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Research Article

Jet Bundles on Projective Space II

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Abstract

In previous papers the structure of the jet bundle as *P*-module has been studied using different techniques. In this paper we use techniques from algebraic groups, sheaf theory, generliazed Verma modules, canonical filtrations of irreducible SL(V)-modules and annihilator ideals of highest weight vectors to study the canonical filtration $U_l(\mathfrak{g})L^d$ of the irreducible SL(V)-module $H^0(X, \mathcal{O}_X(d))^*$ where $X = \mathbb{G}(m, m+n)$. We study $U_l(\mathfrak{g})L^d$ using results from previous papers on the subject and recover a well known classification of the structure of the jet bundle $\mathcal{P}^l(\mathcal{O}(d))$ on projective space $\mathbb{P}(V^*)$ as *P*-module. As a consequence we prove formulas on the splitting type of the jet bundle on projective space as abstract locally free sheaf. We also classify the *P*-module of the first order jet bundle $\mathcal{P}^1_X(\mathcal{O}_X(d))$ for any $d \geq 1$. We study the incidence complex for the line bundle $\mathcal{O}(d)$ on the projective line and show it is a resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$ - the incidence scheme of $\mathcal{O}(d)$. The aim of the study is to apply it to the study of syzygies of discriminants of linear systems on projective space and grassmannians.

Keywords: Algebraic group; Jet bundle; Grassmannian; *P*-module; Generalized verma module; Higher direct image; Annihilator ideal; Canonical filtration; Discriminant; Koszul complex; Regular sequence; Resolution

Introduction

In a series of papers of Maakestad [1-4], the structure of the jet bundle as *P*-module has been studied using different techniques. In this paper we continue this study using techniques from algebraic groups, sheaf theory, generalized Verma modules, canonical filtrations of irreducible SL(V)-modules and annihilator ideals of highest weight vectors and study the canonical filtration $U_l(\mathfrak{g})L^d$ of the SL(V)-module $H^0(X, \mathcal{O}_X(d))^*$ where $X = \mathbb{G}(m, m+n)$ is the grassmannian of m-planes in an $m+n$-dimensional vector space. Using results obtained in studies of Maakestad [1] we classify $U_l(\mathfrak{g})L^d$ and as a corollary we recover a well known result on the structure of the jet bundle $\mathcal{P}^l(\mathcal{O}(d))$ on $\mathcal{P}(V^*)$ as *P*-module. As a consequence we get well known formulas on the splitting type of the jet bundle on projective space as abstract locally free sheaf. We also classify the *P*-module of the first order jet bundle $\mathcal{P}^1_X(\mathcal{O}_X(d))$ on any grassmannian $X = \mathbb{G}(m, m+n)$ (Corollary 3.10).

In the first section of the paper we study the jet bundle $\mathcal{P}^l_{G/H}(\mathcal{E})$ of any locally free G-linearized sheaf ε on any quotient G/H. Here G is an affine algebraic group of finite type over an algebraically closed field K of characteristic zero and $H \subseteq G$ is a closed subgroup. There is an equivalence of categories between the category of finite dimensional H-modules and the category of finite rank locally free $\mathcal{O}_{G/H}$-modules with a G-linearization. The main result of this section is Theorem 2.3 where we give a classification of the H_l-modules structure of the fiber $\mathcal{P}^l_{G/H}(\mathcal{E})(x)^*$ where $H_l \subseteq H$ is a Levi subgroup. Here $x\, G/H$ is the distinguished K-rational point defined by the identity $e \in G$. We also study the structure of $\mathcal{P}^l_X(\mathcal{O}_X(d))(x)^*$ as H_l-module where $X = \mathbb{G}(m, m+n)$ is the grassmannian of m-planes in an $m+n$-dimensional vector space (Corollary 2.5 and 2.8).

In the second section we study the canonical filtration $U_l(\mathfrak{g})L^d$ for the irreducible SL(V)-module $H^0(\mathbb{G}, \mathcal{O}\mathbb{G}(d))^*$. Here $\mathbb{G} = \mathbb{G}(m, m+n)$. We prove in Theorem 3.5 there is an isomorphism

$$U_l(\mathfrak{g})L^d \cong L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of *P*-modules when $\mathbb{G} = \mathbb{G}(1, n+1) = \mathcal{P}^n$ is projective n-space. As a result we recover in Corollary 3.6 the structure of the fiber $\mathcal{P}^l_{\mathbb{G}}(\mathcal{O}_{\mathbb{G}}(d))(x)^*$ as *P*-module. This result was proved in another paper [5] using different techniques. We also recover in Corollary 3.8 a known formula on the structure of the jet bundle on projective space as abstract locally free sheaf [2,6-10].

In the third section we study the incidence complex

$$\wedge^{\bullet}\mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))^*_Y$$

of the line bundle $\mathcal{O}(d)$ on the projective line. Using Koszul complexes and general properties of jet bundles we prove it is a locally free resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$ - the incidence scheme of $\mathcal{O}(d)$.

In Appendix A and B we study SL(V)-modules, automorphisms of SL(V)-modules and give an elementary proof of the Cauchy formula.

Hence the paper initiates a general study of the canonical filtration $U_l(g)L^d$ for any line bundle $\mathcal{O}(d)$ with $d \geq 1$ on any grassmannian $\mathbb{G}(m, m+n)$ as *P*-module. In Section 3 we show some of the complications arising in this study by giving explicit examples.

The study of the jet bundle $\mathcal{P}^l_X(\mathcal{O}_X(d))$ of a line bundle $\mathcal{O}_X(d)$ on the grassmannian $X = \mathbb{G}(m, m+n)$ is motivated partly by its relationship with the discriminant $D^l\mathcal{O}_X(d)$ of the line bundle $\mathcal{O}_X(d)$. There is by studies of Maakestad [11] for all $1 \leq l < d$ an exact sequence of locally free $\mathcal{O}_X$-modules

$$0 \to \mathcal{Q} \to H^0(X, \mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}^l_X(\mathcal{O}_x(d)) \to 0$$

giving rise to a diagram of maps of schemes

$$\begin{tikzcd}
\mathbb{P}(\mathcal{Q}^*) \arrow[r, "i"] \arrow[d, "\pi"] & \mathbb{P}(W^*) \times X \arrow[d, "p"] \\
D^1(\mathcal{O}_X(d)) \arrow[r, "j"] & \mathbb{P}(W^*)
\end{tikzcd},$$

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Where $W = H^0(X, \mathcal{O}_X(d))$, π is the restriction of the projection map and i, j are closed immersions. By definition $D^l(\mathcal{O}_X(d)) := \pi(\mathbb{P}(\mathcal{Q}^*))$ is the schematic image of $\mathcal{P}(\mathcal{Q}^*)$ via π. The K-rational points of $\mathcal{P}(\mathcal{Q}^*)$ are pairs of K-rational points (s, x) with the property that $T^l(x)(s) = 0$ in $\mathcal{P}_X^l(\mathcal{O}_X(d))(x)$. The scheme $\mathcal{P}(\mathcal{Q}^*)$ is the *incidence scheme* of the l'th Taylor morphism

$$T^l : H^0(X,\mathcal{O}_X(d)) \otimes \mathcal{O}_X \to \mathcal{P}_X^l(\mathcal{O}_x(d)).$$

The map π is a surjective generically finite morphism between irreducible schemes. There is by literature of Maakestad [11] a Koszul complex of locally free sheaves on $Y = \mathcal{P}(W^*) \times X$

$$0 \to \mathcal{O}(-r)_Y \otimes \wedge^r \mathcal{P}_X^l(\mathcal{O}_X(d))_Y^* \to \cdots \to \mathcal{O}(-1)_Y \otimes \mathcal{P}_X^l(\mathcal{O}_X(d))_Y^* \to \quad (1.0.1)$$
$$\mathcal{O}_Y \to \mathcal{O}_{\mathbb{P}(\mathcal{Q}^*)} \to 0$$

which is a resolution of the ideal sheaf of $\mathcal{P}(\mathcal{Q}^*)$ when it is locally generated by a regular sequence. The complex 1 might give information on a resolution of the ideal sheaf of $D^l(\mathcal{O}_X(d))$. A resolution of the ideal sheaf of $D^l(\mathcal{O}_X(d))$ will give information on its syzygies. By literature of Maakestad [11] the first discriminant $D^1(\mathcal{O}\mathcal{P}(d))$ on the projective line $\mathcal{P} = \mathcal{P}^1$ is the *classical discriminant of degree d polynomials*, hence it is a determinantal scheme. By the results of Lascoux [12], we get an approach to the study of the syzygies of $D^1(\mathcal{O}\mathcal{P}(d))$. Hence we get two approaches to the study of syzygies of discriminants of line bundles on projective space and grassmannians: One using Taylor maps, incidence schemes, jet bundles and generalized Verma modules. Another one using determinantal schemes.

Jet Bundles on Quotients

In this section we study the jet bundle of any finite rank G-linearized locally free sheaf $\mathcal{E}$ on the grassmannian $G/P=\mathbb{G}(m, m + n)$ as P_l-module, where $P_l \subseteq P$ is a maximal linearly reductive subgroup.

Let K be an algebraically closed field of characteristic zero and let V be a K-vector space of dimension n. Let $H \subseteq G \subseteq GL(V)$ be closed subgroups. The following holds: There is a quotient morphism

$$\pi : G \to G/H \quad (2.0.2)$$

and G/H is a smooth quasi projective scheme of finite type over K. Moreover

$$H \subseteq G \text{ is parabolic if and only if } G/H \text{ is projective.} \quad (2.0.3)$$

For a proof refer to literature of Jantzen [13]. Let $X = G/H$ and let $\underline{mod}^G(\mathcal{O}_{G/H})$ be the category of locally free $\mathcal{O}_{G/H}$-modules with a G-linearization. Let $\underline{mod}(H)$ be the category of finite dimensional H-modules. It follows from Jantzen [13], there is an exact equivalence of categories

$$\underline{mod}(H) \cong \underline{mod}^G(\mathcal{O}_{G/H}).$$

Let $\mathcal{E} \in \underline{mod}^G(\mathcal{O}_{G/H})$ be a locally free $\mathcal{O}_{G/H}$-module.

Let $Y = G/H \times G/H$ and $p, q : Y \to G/H$ be the canonical projection maps. The scheme G/H is smooth and separated over $\mathrm{Spec}(K)$ hence the diagonal morphism

$$\Delta : G/H \to Y$$

is a closed immersion of schemes. Let $\mathcal{I} \subseteq \mathcal{O}_Y$ be the ideal of the diagonal and let $\mathcal{O}_{\Delta^l} = \mathcal{O}_Y / \mathcal{I}^{l+1}$ be the structure sheaf of the *n'th infinitesimal neigborhood of the diagonal.*

Definition 2.1. *Let $\mathcal{E}$ be a locally free finite rank $\mathcal{O}_{G/H}$-module. Let*

$$\mathcal{P}_{G/H}^l(\mathcal{E}) = p_*(\mathcal{O}_{\Delta^l} \otimes q^*\mathcal{E})$$

be the l'th jet bundle of $\mathcal{E}$.

Proposition 2.2. *There is for all $l \geq 1$ an exact sequence of locally free $\mathcal{O}_{G/H}$-modules*

$$0 \to Sym^l(\Omega_{G/H}^1) \otimes \mathcal{E} \to \mathcal{P}_{G/H}^l(\mathcal{E}) \xrightarrow{\phi} \mathcal{P}_{G/H}^{l-1}(\mathcal{E}) \to 0 \quad (2.2.1)$$

with G-linearization.

Proof. By literature of Maakestad [4] sequence 2.2.1 is an exact sequence of locally free $\mathcal{O}_{G/H}$-modules. The scheme Y is equipped with the diagonal G-action. It follows p_* and q^* preserve G-linearizations. We get a diagram of exact sequences of $\mathcal{O}_Y$-modules with a G-linearization

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{I}^{l+1} \otimes q^*\mathcal{E} & \longrightarrow & \mathcal{O}_Y \otimes q^*\mathcal{E} & \longrightarrow & \mathcal{O}_{\Delta^l} \otimes q^*\mathcal{E} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{I}^{l} \otimes q^*\mathcal{E} & \longrightarrow & \mathcal{O}_Y \otimes q^*\mathcal{E} & \longrightarrow & \mathcal{O}_{\Delta^{l-1}} \otimes q^*\mathcal{E} & \longrightarrow & 0 \end{array}$$

Since p_* preserves G-linearization we get a morphism

$$\phi : \mathcal{P}_{G/H}^l(\mathcal{E}) \to \mathcal{P}_{G/H}^{l-1}(\mathcal{E})$$

preserving the G-linearization, and the Proposition is proved.

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{h} = Lie(H)$. Let $H_l \subseteq H$ be a Levi subgroup of H. It follows H_l is a maximal linearly reductive subgroup of H. The group H_l is not unique but all such groups are conjugate under automorphisms of H. Let $x \in G/H$ be the K-rational point defined by the identity $e \in G$.

Theorem 2.3. *There is for all $l \geq 1$ an isomorphism*

$$\mathcal{P}_X^l(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus_{i=}^l \mathrm{Sym}^i(\mathfrak{g}/\mathfrak{h})) \quad (2.3.1)$$

of L-modules.

Proof. Dualize the sequence 2.2.1 and take the fiber at x to get the exact sequence

$$0 \to \mathcal{P}_X^{l-1}(\mathcal{E})(x)^* \to \mathcal{P}_X^l(\mathcal{E})(x)^* \to \mathcal{E}(x)^* \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{h}) \to 0$$

of H-modules (and H_l-modules). This sequence splits since H_l is linearly reductive and the Theorem follows by induction on l.

Hence the study $\mathcal{P}_X^l(\mathcal{E})(x)^*$ as H_l-module is reduced to the study of $\mathcal{E}(x)^*$ and $\mathrm{Sym}^l(\mathfrak{g}/\mathfrak{h})$.

Let $W \subseteq V$ be K-vector spaces of dimension m and $m + n$ and let $G = SL(V)$ and $P \subseteq G$ the subgroup fixing W. It follows $G/P = \mathbb{G}(m, m + n)$ is the grassmannian of m-planes in V. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$. Fix a basis $e_1, .., e_m$ for W and $e_1, .., e_m, e_{m+1}, .., e_{m+n}$ for V. It follows the K-rational points of P are matrices M on the form

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$, A an $m \times m$-matrix and B an $n \times n$-matrix. Let $P_l \subseteq P$ be the subgroup defined as follows: The K-rational points of P_l are matrices M on the form

$$M = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$ and similarly A an $m \times m$-matrix and B an $n \times n$-matrix. It follows P_l is a Levi subgroup of P, hence it is a maximal linearly reductive subgroup.

Proposition 2.4. *There is a canonical isomorphism*

$$\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(W, V/W)$$

of P-modules.

Proof. By definition $\mathfrak{g} = \mathrm{sl}(V)$, hence $\varphi \in \mathfrak{g}$ is a map

$$\varphi : V \to V$$

with $tr(\varphi) = 0$. Let $i : W \to V$ be the inclusion map and $p : V \to V / W$ the projection map. Define the following map:

$$\mathrm{J}' : \mathfrak{g} \to \mathrm{Hom}(W, V / W)$$

by

$$j'(\varphi) = p \circ \varphi \;\; i.$$

It follows $j(\mathrm{p}) = 0$ hence we get a well defined map

$$j : \mathrm{g/p} \to \mathrm{Hom}(W, V / W)$$

defined by

$$j(\bar{\phi}) = p \circ \phi \circ i.$$

One checks g/p and Hom(W, V/W) are P-modules and j a morphism of P-modules. It is an isomorphism and the Proposition follows.

Corollary 2.5. *On* $X = \mathbb{G}(m, m + n)$ *there is an isomorphism*

$$\mathcal{P}_X^l(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus_{i=0}^l \mathrm{Sym}^i(\mathrm{Hom}(W, V / W))$$

of P_l *-modules.*

Proof. The proof follows from Theorem 2.3 and Proposition 2.4.

There is an isomorphism of P-modules

$$\mathrm{Hom}(W, V / W) \cong W^* \otimes V / W$$

hence the decomposition into irreducible components of the module $\mathrm{Sym}^i(W^* \otimes V / W)$ as P_l -module may be done using the *Cauchy formula* (Appendix B).

Let $\lambda - |i|$ denote λ is a partition of the integer i If $\lambda = \{\lambda_1, .., \lambda_d\}$ is a partition of an integer l, let μ (λ) denote the following partition:

$$\mu(\lambda)_i = l - \lambda_{d+1-i}.$$

Let for any partition λ of an integer l and any vector space W, $\mathbb{S}_\lambda(W)$ denote the *Schur-Weyl module* of λ.

Corollary 2.6. *There is an isomorphism*

$$\mathcal{P}_X^l(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus_{i=0}^l (\bigoplus_{\lambda - |i|} \mathbb{S}_\lambda(W^*) \otimes \mathbb{S}_{\mu(\lambda)}(V / W)))$$

of $\mathrm{SL}(W) \times \mathrm{SL}(V/W)$*-modules.*

Proof. By Corollary 2.5 there is an isomorphism

$$\mathcal{P}_X^l(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus_{i=0}^l \mathrm{Sym}^i(\mathrm{Hom}(W, V / W))$$

of P_l -modules and $\mathrm{SL}(W) \times \mathrm{SL}(V/W)$-modules, since $\mathrm{SL}(W) \times \mathrm{SL}(V/W) \subseteq P_l$ is a closed subgroup. Since

$$\mathrm{Sym}^i(\mathrm{Hom}(W, V / W)) \cong \mathrm{Sym}^i(W^* \otimes V / W)$$

the result follows from the Cauchy formula (Appendix B or [14]).

Example 2.7. *Calculation of the cohomology group* $\mathrm{H}^i(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*)$.

In the following we use the notation introduced in litertature of Jantzen [13]. Let $P_{semi} = \mathrm{SL}(m) \times \mathrm{SL}(n) \subseteq P$ be the semi simplification of P. We get a vector bundle

$$\pi : G / P_{semi} \to G / P = \mathbb{G}(m, m + n).$$

Let $X = G / P$ and $Y = G / P_{semi}$ Given any finite dimensional P-module W, let $\mathcal{L}_X(W)$ denote its corresponding $\mathcal{O}_X$-module. Let W_{semi} denote the restriction of W to P_{semi}. By the results of Perkinson [13] it follows there is an isomorphism

$$\pi^* \mathcal{L}_X(W) \cong \mathcal{L}_Y(W_{semi})$$

of locally free sheaves. This will help calculating the higher cohomology group

$$\mathrm{H}^i(X, \mathcal{L}_X(W))$$

since P_{semi} is semi simple and π is a locally trivial fibration. If W is the P-module corresponding to the dual of the j'th exterior power of the jet bundle $\wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*$ we can use this construction to calculate the cohomology group

$$\mathrm{H}^i(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*).$$

Such a calculation will be by the results of Maakestad [11], Example 5.12 give information on resolutions of the ideal sheaf of $D^l(\mathcal{O}_X(d))$ since the push down of the Koszul complex 1.0.1 is the locally trivial sheaf

$$\mathcal{O}(-j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*).$$

To describe the locally trivial sheaf $\mathcal{O}(-j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*)$ for all i, j we need to calculate the dimension $h^i(X, \wedge^j \mathcal{P}_X^l(\mathcal{O}_X(d))^*)$ and this calculation may be done using the approach indicated above.

Let $m = 2$, $n = 4$ and $X = \mathbb{G}(2, 4)$.

Corollary 2.8. *There is an isomorphism*

$$\mathcal{P}_X^l(\mathcal{E})(x)^* \cong \mathcal{E}(x)^* \otimes (\oplus_{i=0}^l \oplus_{j=0}^n \mathrm{Sym}^{2j+m}(W^*) \otimes \mathrm{Sym}^{2j+m}(V / W))$$

of $\mathrm{SL}(2) \times \mathrm{SL}(2)$*-modules. Here* $(n, m) = (\frac{i}{2}, 0)$ *if* $i = 2n$ *and* $(\frac{i-1}{2}, 1)$ *if* $i = 2n + 1$.

Proof. This follows from Corollary 2.5 and Proposition 5.1.

On Canonical Filtrations and Jet Bundles on Projective Space

In this section we study the canonical filtration for the dual of the SL(V)-module of global sections of an invertible sheaf on the grassmannian. We classify the canonical filtration on projective space and as a result recover known formulas on the splitting type of the jet bundle as abstract locally free sheaf.

Let $W \subseteq V$ be vector spaces over K of dimension m and $m + n$. Let W have basis $e_1, .., e_m$ and V have basis $e_1, .., e_{m+n}$. Let V^* have basis $x_1, .., x_{m+n}$. Let $G = \mathrm{SL}(V)$ and $P \subseteq G$ the parabolic subgroup of elements fixing W. It follows there is a quotient morphism

$$\pi : G \to G / P$$

and $G / P \cong \mathbb{G}(m, m + n)$ is the grassmannian of m-planes in V. Let $\mathbb{P} = \mathbb{G}(1, n+1) = \mathbb{P}(V^*)$. Let $L^d = \mathrm{Sym}^d(\wedge^m W)$. There is an inclusion of P-modules $L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$. Since K has characteristic zero there is an inclusion of G-modules

$$\mathrm{H}^0(G / P, \mathcal{O}_{G/P}(d))^* \subseteq \mathrm{Sym}^d(\wedge^m V^*)^* \cong \mathrm{Sym}^d(\wedge^m V).$$

Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra og $\mathfrak{g}$ and let $\mathrm{U}_l(\mathfrak{g})$ be the l'th term to its canonical filtration.

By the Corollary 3.11 in studies of Maakestad [15] there is for all $1 \le l \le d$ an exact sequence of P-modules

$$0 \to \mathcal{P}_{\mathbb{G}}^l(\mathcal{O}_{\mathbb{G}}(d))(x)^* \to \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(d))^* \to \mathrm{H}^0(\mathbb{G}, \mathfrak{m}^{l+1}\mathcal{O}_{\mathbb{G}}(d))^* \to 0.$$

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Since the grassmannian is projectively normal in the Plucker embedding we get an inclusion

$$\mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^* \subseteq \mathrm{Sym}^d(\wedge^m V)$$

of P-modules. The highest weight vector for $\mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^*$ is the line $L^d = \mathrm{Sym}^d(\wedge^m W)$. Let $ann(L^d) \subseteq \mathrm{U}(\mathfrak{g})$ be the left annihilator ideal of L^d. It is the ideal generated by elements $x \in \mathrm{U}(\mathfrak{g})$ with the property $x(L^d) = 0$. Let $ann_l(L^d)$ be its canonical filtration. We get an exact sequence of G-modules

$$0 \to ann(L^d) \otimes L^d \to \mathrm{U}(\mathfrak{g}) \otimes L^d \to \mathrm{H}^0(X,\mathcal{O}_X(d))^* \to 0$$

and an exact sequence of P-modules

$$0 \to ann_l(L^d) \otimes L^d \to \mathrm{U}_l(\mathfrak{g}) \otimes L^d \to \mathrm{U}_l(\mathfrak{g})L^d \to 0$$

for all $l \geq 1$. The G-module $\mathrm{U}(\mathfrak{g}) \otimes L^d$ is the *generalized Verma module* corresponding to the P-module defined by $L^d = \mathrm{Sym}^d(\wedge^m V)$. There is an inclusion of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{H}^0(\mathbb{G},\mathcal{O}_{\mathbb{G}}(d))^*.$$

Definition 3.1. *Let* $\{\mathrm{U}_l(\mathfrak{g})L^d\}_{l\geq 1}$ *be the canonical filtration for* $\mathrm{H}^0(\mathbb{G},\mathcal{O}\mathbb{G}(d))^*$.

Lemma 3.2. *Assume* $y \in \mathfrak{g}$ *and* $x_1 \cdots x_i \in \mathrm{U}_i(\mathfrak{g})$ *with* $x_i \in \mathfrak{g}$. *The following holds:*

$$y(x_1 \cdots x_i) = (x_1 \cdots x_i)y + w$$

where $w \in \mathrm{U}_{i-1}(\mathfrak{g})$ $\omega \in U_{i-1}(\mathfrak{g})$.

Proof. The proof is by induction.

The Lie algebra $\mathfrak{p}$ is the sub Lie algebra of $\mathfrak{g} = \mathrm{sl}(V)$ given by matrices M of the following type:

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where A is an $m \times m$-matrix, B and $n \times n$-matrix and $tr(A) + tr(B) = 0$. Let $\mathfrak{p}_L$ be the sub Lie algebra of $\mathfrak{p}$ consisting of matrices $M \in \mathfrak{p}$ of the following type:

$$M = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $tr(A) + tr(B) = 0$.

Proposition 3.3.

The sub Lie algebra $\mathfrak{p}_L \subseteq \mathfrak{p}$ *is a sub P-module of* $\mathfrak{p}$. (3.3.1)

There is an exact sequence of P-modules

$$0 \to \mathfrak{p}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p} \to 0. \tag{3.3.2}$$

and $\mathfrak{p}/\mathfrak{p}_L$ *is the trivial P-module.*

The following holds:

$$dim_K(L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)) = \binom{mn+k}{mn}. \tag{3.3.3}$$

There is a filtration of P-modules

$$0 = G_{l+1} \subseteq G_l \subseteq \cdots \subseteq G_0 = L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \tag{3.3.4}$$

with quotients

$$G_i / G_{i+1} \cong L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}((\mathfrak{g}/\mathfrak{p} \otimes L)$$

for $1 \leq i \leq k$.

Assume $dim_k(W) = 1$ *and let* $W = L$. *There is an exact sequence of P-modules*

$$0 \to \mathfrak{p}_L \otimes L \to \mathfrak{g} \otimes L \to V \to 0 \tag{3.3.5}$$

giving an isomorphism of P-modules $\mathfrak{g}/\mathfrak{p}_L \otimes L \cong V$.

Proof. We prove 3.3.1: In the following A, a are square matrices of size m and b, B square matrices of size n. The K -rational points of the group P are matrices g on the form

$$g = \begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where $det(A)det(B) = 1$. Assume $x \in \mathfrak{p}$ is the following element:

$$x = \begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$$

with $tr(a) + tr(b) = 0$. It follows $g(x) = gxg^{-1}$ has $tr(gxg^{-1}) = tr(gg^{-1}x) = tr(x) = 0$ hence $gxg^{-1} \in \mathfrak{p}$ and $\mathfrak{p}$ is a P-module. Assume $x \in \mathfrak{p}_L$ ie $tr(a) = tr(b) = 0$. It follows

$$gxg^{-1} = \begin{pmatrix} aAa^{-1} & * \\ 0 & bBb^{-1} \end{pmatrix}$$

and $tr(aAa^{-1}) + tr(aa^{-1}A) = tr(A) = 0$ hence $g(x) \in \mathfrak{p}_L$ and 3.3.1 is proved.

We prove 3.3.2: By 3.3.1 it follows $\mathfrak{p}_L \subseteq \mathfrak{p}$ is a sub P-module. One checks $\mathfrak{p}/\mathfrak{p}_L$ is a trivial P-module. We clearly get an exact sequence of P-modules and 3.3.2 is proved.

We prove 3.3.3: Since

$$dim_K(\mathfrak{g}) = (m + n)^2 - 1 = n^2 + 2mn + m^2 - 1$$

and

$$dim_K(\mathfrak{p}_L) = m^2 + mn + n^2 - 2$$

it follows $dim_k(\mathfrak{g}/\mathfrak{p}_L) = mn + 1$. It follows

$$dim_K(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes l) = \binom{mn+1+l-1}{mn+1-1} = \binom{mn+l}{mn}.$$

We prove 3.3.4: Since $\mathfrak{p}/\mathfrak{p}_L$ is a trivial P-module there are isomorphisms of P-modules

$$L^{d-(k-i)} \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong L^{d-k} \otimes L^i \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong$$
$$L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

for all $1 \leq i \leq k$. We get an injection

$$j : L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

defined by

$$j(L^{d-k} \otimes \overline{y_1} \otimes L \cdots \overline{y_i} \otimes L \otimes \overline{x_1} \otimes L \cdots \overline{x_{k-i}} \otimes L) = L^{d-k} \otimes \overline{y_1} \otimes L \cdots \overline{y_i} \otimes L\overline{x_1} \otimes L \cdots \overline{x_{k-i}}.$$

The injection j gives rise to an injection

$$L^{d-(k-i)} \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \cong L^{d-k} \otimes \mathrm{Sym}^i(\mathfrak{p}/\mathfrak{p}_L \otimes L) \otimes \mathrm{Sym}^{k-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to^j$$
$$L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules for all $1 \leq i \leq k$. The exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p}_L \to \mathfrak{g}/\mathfrak{p} \to 0$$

gives rise to a filtration of P-modules

$$0 = F_{l+1} \subseteq F_l \subseteq \cdots \subseteq F_0 = \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

with quotients

$$F_i / F_{i+1} \cong L^i \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p} \otimes L).$$

Put $G_i = L^{d-1} \otimes F_i$. It follows

$$G_i = L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

There is an isomorphism

$$G_i / G_{i+1} \cong L^{d-(l-i)} \otimes \mathrm{Sym}^{l-i}(\mathfrak{g}/\mathfrak{p} \otimes L)$$

and claim 3.3.4 is proved.

We prove 3.3.5: Let $V = K\{e_0, .., e_n\}$ and $L = W = e_0$. It follows $P \subseteq G = \mathrm{SL}(V)$ is the group whose K-rational points are the following:

$$g = \begin{pmatrix} a & * \\ 0 & B \end{pmatrix}$$

with $a = \frac{1}{det(B)}$. Also B is an $n \times n$-matrix with coefficients in K. By definition the maps in the sequence are maps of P-modules. It follows $\mathfrak{p} = Lie(P)$ is the Lie algebra whose elements x are matrices on the following form:

$$x = \begin{pmatrix} -tr(B) & * \\ 0 & B \end{pmatrix}$$

where B is any $n \times n$-matrix with coefficients in K. The sub Lie algebra $\mathfrak{p}_L \subseteq \mathfrak{p}$ is the Lie algebra of matrixes $x \in \mathfrak{p}$ on the following form:

$$x = \begin{pmatrix} 0 & * \\ 0 & B \end{pmatrix}$$

where B is any $n \times n$-matrix with $tr(B) = 0$. Let $x_i \in \mathfrak{g}$ be the following element: Let the first column vector of x_i be the vector e_i and let the rest of the entries be such that $tr(x_i) = 0$. It follows $x_i \otimes e_0 \in \mathfrak{g} \otimes L$ and $x_i(e_0) = e_i$ hence the vertical map is surjective. One easily checks the sequence is exact and 3.3.5 is proved.

We get two P-modules: $\mathfrak{p}_L \subseteq \mathfrak{p}$ and $L^i = \mathrm{Sym}^i(\wedge^m W) \subseteq \mathrm{Sym}^i(\wedge^m V)$. We get for all $1 \leq k \leq d$ a P-module

$$L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

There is an injection of P-modules

$$i : L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to \mathrm{Sym}^d(\wedge^m V)$$

defined by

$$i(L^{d-k} \otimes \overline{x_1} \otimes L \cdots \overline{x_k} \otimes L) = L^{d-k} x_1(L) \cdots x_k(L).$$

There are natural embeddings of P-modules

$$U_k(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V).$$

Assume in the following $m = 1$ and $L = W$. It follows $\mathbb{G} = \mathcal{P}(V^*) = \mathcal{P}$ is projective n-space.

Proposition 3.4. *Let* $x_1 \cdots x_k(L^d) \in U_k(\mathfrak{g})L^d$. *The following formula holds:*

$$x_1 \cdots x_k(L^d) = \alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega$$

where $\omega \in L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$.

Proof. we prove the result by induction on k. Assume k=1 and let $x(L^d) \in \mathrm{U}_1(\mathfrak{g})L^d$. It follows $x(L^d) = dL^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ and the claim holds for k =1. Assume the result is true for k. Hence

$$x_1 \cdots x_k(L^d) = \alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega$$

with $\omega \in L^{d-(k-1)} \otimes \mathrm{Sym}^{k-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. Assume

$$\omega = \sum_i \alpha_i L^{d-(k-1)} x_1^i(L) \cdots x_{k-1}^i(L).$$

We get

$$x_0 x_1 \cdots x_k(L^d) = x_0(\alpha L^{d-k} x_1(L) \cdots x_k(L) + \omega) =$$

$$\alpha(d-k)L^{d-(k+1)} x_0(L) x_1(L) \cdots x_k(L) +$$

$$\sum_j \alpha L^{d-k} x_1(L) \cdots x_0(x_j(L)) \cdots x_k(L) +$$

$$\sum_i \alpha_i (d-(k-1)) L^{d-k} x_0(L) x_1^i(L) \cdots x_{k-1}^i(L) +$$

$$\sum_i \sum_l \alpha_i L^{d-(k-1)} x_1^i(L) \cdots x_0(x_l^i(L)) \cdots x_{k-1}^i(L).$$

Let $z_j(L) = x_0(x_j(L))$ and $z_l^i(L) = x\,(x_l^i(L))$. Such elements exist since $\mathfrak{g}/\mathfrak{p}_L \otimes L \cong V$ as P-module. Let

$$\omega = \sum_j \alpha L^{d-k} x_1(L) \cdots z_j(L) \cdots x_k(L) +$$

$$\sum_i \alpha_i (d-(k-1)) L^{d-k} x_0(L) \cdots x_1^i(L) \cdots x_{k-1}^i(L) +$$

$$\sum_i \sum_l \alpha_i L^{d-(k-1)} x_1^i(L) \cdots z_l^i(L) \cdots x_{k-1}^i(L).$$

it follows $\omega \in L^{d-k} \otimes \mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. Moreover

$$x_0 x_1 \cdots x_k(L^d) = \tilde{\alpha} L^{d-(k+1)} x_0(L) \cdots x_k(L) + \omega$$

where $\tilde{\alpha} = (d-k)\alpha$. The Proposition is proved.

Theorem 3.5. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathrm{U}_l(\mathfrak{g})L^d \cong L^{d\ l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. There are embeddings of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(V)$$

and

$$L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(V).$$

Recall from studies of Maakestad [1] it follows $dim_K(\mathrm{U}_l(\mathfrak{g})L^d) = \binom{l+n}{n}$ where $dim_k(V) = n$+1. Assume $z = x_1 \cdots x_l(L^d) \in \mathrm{U}_l(\mathfrak{g})L^d$. It follows from Proposition 3.4

$$z = \alpha L^{d-l} x_1(L) \cdots x_l(L) + \omega$$

where

$$\in L^{d-(l-1)} \mathrm{Sym}^{l-1}(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Since

$$\alpha L^{d-l} x_1(L) \cdots x_l(L) \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

it follows $z \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ Hence we get an inclusion of P-modules $\mathrm{U}_l(\mathfrak{g})L^d \subseteq L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$.

Since

$$dim_K(\mathrm{U}_l(\mathfrak{g})L^d) = dim_K(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L))$$

the Theorem follows.

Corollary 3.6. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}_{\mathbb{P}}(d))(x) \cong (L^*)^{d-l} \otimes \mathrm{Sym}^1(V^*)$$

of P-modules.

Proof. There is by studies of Maakestad [1], Theorem 3.10 an isomorphism

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))(x)^* \cong \mathrm{U}_l(\mathfrak{g})L^d$$

of P-modules. From this isomorphism and Theorem 3.5 the Corollary follows since

$$(L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L))^* \cong (L^*)^{d-l} \otimes \mathrm{Sym}^l(V^*)$$

as P-modules.

Note: Corollary 3.6 is proved in literature of Maakestad [5] Theorem 2.4 using more elementary techniques.

Let $Y = \mathrm{Spec}(K)$ and $\pi : \mathcal{P}(V^*) \to Y$ be the structure morphism. Let $\mathcal{P} = \mathcal{P}(V^*)$. Since $\mathrm{Sym}^1(V^*)$ is a finite dimensional $\mathrm{SL}(V)$-module it follows it is a free $\mathcal{O}_Y$-module with an $\mathrm{SL}(V)$-linearization. It follows $\pi^*\mathrm{Sym}^1(V^*)$ is a locally free $\mathcal{O}\mathcal{P}$-module with an $\mathrm{SL}(V)$-linearization since π^* preserves the $\mathrm{SL}(V)$-linearization.

Proposition 3.7. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$$

of locally free $\mathcal{O}\mathcal{P}$*-modules with an* $\mathrm{SL}(V)$*-linearization.*

Proof. Let $P \subseteq \mathrm{SL}(V)$ be the subgroup fixing the line $L \in V$ There is an exact equivalence of categories

$$\underline{mod}(P) \cong \underline{mod}^G(\mathcal{O}_{G/P}). \tag{3.7.1}$$

The P-module corresponding to $\mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$ is $(L)^{d-l} \otimes \mathrm{Sym}^l(V^*)$. By the equivalence 3.7.1 and Corollary 3.6 we get an isomorphism

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$$

of locally free sheaves with $\mathrm{SL}(V)$-linearization and the Proposition is proved.

We get a formula for the splitting type of $\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}(d))$ on projective space:

Corollary 3.8. *There is for all* $1 \leq l \leq d$ *an isomorphism*

$$\mathcal{P}^l_{\mathbb{P}}(\mathcal{O}(d)) \cong \oplus^{\binom{n+l}{n}} \mathcal{O}_{\mathbb{P}}(d-l)$$

of locally free sheaves.

Proof. The P-modules $\mathrm{Sym}^1(V^*)$ corresponds to the free $\mathcal{O}\mathcal{P}$-module $\oplus^{\binom{n+l}{n}} \mathcal{O}_{\mathbb{P}}$. The Corollary now follows from Proposition 3.7.

Let $X = \mathbb{G}(m, m+n)$ and consider the P-modules

$$L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$\mathrm{U}_1(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V).$$

Proposition 3.9. *There is an isomorphism*

$$\mathrm{U}_1(\mathfrak{g})L^d \cong L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. Pick an element $x(L^d) = dL^{d-1}x(L) \in U_1(\mathfrak{g})L^d$. It follows $dL^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ hence there is an inclusion

$$\mathrm{U}_1(\mathfrak{g})L^d \subseteq L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Let $L^{d-1}x(L) \in L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$. It follows

$$L^{d-1}x(L) = \frac{1}{d}x(L^d) \in \mathrm{U}_1(\mathfrak{g})L^d$$

hence there is an inclusion $L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ and the Proposition is proved.

Corollary 3.10. *There is an isomorphism*

$$\mathcal{P}^1_X(\mathcal{O}_X(d))(x)^* \cong L^{d-1} \otimes \mathrm{Sym}^1(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules.

Proof. There is by studies of Maakestad [1], Theorem 3.10 an isomorphism

$$\mathcal{P}^1_X(\mathcal{O}_X(d))(x)^* \cong \mathrm{U}_1(\mathfrak{g})L^d$$

of P-modules. The Corollary follows from this fact and Proposition 5.1.

Note: By studies of Maakestad [11], Example 5.12 there is a double complex

$$\mathcal{O}_X(j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$$

of sheaves on $\mathcal{P}(W^*)$ where $W = \mathrm{H}^0(X, \mathcal{O}_X(d))$ and $X = \mathbb{G}(m, m+n)$. This double complex might give rise to a resolution of the ideal sheaf of the l'th discriminant $D^l(\mathcal{O}_X(d)) \subseteq \mathbb{P}(W^*)$ of the line bundle $\mathcal{O}_X(d)$. By the literature of Maakestad, Theorem 5.2 it follows knowledge on the P-module structure of $\mathcal{P}^l_X(\mathcal{O}_X(d))$ gives information on the $\mathrm{SL}(V)$-module structure of the higher cohomology groups $\mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$ for all $i \geq 0$. This again gives information on the dimension $h^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*)$. We get a description of the locally free sheaf

$$\mathcal{O}_X(j) \otimes \mathrm{H}^i(X, \wedge^j \mathcal{P}^l_X(\mathcal{O}_X(d))^*).$$

for all i, j.

Example 3.11. *Canonical filtration for the grassmannian* $\mathbb{G}(2,4)$.

Consider the example where $m = n = 2$ and $X = \mathbb{G}(2,4)$. We get two inclusions

$$L^{d-2} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^2 V)$$

and

$$\mathrm{U}_2(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^2 V).$$

We may choose a basis for $\mathfrak{p} \subseteq \mathfrak{g}$ on the following form:

$$\mathfrak{p} = \mathfrak{p}_L \oplus L_x$$

where L_x is the line spanned by the following vector x:

$$x = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0. \end{pmatrix}$$

Let $\mathfrak{n} \subseteq \mathfrak{g}$ be the sub Lie algebra spanned by the following vectors:

$$x_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$x_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$x_4 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

Let $\tilde{\mathfrak{n}}$ be the vector space spanned by the vectors x_1, x_2, x_4, x_4 and x. It follows $\mathrm{U}_2(\mathfrak{g})L^d = \mathrm{U}_2(\tilde{\mathfrak{n}})L^d \subseteq \mathrm{Sym}^d(\wedge^2 V)$. The vector space V has a basus e_1, e_2, e_3 and e_4. The vector space W has basis e_1, e_2. It follows $\wedge^2 W$ has a basis given by $e_1 \wedge e_2 = e[12]$ and $\wedge^2 V$ has basis given by $e[12]$, $e[13]$, $e[14]$, $e[23]$, $e[24]$, $e[34]$. By definition $L = e[12]$. We get the following calculation:

$$x_1(L) = -e[23], x_2(L) = e[13], x_3(L) = -e[24]$$

$$x_4(L) = e[14], x(L) = e[12].$$

A basis for the P-module $L^{d-2} \otimes \mathrm{Sym}^2(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ are the following vectors:

$$L^{d-2}x(L)x(L) = L^{d-2}e[12]^2$$

$$L^{d-2}x_2(L)x(L) = L^{d-2}e[12]e[13]$$

$$L^{d-2}x_4(L)x(L) = L^{d-2}e[12]e[14]$$

$$L^{d-2}x_1(L)x(L) = -L^{d-2}e[12]e[23]$$

$$L^{d-2}x_3(L)x(L) = -L^{d-2}e[12]e[24]$$

$$L^{d-2}x_2(L)x_2(L) = L^{d-2}e[13]^2$$

$$L^{d-2}x_2(L)x_4(L) = L^{d-2}e[13]e[14]$$

$$L^{d-2}x_1(L)x_2(L) = -L^{d-2}e[13]e[23]$$

$$L^{d-2}x_2(L)x_3(L) = -L^{d-2}e[13]e[24]$$

$$L^{d-2}x_4(L)x_4(L) = -L^{d-2}e[14]^2$$

$$L^{d-2}x_1(L)x_4(L) = -L^{d-2}e[14]e[23]$$

$$L^{d-2}x_3(L)x_4(L) = -L^{d-2}e[14]e[24]$$

$$L^{d-2}x_1(L)x_1(L) = L^{d-2}e[23]^2 e$$

$$L^{d-2}x_1 \quad L \quad x_3 \quad L \qquad L^{d-2}e \qquad e$$

$$L^{d-2}x_3(L)x_3(L) = L^{d-2}e[24]^2$$

Let $a = d(d-1)$. A basis for the P-module $\mathrm{U}_2(\mathfrak{g})L^d = \mathrm{U}_2(\tilde{\mathfrak{n}})L^d$ are the following vectors:

$$x^2(L^d) = L^{d-2}e[12]^2$$

$$x_2 x(L^d) = aL^{d-2}e[12]e[13] + dL^{d-1}e[13]$$

$$x_4 x(L^d) = aL^{d-2}e[12]e[14] + dL^{d-1}e[14]$$

$$x_1 x(L^d) = aL^{d-2}e[12]e[23] - dL^{d-1}e[23]$$

$$x_3 x(L^d) = aL^{d-2}e[12]e[24] - dL^{d-1}e[24]$$

$$x_2^2(L^d) = aL^{d-2}e[13]^2$$

$$x_2 x_4(L^d) = aL^{d-2}e[13]e[14]$$

$$x_1 x_2(L^d) = aL^{d-2}e[13]e[23]$$

$$x_2 x_3(l^D) = -aL^{d-2}e[13]e[24] - dL^{d-1}e[34]$$

$$x_4^2(L^d) = L^{d-2}e[14]^2$$

$$x_1 x_4(L^d) = -aL^{d-2}e[14]e[23] + dL^{d-1}e[34]$$

$$x_3 x_4(L^d) = -aL^{d-2}e[14]e[24]$$

$$x_1^2(L^d) = aL^{d-2}e[23]^2$$

$$x_1 x_3(L^d) = aL^{d-2}e[23]e[24]$$

$$x_3^2(L^d) = aL^{d-2}e[24]^2.$$

In the case where $W \subseteq V$ have dimensions m and $m + n$ we get embeddings of P-modules

$$\mathrm{U}_l(\mathfrak{g})L^d \subseteq \mathrm{Sym}^d(\wedge^m V)$$

and

$$L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \subseteq \mathrm{Sym}^d(\wedge^m V).$$

There is no equality

$$\mathrm{U}_l(\mathfrak{g})L^d = L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$$

of P-modules as submodules of $\mathrm{Sym}^d(\wedge^m V)$ in general as Example 3.11 shows.

Since $\mathrm{U}_l(\mathfrak{g})L^d$ and $L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ by Theorem 3.5 and Proposition 3.3 are isomorphic when $m = 1$ and $1 \le l \le d$, have the same dimension over K and both have natural filtrations of P-modules we may conjecture they are isomorphic as P-modules for all $m,n \ge 1$. Note: There is a canonical line $L^d \in \mathrm{U}_l(\mathfrak{g})L^d$ for all l. There is similarly a canonical line

$$L^d \cong L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{p}/\mathfrak{p}_L \otimes L) \in L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L).$$

Hence the two P-modules $\mathrm{U}_l(\mathfrak{g})L^d$ and $L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L)$ look similar.

In general the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ decompose

$$\mathrm{Sym}^d(\wedge^m V) \cong \oplus_i V_{\lambda_i}^{a_i}$$

where V_{λ_i} are irreducible $\mathrm{SL}(V)$-modules and $a_i \ge 1$ are integers (Proposition 5.4 for the situation of $\mathbb{G}(2,4)$. One may ask if there is a non-trivial automorphism

$$\phi \in \mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^m V))$$

with the property that the morphism

$$\phi : \mathrm{Sym}^d(\wedge^m V) \to \mathrm{Sym}^d(\wedge^m V)$$

induce an isomorphism

$$\tilde{\phi} : L^{d-l} \otimes \mathrm{Sym}^l(\mathfrak{g}/\mathfrak{p}_L \otimes L) \to \mathrm{U}_l(\mathfrak{g})L^d$$

of P-modules. In general the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ has lots of automorphisms. When $m = 2$ and $dim_k(V) = 4$ it follows by Corollary 5.4 there is for every $d \ge 1$ an equality

$$\mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^2 V)) = \prod_{i=0}^{l} K^*$$

where $l = k$ if $d = 2k$ or $d = 2k + 1$. For $m = n = 2$ the $\mathrm{SL}(V)$-module $\mathrm{Sym}^d(\wedge^m V)$ is by Proposition 5.4 multiplicity free. The module $\mathrm{Sym}^d(\wedge^m K^{m+n})$ is not multiplicity free in general when $m, n > 2$.

Jet Bundles and Incidence Complexes on the Projective Line

In this section we construct a resolution by locally free sheaves of the ideal sheaf of the l'th incidence scheme $I^l(\mathcal{O}_{\mathbb{P}}(d)) \subseteq \mathbb{P}(W^*)\times\mathbb{P}$. Here $\mathcal{OP}(d)$ is an invertible sheaf on the projective line $\mathcal{P} = \mathcal{P}^1$ and $W = \mathrm{H}^0(\mathbb{P},\mathcal{O}_p(d))$. There is on $Y = \mathcal{P}(W^*) \times \mathcal{P}^1$ a morphism $\varphi(\mathcal{O}(d))$ of locally free sheaves

$$\phi(\mathcal{O}(d)) : \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y$$

Its zero scheme $Z(\phi(\mathcal{O}(d))) = I^l(\mathcal{O}(d)) \subseteq Y$ is the l'th incidence scheme of $\mathcal{O}(d)$. The Koszul complex of the morphism $\varphi(\mathcal{O}(d))$

$$0 \to \wedge^l \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))^*_Y \to \cdots \to \wedge^2 \mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))^*_Y \to$$

$$\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))^*_Y \to \mathcal{O}_Y \to \mathcal{O}_{I^l(\mathcal{O}(d))} \to 0$$

- called the *incidence complex of* $\mathcal{O}(d)$ - is a resolution of the ideal sheaf of $I^l(\mathcal{O}(d))$. This follows from the fact that the ideal sheaf of $I^l(\mathcal{O}(d))$ is locally generated by a regular sequence. We also calculate the higher direct images of the terms

$$\mathcal{O}(-j)_Y \otimes \wedge^j \mathcal{P}^l(\mathcal{O}(d))^*_Y$$

appearing in the incidence complex.

The aim of the construction is to use it to construct a resolution of the ideal sheaf of the discriminant $D^l(\mathcal{O}(d))$ where $\mathcal{O}(d)$ is a line bundle on projective space or a grassmannian.

Example 4.1. *The Koszul complex of a map of locally free modules.*

Let A be an arbitrary commutative ring with unit and let $\varphi : E \to F$ be a map A-modules.

Define the following map:

$$d^0 : E \otimes_A F^* \to A$$

by

$$d^0(x \otimes f) = f(\phi(x)).$$

Let $I\ A$ be the image of d^1. We let I_φ be the *ideal of* φ. Define the following map

$$d^p : \wedge^p E \otimes F^* \to \wedge^{p-1} E \otimes F^*$$

by

$$d^p(x_1 \otimes f_1 \wedge \cdots \wedge x_p \otimes f_p) = \sum_{r=1}^{p} (-1)^{r-1} f_r(\phi(x_r)) x_1 \otimes f_1 \wedge \cdots \wedge \widetilde{x_r \otimes f_r} \wedge \cdots \wedge x_p \otimes f_p.$$

Lemma 4.2. *The following holds for all $p \geq 1$: $d^p \circ d^{p-1} = 0$.*

Proof. We get

$$d^{p-1} d^p(x_1 \otimes f_1 \wedge \cdots \wedge x_p \otimes f_p) =$$

$$\sum_{r=1}^{p} (-1)^{r-1} f_r(\phi(x_r))$$

$$\sum_{l \neq r} (-1)^{l-1} f_l(\phi(x_l)) x_1 \otimes f_1 \wedge \cdots \wedge \widetilde{x_l \otimes f_l} \wedge \cdots \wedge \widetilde{x_r \otimes f_r} \wedge \cdots \wedge x_p \otimes f_p = 0$$

and the claim of the Lemma follows.

Assume E, F are locally free of finite rank and let $r = rk(E \otimes F^*)$. We get a complex of locally free A-modules

$$0 \to \wedge^r E \otimes F^* \to \cdots \to \wedge^2 E \otimes F^* \to E \otimes F^* \to A \to A / I_\phi \to 0$$

called the *Koszul complex of the map* φ

Example 4.3. *The Koszul complex of a regular sequence.*

Let $\underline{x} = \{x_1,..,x_n\}$ be a regular sequence of elements in A and let $E = Ae$ be the free A-module on the element e. Let $F = A\{e_1, .., e_n\}$ be a free rank n module on $e_1, .., e_n$. Let $y_i = e_i^*$. Define

$$\varphi : E \to F$$

by

$$\phi(e) = x_1 e_1 + \cdots + x_n e_n.$$

Let $e \otimes y_i = z_i$. It follows

$$d^p : \wedge^p E \otimes F^* \to \wedge^{p-1} E \otimes F^*$$

looks as follows:

$$d^p(z_{i_1} \wedge \cdots \wedge z_{i_p}) =$$

$$\sum_{r=1}^{p} (-1)^{p-1} y_{i_r}(\phi(e)) z_{i_1} \wedge \cdots \wedge \widetilde{z_{i_r}} \wedge \cdots \wedge z_{i_p} =$$

$$\sum_{r=1}^{p} (-1)^{r-1} x_{i_r} z_{i_1} \wedge \cdots \wedge \widetilde{z_{i_r}} \wedge \cdots \wedge z_{i_p}.$$

Hence the complex $\wedge^{\bullet} E \otimes F^*$ equals the Koszul complex $K_{\bullet}(\underline{x})$ of the regular sequence $\underline{x}$. It is an exact complex since $\underline{x}$ is a regular sequence.

Example 4.4. *The Koszul complex of a morphism of locally free sheaves.*

The construction of the differential in the Koszul complex of a map of modules is intrinsic, hence we may generalize to morphisms of locally free sheaves. Let Y be an arbitrary scheme and let $\varphi : \mathcal{E} \to \mathcal{F}$ be a map of locally free $\mathcal{O}_Y$-modules. Let

$$d^0 : \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{O}_Y$$

be defined locally by

$$d^0(s \otimes v) = v(\phi(s)).$$

Let $\mathcal{I}_\phi = Im(d^0) \subseteq \mathcal{O}_Y$ be the ideal sheaf defined by d^1. Since $\mathcal{I}_\varphi$ is quasi coherent sheaf of ideals it follows the ideal sheaf $\mathcal{I}_\varphi$ corresponds to a subscheme $Z(\varphi) \subseteq Y$ - the *zero scheme of* φ. Let $U \subseteq Y$ be an open subset and define the following map:

$$d^p : \wedge^p(\mathcal{E} \otimes \mathcal{F}^*)(U) \to \wedge^{p-1}(\mathcal{E} \otimes \mathcal{F}^*)(U)$$

by

$$d^p(s_1 \otimes v_1 \wedge \cdots \wedge s_p \otimes v_p) = \sum_{r=1}^{p} (-1)^{r-1} v_r(\phi(s_r)) s_1 \otimes v_1 \wedge \cdots \wedge \widetilde{s_r \otimes v_r} \wedge \cdots \wedge s_p \otimes v_p.$$

This gives a well defined map of locally free sheaves since we have not chosen a basis for the module $\wedge^p(\mathcal{E} \otimes \mathcal{F}^*)(U)$ to give a definition. By Lemma 4.2 it follows $d^p \circ d^{p+1} = 0$ for all $p \geq 1$ hence we get a complex of locally free sheaves. The sequence of maps of locally free sheaves

$$0 \to \wedge^r \mathcal{E} \otimes \mathcal{F}^* \to \cdots \to \wedge^2 \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{E} \otimes \mathcal{F}^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\phi)} \to 0$$

is called the *Koszul complex of* ϕ. Here $r = rk(\mathcal{E} \otimes \mathcal{F})$.

Example 4.5. *Koszul complexes and local complete intersections.*

Assume $\phi : \mathcal{L} \to \mathcal{F}$ is a map of locally free $\mathcal{O}_Y$-modules where $\mathcal{L}$ is a line bundle. Let $Z(\phi) \subseteq Y$ be the subscheme defined by ϕ - the zero scheme of ϕ. Let $r = rk(\mathcal{F})$. Choose an open affine cover U_i of Y where $\mathcal{F}$ and $\mathcal{L}$ trivialize, i.e

$$\mathcal{F}(U_i) = \mathcal{O}(U_i)\{f_{i1},..,f_{ir}\}$$

and

$$\mathcal{L}(U_i) = \mathcal{O}(U_i)e_i.$$

Let $\mathcal{O}(U_i) = A_i$, $L_i = \mathcal{L}(U_i)$ and $F_i = \mathcal{F}(U_i)$. Assume the image

$$\phi(U_i): L_i \to F_i$$

has

$$\phi(U_i)(e_i) = x_{i1}f_{i1} + \cdots + x_{ir}f_{ir}$$

where $\{x_{i1},..,x_{ir}\} \subseteq A_i$ is a regular sequence. Let $I_i = \underline{x_i} = \{x_{i1},..,x_{ir}\}$. It follows from Example 4.3 the Koszul complex

$$0 \to \wedge^r(L_i \otimes F_i^*) \to \cdots \to \wedge^2(L_i \otimes F_i^*) \to L_i \otimes F_i^*$$

$$\to A_i \to A_i / I_i \to 0$$

is a resolution of the ideal I_i since I_i is generated by a regular sequence. The complex $\wedge^\bullet L_i \otimes F_i^*$ is isomorphic to the Koszul complex $K_\bullet(\underline{x_i})$ on the regular sequence $\underline{x_i}$. It follows the global complex

$$0 \to \mathcal{L}^{\otimes r} \wedge^r \mathcal{F}^* \to \cdots \to \mathcal{L}^{\otimes 2} \wedge^2 \mathcal{F}^* \to \mathcal{L} \otimes \mathcal{F}^* \to \mathcal{O}_Y \to \mathcal{O}_{Z(\phi)} \to 0$$

is a resolution of the ideal sheaf $\mathcal{I}_{Z(\varphi)}$ of $Z(\varphi) \subseteq Y$ since it is locally isomorphic to the Koszul complex $K_\bullet(\underline{x_i})$ for all i.

Since the ideal I_i is generated by a regular sequence of length r it follows $dim(A_i / I_i) = dim(A_i) - r$. If Y is irreducible of dimension d it follows $Z(\varphi) \subseteq Y$ is a local complete intersection of dimension $d - r$.

Example 4.6. *The incidence complex of $\mathcal{O}(d)$ on the projective line.*

Let $\mathbb{P} = \mathbb{P}^1_K$ where K is a field of characteristic zero and let $\mathcal{O}(d) \in \mathrm{Pic}(\mathcal{P}) = \mathrm{Z}$ be a line bundle where $d \in \mathrm{Z}$. Let

$$W = \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) = K\{e_0,..,e_d\}$$

where $e_i = x_0^{d-i}x_1^i$. Let $y_i = e_i^*$. Let $Y = \mathcal{P}(W^*) \times \mathcal{P}$ and consider the following diagram

$$\begin{array}{ccc} Y & \xrightarrow{p} & \mathbb{P} \\ \downarrow q & & \downarrow \pi \\ \mathbb{P}(W^*) & \xrightarrow{\pi} & \mathrm{Spec}(K) \end{array} .$$

There is a sequence of locally free $\mathcal{O}_Y$-modules

$$\mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathrm{H}^0(\mathbb{P}.\mathcal{O}(d)) \otimes \mathcal{O}_Y \xrightarrow{T^l_Y} \mathcal{P}^l(\mathcal{O}(d))_Y$$

and let φ ($\mathcal{O}(d)$) be the composed map

$$\phi(\mathcal{O}(d)): \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y. \qquad (12)$$

It follows by studies of Maakestad [11], the zero scheme $Z(\varphi(\mathcal{O}(d)))$ equals the incidence scheme $I^l(\mathcal{O}(d))$ of the line bundle $\mathcal{O}(d)$. By definition $\mathbb{P}(W^*) = \mathrm{Proj}(K[y_0,..,y_d])$ where $y_i = e_i^*$. It has an open cover on the following form: $D(y_i) = \mathrm{Spec}(K[u_0,..,u_d])$ where we let $u_j = \frac{y_j}{y_i}$. Let $y_j / y_j = 1$. Let

$$F(t) = u_0 + u_1t + \cdots + u_dt^d \in K[u_0,..,u_d,t].$$

Restrict the map 4.6.1 to the open set $U_{i0} = D(y_i) \times D(x_0) \subseteq Y$. We get the following two maps of modules:

$$\alpha: \mathcal{O}_{\mathbb{P}(W^*)}(-1)|_{U_{i0}} \to \mathcal{O}_{U_{i0}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d))$$

$$\alpha: K[y_i,t]\frac{1}{y_i} \to K[u_i,t] \otimes K\{e_0,..,e_d\}$$

defined by

$$\alpha(1/y_i) = \sum_{k=0}^d u_k \otimes e_k = \sum_{k=0}^d u_k \otimes x_0^{d-k}x_1^k = \sum_{k=0}^d u_k \otimes t^k x_0^d.$$

We get the map

$$T^l_{U_{i0}}: \mathcal{O}_{U_{i0}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) \to \mathcal{P}^l(\mathcal{O}(d))|_{U_{i0}}$$

defined by

$$T^l(1 \otimes x_0^{d-i}x_1^i) = T^l(1 \otimes t^i x_0^d) = (t + dt)^i \otimes x_0^d.$$

The composed map

$$\phi(\mathcal{O}(d))_{U_{i0}}: K[u_i,t]\frac{1}{y_i} \to K[u_i,t]\{dt^j \otimes x_0^d\}$$

is the map

$$\phi(\mathcal{O}(d))(\frac{1}{y_i}) = \sum_{k=0}^d u_k(t + dt)^k \otimes x_0^d =$$

$$\sum_{k=0}^l \frac{F^{(k)}(t)}{k!} dt^k \otimes x_0^d \in K[u_i,t]\{1 \otimes x_0^d,..,dt^l \otimes x_0^d\}.$$

Let $U_{i1} = D(y_i) \times D(x_1) \subseteq Y$ and let $\frac{x_0}{x_1} = s$. Let

$$G(s) = u_d + u_{d-1}s + u_{d-2}s^2 + \cdots + u_0s^d \in K[u_0,..,u_d,s].$$

Restrict the map 4.6.1 to the open set U_{i1}

We get the following two maps of modules:

$$\alpha: \mathcal{O}_{\mathbb{P}(W^*)}(-1)|_{U_{i1}} \to \mathcal{O}_{U_{i1}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d))$$

$$\alpha: K[y_i,s]\frac{1}{y_i} \to K[u_i,s] \otimes K\{e_0,..,e_d\}$$

defined by

$$\alpha(1/y_i) = \sum_{k=0}^d u_k \otimes e_k = \sum_{k=0}^d u_k \otimes x_0^{d-k}x_1^k = \sum_{k=0}^d u_k \otimes s^{d-k}x_1^d.$$

We get the map

$$T^l_{U_{i1}}: \mathcal{O}_{U_{i1}} \otimes \mathrm{H}^0(\mathbb{P}, \mathcal{O}(d)) \to \mathcal{P}^l(\mathcal{O}(d))|_{U_{i1}}$$

defined by

$$T^l(1 \otimes x_0^{d-i}x_1^i) = T^l(1 \otimes s^{d-i}x_1^d) = (s + ds)^{d-i} \otimes x_1^d.$$

The composed map

$$\phi(\mathcal{O}(d))_{U_{i1}}: K[u_i,s]\frac{1}{y_i} \to K[u_i,s]\{ds^j \otimes x_1^d\}$$

is the map

$$\phi(\mathcal{O}(d))(\frac{1}{y_i}) = \sum_{k=0}^d u_{d-k}(s + ds)^k \otimes x_1^d =$$

$$\sum_{k=0}^l \frac{G^{(k)}(s)}{k!} ds^k \otimes x_1^d \in K[u_i,s]\{1 \otimes x_1^d,..,ds^l \otimes x_1^d\}.$$

It follows the ideal sheaf $\mathcal{I}_{I^l(\mathcal{O}(d))}$ of $I^l(\mathcal{O}(d))$ is generated by

$$\{\frac{F^{(l)}(t)}{l!}, \frac{F^{(l-1)}(t)}{(l-1)!},..,F(t)\}$$

on U_{i0} and by

$$\{\frac{G^{(l)}(s)}{l!}, \frac{G^{(l-1)}(s)}{(l-1)!},..,G(s)\}$$

on U_{i1}. Let $z_i = \frac{F^{(i)}(t)}{(i)!}$ and $w_i = \frac{G^{(i)}(s)}{(i)!}$ for $i = 0, .., l$.

Lemma 4.7. *Assume B is a commutative ring of characteristic zero and let*

$$f(t) = a_0 + a_1t + \cdots + a_dt^d \in B[t]$$

be an arbitrary degree d polynomial with $a_d \neq 0$. Let $f^{(i)}(t)$ denote the formal derivative with respect to t. It follows

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$$\frac{f^{(k)}(t)}{l!} = \sum_{i=k}^{d} \binom{i}{k} a_i t^{i-k}.$$

Proof. The proof is by induction. It is clearly true for $l = 1$. Assume it is true for $l > 1$. Consider $k = l + 1$. We get

$$\frac{f^{(l+1)}(t)}{(l+1)!} = \frac{1}{l+1}\frac{\partial}{\partial_t}\frac{f^{(l)(t)}}{l!} =$$

$$\frac{1}{l+1}(\binom{l+1}{l}a_{l+1} + \binom{l+2}{l}2a_{l+2}t + \cdots + \binom{d}{l}(d-l)a_d t^{d-(l+1)}) =$$

$$\binom{l+1}{l+1}a_{l+1} + \binom{l+2}{l+1}a_{l+2}t + \cdots + \binom{d}{l+1}a_d t^{d-(l+1)} =$$

$$\sum_{i=l+1}^{d}\binom{i}{l+1}a_i t^{i-(l+1)}$$

and the claim of the Lemma follows.

Lemma 4.8. *The sequence $\{z_l, .., z_0\}$ is a regular sequence in $K[u_i, t]$. The sequence $\{w_l, .., w_0\}$ is a regular sequence in $K[u_i, s]$.*

Proof. Let $z_i = \frac{F^{(i)}(t)}{i!}$ and $w_j = \frac{G^{(j)}}{j!}$. Assume $l < i$ and consider the sequence $z_l, z_{l-1}, .., z_0 \subseteq A[t] = K[u_0, .., u_d][t]$. Since $A[t]$ is a domain it follows z_l is a non zero divisor in $A[t]$. We see from Lemma 4.7

$$A[t]/w_l \cong K[u_0, .., u_{l-1}, u_{l+1}, .., u_d, t]$$

which is a domain, hence w_{l-1} is a non zero divisor in $A[t]/w_l$. By induction it follows $z_l, .., z_0$ is a regular sequence in $A[t]$. Assume $i \leq l$. It follows the sequence $z_l, .., z_{i+1}$ is a regular sequence in $A[t]$. We see from Lemma 4.7 z_l is non zero in

$$A[t]/(z_l, .., z_{i+1}) = K[u_0, .., u_i, u_{l+1}, .., u_d, t]$$

and $K[u_0, .., u_i, u_{l+1}, .., u_d, t]$ is a domain. It follows z_i is a non zero divisor in $A[t]/(z_l, .., z_{i+1})$. It follows $z_l, .., z_0$ is a regular sequence in $A[t]$ and the claim follows. A similar argument proves $w_l, .., w_0$ is a regular sequence in $A[s]$ and the Lemma is proved.

One may prove using similar methods for any permutation $\sigma \in S_{l+1}$ the sequences

$$z_{(l)}, .., z_{\sigma(0)}$$

and

$$w_{(l)}, .., w_{\sigma(0)}$$

are regular sequences.

It follows the ideal sheaf $\mathcal{I}_{I^l(\mathcal{O}(d))}$ is locally generated by a regular sequence.

The morphism

$$\phi(\mathcal{O}(d)) : \mathcal{O}_{\mathbb{P}(W^*)}(-1)_Y \to \mathcal{P}^l(\mathcal{O}(d))_Y$$

gives by Example 4.3 rise to a Koszul complex

$$\wedge^{\bullet}\mathcal{O}_{\mathbb{P}(W^*)}(-1) \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*$$

of locally free sheaves of $Y = \mathcal{P}(W^*) \times \mathcal{P}^1$.

Definition 4.9. Let the complex

$$0 \to \wedge^l\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \cdots \to \wedge^2\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \quad (4.9.1)$$
$$\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^* \to \mathcal{O}_Y \to \mathcal{O}_{I^l(\mathcal{O}(d))} \to 0$$

be the *incidence complex of* $\mathcal{O}(d)$.

Since the ideal sheaf of $I^l(\mathcal{O}(d))$ by the discussion above is locally generated by a regular sequence it follows from Example 4.3 the complex 4.9.1 is a resolution.

In framework of Maakestad [5], Theorem 5.10 one calculates the higer direct images

$$\mathrm{R}^i q_*(\wedge^j\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*)$$

for all i, j. We get the following calculations:

Let $V = K\{e_0, e_1\}$ and $\mathcal{P} = \mathcal{P}(V^*)$. Let $W = \mathrm{H}^0(\mathcal{P}, \mathcal{O}(d)) = \mathrm{Sym}^d(V^*)$ and consider the diagram

$$\begin{tikzcd} Y = \mathbb{P}(W^*) \times \mathbb{P} \arrow[r, "p"] \arrow[d, "q"] & \mathbb{P} \arrow[d, "\pi"] \\ \mathbb{P}(W^*) \arrow[r, "\pi"] & \mathrm{Spec}(K) \end{tikzcd} .$$

By the results of this paper it follows there is an isomorphism

$$\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}(d)) \cong \mathcal{O}_{\mathbb{P}}(d-l) \otimes \pi^*\mathrm{Sym}^l(V^*)$$

a sheaves with an $\mathrm{SL}(V)$-linearization. We get

$$\wedge^j\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}_{\mathbb{P}}(d)) \cong \mathcal{O}_{\mathbb{P}}(j(d-l)) \otimes \pi^* \wedge^j \mathrm{Sym}^l(V^*).$$

By the equivariant projection formula for higher direct images we get

$$\mathrm{R}^i q_*(\wedge^j\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*) \cong \mathcal{O}_{\mathbb{P}(W^*)}(-j) \otimes \mathrm{H}^i(\mathbb{P}, \wedge^j\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}_{\mathbb{P}}(d))^*).$$

Let

$$\pi : \mathcal{P} \to \mathrm{Spec}(K).$$

It follows

$$\wedge^j\mathcal{P}_{\mathbb{P}}^l(\mathcal{O}_{\mathbb{P}}(d))^* \cong \mathcal{O}_{\mathbb{P}}(j(l-d)) \otimes \pi^* \wedge^j \mathrm{Sym}^l(V^*).$$

We get

$$\mathrm{H}^i(\mathbb{P}, \wedge^j\mathcal{P}^l(\mathcal{O}(d))^*) \cong \mathrm{R}^i\pi_*(\pi^*(\wedge^j\mathrm{Sym}^l(V)) \otimes \mathcal{O}_{\mathbb{P}}(j(l-d))) \cong$$

$$\wedge^j(\mathrm{Sym}^l(V)) \otimes \mathrm{H}^i(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(j(l-d))).$$

We get the following Theorem:

Theorem 4.10. *The following holds:*

$$\mathrm{R}^i p_*(\mathcal{O}(-j) \otimes \wedge^j\mathcal{P}^l(\mathcal{O}(d))^*) = 0 \; if \; i = 0 \; or \; i = 1 \; and \; j(d-l) < 2.$$

$$\mathrm{R}^1 p_*(\mathcal{O}(-j) \otimes \wedge^j\mathcal{P}^l(\mathcal{O}(d))^*) = \mathcal{O}(-j) \otimes \mathrm{Sym}^{j(d-l)-2}(V) \otimes \wedge^j\mathrm{Sym}^l(V)$$

if $j(d-l) \geq 2$.

Proof. The proof follows from the calculation of the equivariant cohomology of line bundles on projective space [13].

Hence we have complete control on the sheaf

$$\mathrm{R}^i q_*(\wedge^j\mathcal{O}(-1)_Y \otimes \mathcal{P}^l(\mathcal{O}(d))_Y^*)$$

on the projective line and projective space for all i, j. Using the techniques introduced in this paper one may describe resolutions of incidence schemes $I^l(\mathcal{O}(d))$ on more general grassmannians and flag varieties. The hope is we may be able to construct resolutions of the ideal sheaf of $D^l(\mathcal{O}(d))$ using indicence resolutions in a more general situation.

Note: In literature of Lascoux [12] resolutions of ideal sheaves of determinantal schemes are studied and much is known on such resolutions. In studies of Maakestad [11] it is proved $D^l(\mathcal{O}(d)$ is a determinantal scheme for any $d \geq 2$ on the projective line $\mathcal{PP}^1$. Assume $\mathcal{L} \in \mathrm{Pic}^G(G/P)$ is a G-linearized linebundle, G a semi simple linear algebraic group and P a parabolic subgroup. If one can prove $D^l(\mathcal{L})$ is a determinantal scheme we get two approaches to the study of resolutions of ideal sheaves of discriminants: One using jet bundles and incidence schemes, another one using determinantal schemes.

Appendix A: Automorphisms of Representations

Let $W \subseteq V$ be vectorspaces of dimension two and four over the field K. Consider the subgroup $P \subseteq G = \mathrm{SL}(V)$ where P is the parabolic subgroup of elements fixing W. It follows $\pi : G \to G/P = \mathbb{G}(2,4)$ is a principal P-bundle. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$ be the Lie algebras of G and P. In this section we study the decomposition into irreducibles and automorphisms of some G-modules. We also study some P_{semi}-modules where P_{semi} is the semi-simplification of P. It follows P_{semi} equals SL(2) × SL(2). Since $\mathfrak{p} \subseteq \mathfrak{g}$ is a P-sub module it follows the quotient $\mathfrak{g}/\mathfrak{p}$ is a P-module hence a P_{semi} module. We may apply the theory of highest weights since $P_{semi} = \mathrm{SL}(2) \times \mathrm{SL}(2)$ is a semi simple algebraic group.

Proposition 5.1. *The following hold: There is an isomorphism of* SL(2) × SL(2)*-modules*

$$\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) = \oplus_{i=0}^n \mathrm{Sym}^{2i+m}(W^*) \otimes \mathrm{Sym}^{2i+m}(V/W). \tag{5.1.1}$$

for all $k \geq 1$. *Here* $(n,m) = (-,0)$ *if* $k = 2n$ *and* $(n,m) = (\frac{k-1}{2},1)$ *if* $k = 2n + 1$.

Proof. Recall the canonical isomorphism from Lemma 2.4

$$\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(W, V/W) \cong W^* \otimes V/W$$

of P-modules. It follows

$$\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) \cong \mathrm{Sym}^k(W^* \otimes V/W)$$

and its decomposition into irreducible SL(2) × SL(2)-modules can be done using well known formulas [14]. Alternatively one may compute its highest weight vectors and highest weights explicitly using the construction from Section 5.

Let $i : G/P \to \mathcal{P}(\wedge^2 V^*) = \mathcal{P}$ be the Plucker embedding and let $\mathcal{O}_{G/P}(1) = i^*\mathcal{OP}(1)$ be tautological line bundle on G/P and let $\mathcal{O}_{G/P}(d) = \mathcal{O}_{G/P}(1)\otimes^d$. It follows from the Borel-Weil-Bott Theorem [16] $\mathrm{H}^0(\mathbb{G}, \mathcal{O}\mathbb{G}(d))$ is an irreducible SL(V)-module. Let V have basis e_1, e_2, e_3, e_4 and let $\wedge^2 V$ have basis e_{ij} for $1 \leq i \leq j \leq 4$, with $e_{ij} = e_i \wedge e_j$. Consider the element $f \in \mathrm{Sym}^2(\wedge^2 V)$ where

$$f = e_{12}e_{34} - e_{13}e_{24} + e_{14}e_{23}.$$

One checks f is a highest weight vector for SL(V) with highest weight 0, hence it defines the unique trivial character of SL(V). Its dual

$$f^* = x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} \in \mathrm{Sym}^2(\wedge^2 V^*)$$

is the defining equation for $\mathbb{G} = G/P$ as closed subscheme of $\mathcal{P}(\wedge^2 V^*)$.

Proposition 5.2. *The following hold: there is an isomorphism of* SL(V)*-modules*

$$\mathrm{Sym}^d(\wedge^2 V) = \oplus_{i=0}^l \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(d-2i))^*, \tag{5.2.1}$$

where $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$.

Proof. The result is proved using the theory of highest weights. There is a split exact sequence of SL(V)-modules

$$0 \to f^*\mathrm{Sym}^{d-2}(\wedge^2 V^*) \to \mathrm{Sym}^d(\wedge^2 V^*) \to \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(d)) \to 0.$$

Dualize this sequence to get the split exact sequence

$$0 \to f\,\mathrm{Sym}^{d-2}(\wedge^2 V) \to \mathrm{Sym}^d(\wedge^2 V) \to Q_d \to 0.$$

where $Q_d = \mathrm{H}^0(\mathbb{G}, \mathcal{O}\mathbb{G}(d))^*$. Since f is the trivial character it follows there is an isomorphism

$$f\,\mathrm{Sym}^d(\wedge^2 V) \cong \mathrm{Sym}^d(\wedge^2 V)$$

of SL(V)-modules. By the Borel-Weil-Bott Theorem it follows Q_d is an irreducible SL(V)-module. If $d = 2k$ we get by induction the equality

$$\mathrm{Sym}^d(\wedge^2 V^*) = Q_d \oplus Q_{d-2} \oplus \cdots \oplus Q_2 \oplus Q_0,$$

and the claim of the Proposition is proved in the case where $d = 2k$. The claim when $d = 2k + 1$ follows by a similar argument and the Proposition is proved.

Corollary 5.3. *Let* $\mathcal{E} = \oplus_{i=0}^l \mathcal{O}_{\mathbb{G}}(2i-d)$ *where* $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$. *It follows*

$$\mathrm{H}^0(\mathbb{G}, \mathcal{E}) \cong \mathrm{Sym}^d(\wedge^2 V^*)$$

as SL(V)-module.

Proof. We get by Proposition 5.4 isomorphisms of SL(V)-modules

$$\mathrm{H}^0(\mathbb{G}, \mathcal{E}) \cong \mathrm{H}^0(\mathbb{G}, \oplus_{i=0}^l \mathcal{O}_{\mathbb{G}}(d-2i)) \cong$$

$$\oplus_{i=0}^l \mathrm{H}^0(\mathbb{G}, \mathcal{O}_{\mathbb{G}}(d-2i)) \cong \mathrm{Sym}^d(\wedge^2 V)^* \cong \mathrm{Sym}^d(\wedge^2 V^*)$$

and the Corollary is proved.

Corollary 5.4. *There is for every* $d \geq 1$ *an equality*

$$\mathrm{Aut}_{\mathrm{SL}(V)}(\mathrm{Sym}^d(\wedge^2 V)) = \prod_{i=0}^l K^*$$

where $l = k$ *if* $d = 2k$ *or* $d = 2k + 1$.

Proof. This follows from Proposition 5.4 and the Borel-Weil-Bott theorem (BWB). From the BWB theorem it follows $\mathrm{H}^0(\mathbb{G}, \mathcal{OO}\mathbb{G}(d))^*$ is an irreducible SL(V)-module for all $d \geq 1$. From this and Proposition 5.4 the claim of the Corollary follows.

Hence the SL(V)-module $\mathrm{Sym}^d(\wedge^2 V)$ is a multiplicity free SL(V)-module for all $d \geq 1$. This is not true in general for $\mathrm{Sym}^d(\wedge^m K^{m+n})$ when $m, n > 2$.

In general if $\mathbb{S}_\lambda$ and $\mathbb{S}_\mu$ are two *Schur-Weyl modules* [14] there is a decomposition

$$\mathbb{S}_\lambda(\mathbb{S}_\mu(V)) \cong \oplus_i V_{\lambda_i}$$

where V_{λ_i} is an irreducible SL(V)-module for all i. It is an open problem to calculate this decomposition for two arbitrary partitions λ and μ.

Appendix B: The Cauchy Formula

We include in this section an elementary discussion of the Cauchy formula using multilinear algebra. Let $W \subseteq V$ be vector spaces of dimension m and $m + n$ over K and let $P \subseteq \mathrm{SL}(V)$ be the subgroup fixing W. Let $\mathfrak{g} = Lie(G)$ and $\mathfrak{p} = Lie(P)$. There is a canonical isomorphism

$$\mathfrak{g}/\mathfrak{p} \cong \mathrm{Hom}(W, V/W)$$

of P-modules, hence the elements of $\mathfrak{g}/\mathfrak{p}$ may be interpreted as linear maps. The symmetric power $\mathrm{Sym}^k(\mathfrak{g}/\mathfrak{p}) \cong \mathrm{Sym}^k(\mathrm{Hom}(W, V/W))$ is a P-module hence a $P_{semi} = \mathrm{SL}(m) \times \mathrm{SL}(n)$-module and we want to give an explicit construction of its highest weight vectors as P_{semi}-module.

Proposition 6.1. *Let* $U = K^m$. *There is a canonical map of* SL(V)*-modules*

$$\wedge^m(U^*)\otimes\wedge^m U\to \mathrm{Sym}^m(\mathrm{Hom}(U,U))$$

defined by

$$x_1\wedge\cdots\wedge x_m\otimes e_1\wedge\cdots\wedge e_m\to\begin{vmatrix} x_1\otimes e_1 & x_1\otimes e_2 & \cdots & x_1\otimes e_m\\ x_2\otimes e_1 & x_2\otimes e_2 & \cdots & x_2\otimes e_m\\ x_m\otimes e_1 & x_m\otimes e_2 & \cdots & x_m\otimes e_m.\end{vmatrix}$$

Here e_1, .., e_m is a basis for U and x_1, .., x_m is a basis for U^*.

Proof. The proof is left to the reader as an exercise.

Note: in Proposition 6.1 the element $x_i\otimes e_j$ is an element of $U^*\otimes U$ = Hom(U,U). Hence the determinant

$$\begin{vmatrix} x_1\otimes e_1 & x_1\otimes e_2 & \cdots & x_1\otimes e_m\\ x_2\otimes e_1 & x_2\otimes e_2 & \cdots & x_2\otimes e_m\\ x_m\otimes e_1 & x_m\otimes e_2 & \cdots & x_m\otimes e_m.\end{vmatrix}$$

may be interpreted as a polynomial of degree m in the elements $x_i\otimes e_j$, hence it is an element of $\mathrm{Sym}^m(\mathrm{Hom}(U,U))$.

Let $B\subseteq \mathrm{SL}(m,K)\times\mathrm{SL}(n,K)\subseteq\mathrm{SL}(V)=\mathrm{SL}(m+n,K)$ be the following subgroup: B consists of matrices with determinant one of the form

$$\begin{pmatrix} U_1 & 0\\ 0 & U_2\end{pmatrix}$$

where

$$U_1=\begin{pmatrix} a_{11} & 0 & \cdots & 0\\ a_{21} & a_{22} & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mm}\end{pmatrix}$$

and

$$U_2\begin{pmatrix} b_{11} & 0 & \cdots & 0\\ b_{21} & b_{22} & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ b_{n1} & 0 & b_{n2}\cdots & b_{nn}\end{pmatrix}.$$

Let T be a B-module and $v\in T$ a vector with the property that for all $x\in B$ it follows

$$xv=\lambda(x)v$$

where $\lambda\in(\mathrm{Hom}(B,K^*)$ is a character of B. It follows v is a highest weight vector for T as $\mathrm{SL}(m,K)\times\mathrm{SL}(n,K)$-module. The group $B\subseteq\mathrm{SL}(V)$ defines filtrations of W and V/W as follows: Let W have basis e_1, .., e_m and V have basis e_1, .., e_m, f_1, .., f_n. Let $W_1=\{e_m\}$, $W_2=\{e_m, e_{m-1}\}$, and

$W_i=\{e_m, ..e_{m-i+1}\}$.

It follows we get a filtration

$$0=W_0\subseteq W_1\subseteq\cdots\subseteq W_{m-1}=W$$

of the vector space W. Let

$$U_j=W_{m-1}\cup\{f_n,..,f_{n-j+1}\}$$

and let $V_i=(V/W)/U_{n-i}$. We get a surjection

$$V/W\to V_i$$

for $i=1,..,n-1$. It follows $\dim W_i=\dim V_i=d_i$ for all i. Let $x: W\to V/W$ be a linear map of vector spaces. We get an induced map

$$x_i: W_i\to V_i$$

wich is a square d_i matrix for all i. Let $g\in B$ be the element

$$\begin{pmatrix} G_1 & 0\\ 0 & G_2\end{pmatrix}$$

where

$$G_1=\begin{pmatrix} a_1 & 0 & \cdots & 0\\ & a_2 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ & * & \cdots & a_m\end{pmatrix}$$

and

$$G_2\begin{pmatrix} b_1 & 0 & \cdots & 0\\ & b_2 & \cdots & 0\\ \vdots & \vdots & \cdots & \vdots\\ & 0 & \cdots & b_n\end{pmatrix}.$$

The i'th wedge product

$$|x_i|=\wedge^i x_i\in\mathrm{Hom}(\wedge^i W_i,\wedge^i V_i)=\wedge^i(W_i^*)\otimes\wedge^i V_i$$

may be viewed as an element in

$$|x_i|\in\mathrm{Sym}^i(\mathrm{Hom}(W_i,V_i))\subseteq\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$$

via Proposition 6.1.

Proposition 6.2. *The following formula holds:*

$$g|x_i|=\frac{b_1\cdots b_i}{a_{m-i+1}\cdots a_m}|x_i|=\lambda(g)|x_i|$$

for all $g\in B$. Here $\lambda(g)=\frac{b_1\cdots b_i}{a_{m-i+1}\cdots a_m}$ is a character $\lambda\in\mathrm{Hom}(B,K^*)$.

Proof. The proof is left to the reader as an exercise.

Hence the i'th determinant $|x_i|\in\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$ is a highest weight vector for the $\mathrm{SL}(m)\times\mathrm{SL}(n)$-module $\mathrm{Sym}^i(\mathrm{Hom}(W,V/W))$. By the results of studies Brion [17-22], it follows the vectors $x_0^{d_0}x_1^{d_1}\cdots x_i^{d_i}$ with $\sum id_i=k$ are all highest weight vectors for the module

$$\mathrm{Sym}^k(\mathrm{Hom}(W,V/W))\cong\mathrm{Sym}^k(W^*\otimes V/W).$$

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References

1. Maakestad H (2009) On jet bundles and generalized Verma modules II. arXiv:0903: 3291.
2. Maakestad H (2008) Principal parts on the projective line over arbitrary rings. Manuscripta Math 126: 443-464.
3. Maakestad H (2009) Jet bundles on projective space. Travaux Math.
4. Maakestad H (2004) Modules of principal parts on the projective line. Ark Mat 42: 307-324.
5. Maakestad H (2004) A note on the principal parts on projective space and linear representations. Proc of the AMS 133: 349-355.
6. Mumford D, Fogarty J, Kirwan F (1994) Geometric Invariant Theory. Ergebnisse der Mathematik und ihrer Grenzgebiete, A series of modern surveys in mathematics 34.
7. Perkinson D (1996) Principal parts of line bundles on toric varieties. Compositio Math 104: 27-39.
8. Piene R, Sacchiero G (1984) Duality for rational normal scrolls. Comm in Alg 12: 1041-1066.

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9. Di Rocco S, Sommese AJ (2001) Line bundles for which a projectivized jet bundle is a product. Proc Amer Math Soc 129: 1659-1663.
10. Sommese J (1978) Compact complex manifolds possessing a line bundle with a trivial jet bundle. Abh Math Sem Univ Hamburg 47: 79-91.
11. Maakestad H (2009) Discriminants of morphisms of sheaves. arXiv: 0911.4804.
12. Lascoux A (1978) Syzygies des varietes determinantales. Advances in math 30.
13. Jantzen JC (2003) Representations of algebraic groups: Math Surveys and Monographs (2nd edn.). American Math Soc 107.
14. Fulton W, Harris J (1991) Representation theory - a first course. Graduate Texts in Mathematics, Springer Verlag.
15. Maakestad H (2008) On jet bundles and generalized Verma modules. arXiv:0 812: 2751.
16. Borel A (1991) Linear algebraic groups. Graduate Texts in Math 126.
17. Brion M (1996) Theory of invariants and geometry quotient varieties.
18. Bögvad R, Källström R (2007) Geometric interplay between function subspaces and their rings of differential operators. Trans Am Math Soc 359: 2075-2108.
19. Dixmier J (1996) Enveloping algebras. Graduate studies in mathematics, American Math Soc 11.
20. Hartshorne R (1977) Algebraic geometry. Graduate Texts in Mathematics 52.
21. Lakshmibai V, Seshadri CS (1991) Standard Monomial Theory. Proceedings of the Hyderabad Conference on Algebraic Groups 1989.
22. Manivel L (1998) Fonctions symetriques, polynomes de Schubert et lieux de degenerescence. Societe Mathematique de France.

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