Jordan $\delta$-Derivations of Associative Algebras

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Abstract

We described the structure of jordan $\delta$-derivations and jordan $\delta$-prederivations of unital associative algebras. We gave examples of nonzero jordan $\frac{1}{2}$-derivations, but not $\frac{1}{2}$-derivations.

Keywords: $\delta$-derivation; Jordan $\delta$-derivation; Associative algebra; Triangular algebras

Introduction

Let Jordan $\delta$-derivation be a generalization of the notion of jordan derivation [1,2] and $\delta$-derivation [3-14]. Jordan $\delta$-derivation is a linear mappings $j$, for a fixed element of $\delta$ from the main field, satisfies the following condition

$$j(x^2) = \delta(j(x)x + xj(x)).$$ (1)

Note that various generalizations of Jordan derivations have been widely studied [15-17]. If algebra $A$ is a (anti) commutative algebra, then jordan $\delta$-derivation of $A$ is a $\delta$-derivation of $A$.

In this paper we consider jordan $\delta$-derivations of associative unital algebras. Naturally, we are interested in the nonzero mappings with $\neq 0, 1, 1$ and algebras over field with characteristic $p \neq 2$. In the main body of work, we using the following standard notation

$$[a,b] = ab - ba, a^b = ab + ba.$$

Jordan $\delta$-derivations of Associative Algebras

In this chapter, we consider jordan $\delta$-derivations of associative unital algebras. And prove, that jordan $\delta$-derivation of simple unital associative algebra is a $\delta$-derivation. Also, we give the example of non-trivial jordan $\frac{1}{2}$-derivations.

Lemma: Let $A$ be an unital associative algebra and $j$ be a jordan $\delta$-derivation, then $\delta = \frac{1}{2}$ and $j(x) = \frac{1}{2}(x + \alpha x)$, where $[x, [x, a]] = 0$ for any $x \in A$.

Proof: Let $x = 1$ in condition (1), then $j(1) = 0$ or $\delta = \frac{1}{2}$. If $j(1) = 0$, then for $x = y + 1$ in (1), we get

$$j(y^2 - 1 + y) = \delta(j(y^2 - 1) + \frac{1}{2}j(y)(y + y \cdot j(y))).$$

That is, if $j(1) = 0$, then $j(y) = 0$.

If $\delta = \frac{1}{2}$ and $j(1) = a$, then $j(x) = \frac{1}{2}(x + \alpha x)$. Using the identity (1), obtain

$$2x^2a = (x^2a)x + x(x^2a)$$

and

$$x^2a + ax^2 = 2xax.$$

That is $[x, [x, a]] = 0$. Lemma is proved.

It is easy to see, that mapping $j(x) = \frac{1}{2}(x + \alpha x)$, where $[x, [x, a]] = 0$ for any $x \in A$, is a jordan $\frac{1}{2}$-derivation. Using Kaygorodov et al. [6] $\frac{1}{2}$-derivation of unital associative algebra $A$ is a mapping $R_\alpha$, where $R_\alpha$ - multiplication by the element in the center of the algebra $A$.

Below we give an example of an unital associative algebra with a jordan $\frac{1}{2}$-derivation, different from $\frac{1}{2}$-derivation.

Example: Consider the algebra of upper triangular matrices of size $3\times3$ with zero diagonal over a non-commutative algebra $B$. Let $A'$ be an algebra with an adjoined identity for the algebra A. Then, easy to see, that for any elements $X, Y \in A'$, right $[X, Y] = me_{ss}$, for some $m \in B$. So, for $a = t(e_{12} + e_{21})$ and $t \in B$, will be $[A^2, a] \neq 0$, but $[X, [X, a]] = 0$. So, using corollary from Lemma, mapping $j(x) = \frac{1}{2}(ax + xa)$ is a jordan $\frac{1}{2}$- derivation of algebra $A'$, but not $\frac{1}{2}$- derivation of algebra $A$.

Theorem 1: Jordan $\delta$-derivation of simple unital associative algebra $A$ is a $\delta$-derivation.

Proof: Note, that case of $\delta = 1$ was study in Herstein et al. Cusack et al. [1,2]. It is clear, that the case $\delta = \frac{1}{2}$ is more interesting. Using Herstein et al. [18],\[L = A^\delta / Z(A)\] is a simple Lie algebra. Clearly, that $[[a, x], x] = 0$ and $[[a, x], a] = 0$. Using roots system of simple Lie algebra [19], we can obtain, that $a \in Z(A)$, so $[A, A] = 0$. Which implies that the mapping $j$ is a $\delta$-derivation. Theorem is proved.

Jordan $\delta$-pre-derivations of Associative Algebras

Linear mapping $\zeta$ be a prederivation of algebra $A$, if for any elements $x, y, z \in A$:

$$\zeta(xyz) = (x)yz + x\zeta(y)z + xy\zeta(z).$$

Prederivations considered in Burde and Bajo et al. [20, 21]. Jordan $\delta$-prederivation $\zeta$ is a linear mapping, satisfies the following condition

$$\zeta(x^3) = \delta(\zeta(x)x) + x(\zeta(x)x + xx\zeta(x)).$$ (2)

The main purpose of this section is showing that Jordan $\delta$-prederivation of unital associative algebra is a jordan derivation or
jordan $\frac{1}{2}$ - derivation.

Theorem 2: Let ζ be a jordan $\delta$-prederivation of unital associative algebra $A$, then $\zeta$ is a jordan $\frac{1}{2}$ - derivation or jordan derivation.

Proof: Note, that if $\zeta$ is a jordan $\delta$-prederivation, then $\zeta(1)=3\delta(1)$. So, $\zeta(1)=0$ or $\delta=-\frac{1}{3}$. If $\delta=-\frac{1}{3}$, then $\zeta(x^3+3x^2+3x+1)=\frac{1}{3}(x^2+2x+1)\zeta(x^2+2x+1)(x+1)+\zeta(x+1)(x^2+2x+1).$

That is, we have $9\zeta(x^2)+6\zeta(x)=3x^2\zeta(x)+3\zeta(1)x^2-\zeta(x)+x^2\zeta(x)+\zeta(x)$. Replace $x$ by $x+1$, then obtain $2\zeta(x)=x^2\zeta(x)=x^2a$.

So, using (2), we obtain $x^3a=\frac{1}{3}(x^2a)^3+x(x^2a)x+(x^2a)x^2$.

That is $x^3a+ax^3=2ax+3aax^2$.

We easily obtain $[x^3,[x,a]]=0$.

Replace $x$ by $x+1$, then obtain $[x,[x,a]]=0$. Using Lemma, we obtain that $\zeta$ is a jordan $\frac{1}{2}$-derivation. The case $\zeta(1)=0$ is treated similarly, and the basic calculations are omitted. In this case, we obtain $\zeta$ is a jordan derivation (for $\delta=1$) or zero mapping. Theorem is proved.

### Jordan $\delta$-derivations of Triangular Algebras

Let $A$ and $B$ be unital associative algebras over a field $R$ and $M$ be an unital $(A, B)$-bimodule, which is a left $A$-module and right $B$-module. The $R$-algebra

$T=\text{Tri}(A,B,M) = \left\{ \left( \begin{array}{cc} a & m \\ 0 & b \end{array} \right) \mid a \in A, b \in B, m \in M \right\}$

under the usual matrix operations will be called a triangular algebra. This kind of algebras was first introduced by Chase [22]. Actively studied the derivations and their generalization to triangular algebras [15-17, 23].

Triangular algebra is an unital associative algebra and triangular algebras satisfy the conditions of Lemma. So, if $j$ is a jordan $\delta$-derivation of algebra $T$, then $\delta=\frac{1}{2}$ and there is $C$, which $j(X)=-\frac{1}{2}(CX+XC)$, where $C(\begin{array}{cc} a & m \\ 0 & b \end{array})$ for any $X \in T$.

Also, $\left[ \left( \begin{array}{cc} a & m \\ 0 & b \end{array} \right), \left( \begin{array}{cc} x & m \\ 0 & y \end{array} \right) \right]=0$

for any $x \in A, y \in B, m \in M$. Easy to see, mapping $j_A: A \rightarrow A$, satisfying condition $j_A(x)=\frac{1}{2}(ax+xa)$ and $j_B: B \rightarrow B$, satisfying condition $j_B(x)=\frac{1}{2}(bx+xb)$, are jordan $\frac{1}{2}$-derivations, respectively, of algebras $A$ and $B$. Also, for $m=0, y=0$ and $x=1$, we can get $m_1=0$.

On the other hand, for $x=0$ and $y=1$, we can get $mb=am$.

Theorem 3: Let $A$ and $B$ be a central simple algebras, then jordan $\delta$-derivation of triangular algebra $T$ is a $\delta$-derivation.

Proof: $T$ is an unital algebra and we can consider case of $\delta=\frac{1}{2}$. Algebras $A$ and $B$ are central simple algebras, then $a=\alpha\cdot A$ and $b=\beta\cdot B$.

Using (4), we obtain $a=\alpha\cdot A, b=\beta\cdot B$. So, jordan $\frac{1}{2}$-derivation of $T$ is a $\frac{1}{2}$-derivation.

Theorem is proved.

Theorem 4: Let $A$ be a central simple algebra and $M$ be a faithful module right $B$-module, then jordan $\delta$-derivation of triangular $T$ is a $\delta$-derivation.

Proof: $T$ is an unital algebra and we can consider case of $\delta=\frac{1}{2}$. Algebra $A$ is a central simple algebra, then $a=\alpha\cdot A$. Using (4), we obtain $am=mb$. The module $M$ is a faithful module, we have $b=\alpha\cdot B$. So, $\frac{1}{2}$-derivation is a $\frac{1}{2}$-derivation. Theorem is proved.

Comment: Noted, using the example if non-trivial jordan $\frac{1}{2}$-derivation, but not $\frac{1}{2}$-derivation, of unital associative algebra, we can construct new example of non-trivial jordan $\frac{1}{2}$-derivation of triangular algebra. For example, we can consider triangular algebra $T_{\text{Tri}}(A^h, A^h, B^h)$, where $A^h$ is a bimodule over $A^h$. In conclusion, the author expresses his gratitude to Prof. Pavel Kolesnikov for interest and constructive comments.

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