

Jordan δ -Derivations of Associative Algebras

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Abstract

We described the structure of jordan δ -derivations and jordan δ -prederivations of unital associative algebras. We gave examples of nonzero jordan $\frac{1}{2}$ -derivations, but not $\frac{1}{2}$ -derivations.

Keywords: δ -derivation; Jordan δ -derivation; Associative algebra; Triangular algebras

Introduction

Let Jordan δ -derivation be a generalization of the notion of jordan derivation [1,2] and δ -derivation [3-14]. Jordan δ -derivation is a linear mappings j , for a fixed element of δ from the main field, satisfies the following condition

$$j(x^2) = \delta(j(x)x + xj(x)). \quad (1)$$

Note that various generalizations of Jordan derivations have been widely studied [15-17]. If algebra A is a (anti) commutative algebra, then jordan δ -derivation of A is a δ -derivation of A .

In this paper we consider jordan δ -derivations of associative unital algebras. Naturally, we are interested in the nonzero mappings with $\neq 0, 1, 1$ and algebras over field with characteristic $p \neq 2$. In the main body of work, we using the following standard notation

$$[a, b] = ab - ba, a \circ b = ab + ba.$$

Jordan δ -derivations of Associative Algebras

In this chapter, we consider jordan δ -derivations of associative unital algebras. And prove, that jordan δ -derivation of simple associative unital algebra is a δ -derivation. Also, we give the example of non-trivial jordan $\frac{1}{2}$ -derivations.

Lemma: Let A be an unital associative algebra and j be a jordan δ -derivation, then $\delta = \frac{1}{2}$ and $j(x) = \frac{1}{2}(xa + ax)$, where $[x, [x, a]] = 0$ for any $x \in A$.

Proof: Let $x = 1$ in condition (1), then $j(1) = 0$ or $\delta = \frac{1}{2}$. If $j(1) = 0$, then for $x = y + 1$ in (1), we get

$$j(y \cdot 1 + 1 \cdot y) = \delta(j(y) \cdot 1 + 1 \cdot j(y) + j(1) \cdot y + y \cdot j(1)).$$

That is, if $j(1) = 0$, then $j(y) = 0$.

If $\delta = \frac{1}{2}$ and $j(1) = a$, then $j(x) = \frac{1}{2}(xa + ax)$. Using the identity (1), obtain

$$2x^2 \circ a = (x \circ a)x + x(x \circ a)$$

and

$$x^2 a + ax^2 = 2xax.$$

That is $[x, [x, a]] = 0$. Lemma is proved.

It is easy to see, that mapping $j(x) = \frac{1}{2}(xa + ax)$, where $[x, [x, a]] = 0$ for any $x \in A$, is a jordan $\frac{1}{2}$ -derivation. Using Kaygorodov et al. [6] $\frac{1}{2}$ -derivation of unital associative algebra A is a mapping R_a , where R_a -

multiplication by the element in the center of the algebra A .

Below we give an example of an unital associative algebra with a Jordan $\frac{1}{2}$ -derivation, different from $\frac{1}{2}$ -derivation.

Example: Consider the algebra of upper triangular matrices of size 3×3 with zero diagonal over a non-commutative algebra B . Let $A^\#$ be an algebra with an adjoined identity for the algebra A . Then, easy to see, that for any elements $X, Y \in A^\#$, right $[X, Y] = me_{13}$ for some $m \in B$. So, for $a = t(e_{12} + e_{21})$ and $t \in B$, will be $[A^\#, a] \neq 0$, but $[X, [X, a]] = 0$. So, using corollary from Lemma, mapping $j(x) = \frac{1}{2}(ax + xa)$ is a jordan $\frac{1}{2}$ -derivation of algebra $A^\#$, but not $\frac{1}{2}$ -derivation of algebra $A^\#$ and $a \notin Z(A^\#)$.

Theorem 1: Jordan δ -derivation of simple unital associative algebra A is a δ -derivation.

Proof: Note, that case of $\delta = 1$ was study in Herstein et al. Cusack et al. [1,2]. It is clear, that the case $\delta = \frac{1}{2}$ is more interesting. Using Herstein et al. [18], $L = A^{(-)} / Z(A)$ is a simple Lie algebra. Clearly, that $[[a, x], x] = 0$ and $[[x, a], a] = 0$. Using roots system of simple Lie algebra [19], we can obtain, that $a \in Z(A)$, so $[A, a] = 0$. Which implies that the mapping j is a $\frac{1}{2}$ -derivation. Theorem is proved.

Jordan δ -pre-derivations of Associative Algebras

Linear mapping ζ be a prederivation of algebra A , if for any elements $x, y, z \in A$:

$$\zeta(xyz) = (x)yz + x\zeta(y)z + xy\zeta(z).$$

Prederivations considered in Burde and Bajo et al. [20, 21]. Jordan δ -prederivation ζ is a linear mapping, satisfies the following condition

$$\zeta(x^3) = \delta(\zeta(x)xx + x\zeta(x)x + xx\zeta(x)). \quad (2)$$

The main purpose of this section is showing that Jordan δ -prederivation of unital associative algebra is a jordan derivation or

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Jordan $\frac{1}{2}$ -derivation.

Theorem 2: Let ζ be a Jordan δ -prederivation of unital associative algebra A , then ζ is a Jordan $\frac{1}{2}$ -derivation or Jordan derivation.

Proof: Note, that if ζ is a Jordan δ -prederivation, then $\zeta(1) = 3\delta\zeta(1)$. So, $\zeta(1) = 0$ or $\delta = \frac{1}{3}$. If $\delta = \frac{1}{3}$, then

$$\zeta(x^3 + 3x^2 + 3x + 1) = \frac{1}{3}(x^2 + 2x + 1)\zeta(x + 1) + (x + 1)\zeta(x + 1)(x + 1) + \zeta(x + 1)(x^2 + 2x + 1).$$

That is, we have

$$9\zeta(x^2) + 6\zeta(x) = 3x^2\zeta(x) + 3\zeta(1)x + x^2\zeta(x) + x\zeta(x).$$

Replace x by $x + 1$, then obtain

$$2\zeta(x) = x^2\zeta(1) = x^2a.$$

So, using (2), we obtain

$$x^3a = \frac{1}{3}(x^2(x^2a) + x(x^2a)x + (x^2a)x^2).$$

That is

$$x^3a + ax^3 = x^2ax + xax^2.$$

We easily obtain

$$[x^2, [x, a]] = 0.$$

Replace x by $x + 1$, then obtain $[x, [x, a]] = 0$. Using Lemma, we obtain that ζ is a Jordan $\frac{1}{2}$ -derivation. The case $\zeta(1) = 0$ is treated similarly, and the basic calculations are omitted. In this case, we obtain that ζ is a Jordan derivation (for $\delta = 1$) or zero mapping. Theorem is proved.

Jordan δ -derivations of Triangular Algebras

Let A and B be unital associative algebras over a field R and M be an unital (A, B) -bimodule, which is a left A -module and right B -module. The R -algebra

$$T = \text{Tri}(A, B, M) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in A, b \in B, m \in M \right\}$$

under the usual matrix operations will be called a triangular algebra. This kind of algebras was first introduced by Chase [22]. Actively studied the derivations and their generalization to triangular algebras [15- 17, 23].

Triangular algebra is an unital associative algebra and triangular algebras satisfy the conditions of Lemma. So, if j is a Jordan δ -derivation

of algebra T , then $\delta = \frac{1}{2}$ and there is C , which $j(X) = \frac{1}{2}(CX + XC)$, where $C = \begin{pmatrix} a & m \\ 0 & b \end{pmatrix}$ for any $X \in T$.

Also,

$$\left[\left[\begin{pmatrix} a & m \\ 0 & b \end{pmatrix}, \begin{pmatrix} x & m \\ 0 & y \end{pmatrix} \right], \begin{pmatrix} x & m \\ 0 & y \end{pmatrix} \right] = 0$$

for any $x \in A, y \in B, m \in M$. Easy to see, mapping $j_A: A \rightarrow A$, satisfying condition $j_A(x) = \frac{1}{2}(ax + xa)$ and $j_B: B \rightarrow B$, satisfying condition $j_B(x) = \frac{1}{2}(bx + xb)$, are Jordan $\frac{1}{2}$ -derivations, respectively, of algebras

A and B . Also, for $m = 0, y = 0$ and $x = 1_A$ we can get

$$m_x = 0. \tag{3}$$

On the other hand, for $x = 0$ and $y = 1_B$, we can get

$$mb = am. \tag{4}$$

Theorem 3: Let A and B be a central simple algebras, then Jordan δ -derivation of triangular algebra T is a δ -derivation.

Proof: T is an unital algebra and we can consider case of $\delta = \frac{1}{2}$. Algebras A and B are central simple algebras, then $a = \alpha \cdot 1_A$ and $b = \beta \cdot 1_B$.

Using (4), we obtain $a = \alpha \cdot 1_A, b = \alpha \cdot 1_B$. So, Jordan $\frac{1}{2}$ -derivation of T is a $\frac{1}{2}$ -derivation.

Theorem is proved.

Theorem 4: Let A be a central simple algebra and M be a faithful module right B -module, then Jordan δ -derivation of triangular T is a δ -derivation.

Proof: T is an unital algebra and we can consider case of $\delta = \frac{1}{2}$. Algebra A is a central simple algebra, then $a = \alpha \cdot 1_A$. Using (4), we obtain $am = mb$. The module M is a faithful module, we have $b = \alpha \cdot 1_B$. So, Jordan $\frac{1}{2}$ -derivation is a $\frac{1}{2}$ -derivation. Theorem is proved.

Comment: Noted, using the example if non-trivial Jordan $\frac{1}{2}$ -derivation, but not $\frac{1}{2}$ -derivation, of unital associative algebra, we can construct new example of non-trivial Jordan $\frac{1}{2}$ -derivation of triangular algebra. For example, we can consider triangular algebra $\text{Tri}(A^\#, A^\#, A^\#)$, where $A^\#$ is a bimodule over $A^\#$. In conclusion, the author expresses his gratitude to Prof. Pavel Kolesnikov for interest and constructive comments.

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