1 d[g] \tag{32}$$

$$= \int_{G(n) \backslash G} \int_{G(n) \times G(n)} u(g_1^{-1}g) \overline{v(h_1 g)} (\pi_1(g_1)u_1, \pi(h_1^{-1})v_1) dg_1 dh_1 d[g] \tag{33}$$

$$= \int_{G(n) \backslash G} \int_{G(n) \times G(n)} u(g_1^{-1}g) \overline{v(h_1^{-1}g)} (\pi_1(g_1)u_1, \pi_1(h_1)v_1) dg_1 dh_1 d[g] \tag{34}$$

$$= (\mathcal{L}^0(u \otimes u_1), \mathcal{L}^0(v \otimes v_1))_{G(n) \backslash G} \tag{35}$$

where  $G(n) \backslash G$  is equipped with a right  $G$  invariant measure. Eqn. (31) is valid because the integrative Eqn. (29) converges absolutely. In fact, we have

$$\int_{G(n) \times G} |u(h_1 g) v(g)| dg dh_1 < \infty.$$

To see this, recall that  $u(g), v(g) \in \mathcal{HCS}(G)$ . In particular, for any  $N > 0$  and  $a \in A^+$ ,  $k_1, k_2 \in K$ , there exists  $C_{u,v,N} > 0$  such that:

$$|u(k_1 a k_2)| \leq C_{u,N} \|\log a\|^{-N} \Xi_G(k_1 a k_2).$$

Write  $W_N(g) = \|\log a\|^{-N} \Xi_G(k_1 a k_2)$  for  $g = k_1 a k_2$ . Then there also exists  $C_{v,N} > 0$  such that

$$|v(g)| \leq C_{v,N} W_N(g).$$

Fix an  $N$  such that  $W_N(G) \in L^2(G)$ . In particular,  $W_N(G) \in L^2(G)_K$ . Observe that the function

$$h \in G \implies |u(h)|, |v(h)|$$

is bounded by a multiple of  $(L(h)W_N(g), W_N(g))$ , which, by a Theorem of Cowling-Haagerup-Howe [5], is bounded by a multiple of  $\Xi_G(g)$ . Hence

$$\int_{G(n) \backslash G} |u(h_1 g) v(g)| dg dh_1 < \int_{G(n)} \left( \int_G u(h_1 g) \overline{v(g)} dg \right) dh_1 < \int_{G(n)} C \Xi_G(h_1) dh_1 < \infty.$$

Eqn. (29) converges absolutely. Therefore Eqn. (31) holds.

Now we have

$$(u \otimes_{G(n)} u_1, v \otimes_{G(n)} v_1)_{G(n)} = (\mathcal{L}^0(u \otimes u_1), \mathcal{L}^0(v \otimes v_1))_{G(n) \backslash G}$$

It follows that  $\mathcal{L}^0(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1) \cong \mathcal{L}(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1)$ . Realize  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  as  $\mathcal{L}^0(\mathcal{H}_\pi^\infty \otimes \mathcal{H}_1)$ , which is a subspace of  $L^2$ -sections of the Hilbert bundle:  $\mathcal{H}_{1G(n)} \times GG(n) \backslash G$ .

Clearly  $(\cdot)_{G(n)}$  is positive definite. Let  $\overline{\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1}$  be the completion of  $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\cdot)_{G(n)})$ .

Since  $G(m)$  acts on  $\mathcal{H}_\pi^\infty$  and it commutes with  $G(n)$ ,  $G(m)$  acts on  $\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1$  and it preserves  $(\cdot)_{G(n)}$ . So the action of each  $g_2 \in G(m)$  can be extended into a unitary operator on  $\overline{\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1}$ . The group structure is kept in this completion essentially due to the fact that each extension is unique. Therefore  $(\overline{\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1}, (\cdot)_{G(n)})$  completes to a unitary representation of  $G(m)$ .

**Definition 4**

Let  $\overline{\Pi}_u(G)$  be the unitary dual of  $G$ . Suppose that  $m > n$ . Let  $\pi$  be a discrete series representation of  $G(m+n)$ . We denote the functor from  $\pi_1$  to the completion of  $(\mathcal{H}_\pi^\infty \otimes_{G(n)} \mathcal{H}_1, (\cdot)_{G(n)})$  by  $IT\pi$ . If  $IT\pi(\cdot)_0, IT\pi(\pi_1)$  is a unitary representation of  $G(m)$ . Regarding the zero dimensional representation as a unitary representation,  $IT\pi$  defines a functor from unitary representations of  $G(n)$  to unitary representations of  $G(m)$ .

**Conclusion**

One natural question arises. That is, if  $\pi_1$  is irreducible, is  $IT\pi(\cdot)_1$  irreducible? This is beyond the scope of this paper. In fact, this problem is quite difficult. In general,  $IT\pi(\pi_1)$  is not irreducible. However, for a certain holomorphic discrete series representation  $\pi$ ,  $IT\pi(\pi_1)$  will indeed be irreducible. For the time being, it is not clear which discrete series representation  $\pi$  has such a property. This question may be intrinsically related to the cohomology induction [7].

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