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Kaplansky's Type Constructions for Weak Bialgebras and Weak Hopf Algebras

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Abstract

In this paper, we study weak bialgebras and weak Hopf algebras. These algebras form a class wider than bialgebras respectively Hopf algebras. The main results of this paper are Kaplansky's type constructions which lead to weak bialgebras or weak Hopf algebras starting from a regular algebra or a bialgebra. Also we provide a classification of 2-dimensional and 3-dimensional weak bialgebras and weak Hopf algebras. We determine then the stabilizer group and the representative of these classes, the action being that of the linear group.

Keywords: Weak bialgebra; Weak Hopf algebra; Construction; Classification; Automorphisms group

Introduction

Motivated by quantum symmetry and field algebras, the weak coproduct was introduced first by Mack and Schomerus [1,2]. The weak bialgebras were introduced by Böhm et al. [3] with a motivation from operator algebras and quantum field theory. The weak Hopf algebras, called also quantum groupoids, appeared also in dynamic deformation theory of quantum groups [4]. Weak bialgebras and weak Hopf algebras were developed from the algebraic point of view and have been considered by several authors in various settings [5-16]. The weak bialgebras (resp. weak Hopf algebras) constitute a class wider than bialgebras (resp. Hopf algebras) where we do not require the conservation of the unit by the multiplication and the multiplicativity of the counit.

The aim of this work is to provide some constructions of finite-dimensional weak bialgebras starting from any algebras. These constructions are inspired from the Kaplansky's construction of bialgebras [17] (Theorem 1.3). Also, we show that the set of weak bialgebras (resp. weak Hopf algebras) forms an algebraic variety fibred by a linear group action. The orbits under this action correspond to isomorphic classes. Therefore, we determine up to isomorphisms all the n-dimensional weak bialgebras and weak Hopf algebras with $n \leq 3$, for which we compute the corresponding stabilizer subgroups.

In the first Section of the paper we summarize the definitions and the main properties of weak bialgebras and weak Hopf algebras. Section 2 is dedicated to Kaplansky's type constructions, we provide several constructions of weak bialgebras (resp. weak Hopf algebras) starting from any algebra or bialgebra (resp. Hopf algebra). In the last Section, we establish a classification up to isomorphism of n-dimensional weak bialgebras and weak Hopf algebras for $n \leq 3$. Then we compute their stabilizer groups.

Generalities

Throughout this paper $\mathbb{K}$ is an algebraically closed field of characteristic 0. In this Section, we review briefly the algebraic theory of weak bialgebras and weak Hopf algebras, [18-20] for bialgebras and Hopf algebras theory. Let V be a finite-dimensional $\mathbb{K}$ -vector space. In the sequel we use Sweedler's notation for a comultiplication Δ, that is $\Delta(x) = \sum_{(1)(2)} x_{(1)} \otimes x_{(2)}, \forall x \in V$. The summation sign is omitted when there is no ambiguity.

A *bialgebra* is a $\mathbb{K}$ -vector space V equipped with an algebra structure given by a multiplication m and a unit ∈ and a coalgebra structure given by a comultiplication Δ and a counit ε, such that there is a compatibility condition between these two structures expressed by the fact that Δ and ε are algebra homomorphisms, that is for $x, y \in V$

$$\Delta(m(x \otimes y)) = \Delta(x) \bullet \Delta(y) \text{ and } \varepsilon(m(x \otimes y)) = \varepsilon(x)\varepsilon(y) .$$

The multiplication $\bullet$ on $V \otimes V$ is the usual multiplication on the tensor product,

$$\left(x \otimes y\right) \bullet \left(x' \otimes y'\right) = m\left(x \otimes x'\right) \otimes m\left(y \otimes y'\right) \cdot$$

The unit η is completely determined by $\eta(1)$, which we denote by 1. It is assumed also that the unit 1 is preserved by the comultiplication, that is $\Delta(1) = 1 \otimes 1$. A bialgebra is said to be a *Hopf algebra* if the identity map on V has an inverse for the convolution product defined by

$$f \star g := m \circ (f \otimes g) \circ \Delta \cdot \tag{1.1}$$

The unit for the convolution product being $\eta \circ \varepsilon$. For simplicity, the multiplication m is denoted by a dot when there is no confusion. In the following, we recall the definition of weak bialgebra.

Definition 1.1: A weak bialgebra is a quintuple $\mathcal{B} = (V, m, \eta, \Delta, \varepsilon)$, where *m:* $V \otimes V \to V$ (multipli-cation), $\eta : \mathbb{K} \to V$ (unit), Δ: $V \to V \otimes V$ (comultiplication) and $\varepsilon : V \to \mathbb{K}$ (counit) are linear maps satisfying:

(1) The triple (V, m, η) is a unital associative algebra, that is

$$m\left(m\left(x \otimes y\right) \otimes z\right) - m\left(x \otimes m\left(y \otimes z\right)\right) = 0 \quad \forall x, y, z \in V, \tag{1.2}$$

$$m\left(x \otimes 1\right) = m\left(1 \otimes x\right) = x \quad \forall x \in V, \tag{1.3}$$

(2) The triple (V, Δ, ε) is coalgebra, that is

$$(\Delta \otimes id)\Delta(x) = (id \otimes \Delta)\Delta(x) \quad \forall x \in V, \tag{1.4}$$

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$$(\varepsilon \otimes id)\Delta(x) = (id \otimes \varepsilon)\Delta(x) = id(x) \quad \forall x \in V, \tag{1.5}$$

(3) the compatibility condition is expressed by the following three identities:

$$\Delta(m(x \otimes y)) = \sum_{(1)(2)} m(x_{(1)} \otimes y_{(1)}) \otimes m(x_{(2)} \otimes y_{(2)}) \quad \forall x, y \in V, \tag{1.6}$$

$$(\Delta \otimes id)\Delta(1) = (\Delta(1) \otimes 1) \bullet (1 \otimes \Delta(1)) = (1 \otimes \Delta(1)) \bullet (\Delta(1) \otimes 1), \tag{1.7}$$

$$\varepsilon(m(m(x \otimes y) \otimes z)) = \varepsilon(m(x \otimes y_{(1)}))\, \varepsilon(m(y_{(2)} \otimes z)) \quad \forall x, y, z \in V. \tag{1.8}$$

Remark 1.2: *The condition (1.6) means that Δ is an algebra homomorphism. But condition (1.7) shows that ¢ does not necessarily preserve the unit 1. If Δ(1) = 1⊗1 then the condition (1.7) is satisfied. Identity (1.8) is a weak version of the fact that ε is an algebra homomorphism in the bialgebra case. Indeed, if ε is an algebra homomorphism then*

$$\begin{aligned}\varepsilon(m(x \otimes y_{(1)}))\, \varepsilon(m(y_{(2)} \otimes z)) &= \varepsilon(x)\varepsilon(y_{(1)})\varepsilon(y_{(2)})\varepsilon(z) \\ &= \varepsilon(x)\varepsilon(y_{(1)}\varepsilon(y_{(2)}))\varepsilon(z) = \varepsilon(x)\varepsilon(y)\varepsilon(z) \\ &= \varepsilon(m(m(x \otimes y) \otimes z))\end{aligned}$$

When Δ(1) = 1⊗1 then one can derive that the counit is an algebra homomorphism, indeed

$$\varepsilon(m(x \otimes y)) = \varepsilon(m(m(x \otimes 1) \otimes y)) = \varepsilon(m(x \otimes 1))\varepsilon(m(1 \otimes y)) = \varepsilon(x)\varepsilon(y).$$

Hence a bialgebra is always a weak bialgebra.

One may consider a definition of weak bialgebras where counits ε are algebra homomorphisms. In this paper we consider weak bialgebras where counits ε are not necessarily algebra homomorphisms but satisfy the identity (1.8).

Definition 1.3: A weak Hopf algebra is a sextuple $\mathcal{H} = (V, m, \eta, \Delta, \varepsilon, S)$, where $(V, m, \eta, \Delta, \varepsilon)$ is a weak bialgebra and S is an antipode which is an endomorphism of V such as $\forall x \in V$,

$$m(id \otimes S)\Delta(x) = (\varepsilon \otimes id)\Delta(1)(x \otimes 1), \tag{1.9}$$

$$m(S \otimes id)\Delta(x) = (id \otimes \varepsilon)(1 \otimes x)\Delta(1), \tag{1.10}$$

$$m(m \otimes id)(S \otimes id \otimes S)(\Delta \otimes id)\Delta(x) = S(x). \tag{1.11}$$

Remark 1.4: The antipode when it exists is unique and bijective. It is also both algebra and coalgebra anti-homomorphism.

Kaplansky's Type Constructions of Weak Bialgebras

In this section, we provide constructions of finite-dimensional weak bialgebras and weak Hopf algebras starting from any algebra or bialgebra. These constructions are inspired by Kaplansky's constructions for bialgebras [17].

Theorem 2.1: Let $\mathcal{A}$ be any algebra (not necessarily unital) and $\mathcal{B}$ be the result of adjoining to $\mathcal{A}$ two successive unit elements e and 1. On the vector space $\mathcal{B}$ spanned by the vector space $\mathcal{A}$ together with the generators {1,e}, we consider a comultiplication Δ and a counit ε defined by

$$\Delta(1) = (1-e) \otimes (1-e) + e \otimes e, \tag{2.1}$$

$$\Delta(a) = a \otimes a \quad \forall a \in \mathcal{B} \setminus \{1\}, \tag{2.2}$$

$$\varepsilon(a) = 1 \quad \forall a \in \mathcal{B} \setminus \{1\}, \tag{2.3}$$

$$\varepsilon(1) = 2. \tag{2.4}$$

Then $\mathcal{B}$ becomes a weak bialgebra.

Proof: The identities (1.2), (1.3) are satisfied. In the following we check the remaining identities in Definition 1.1. First, we show that Δ is coassociative (1.4). Let, $a \in \mathcal{B}\setminus\{1\}$ we have $(\Delta \otimes id)\Delta(a) = a \otimes a \otimes a = (id \otimes \Delta)\Delta(a)$. We have also $(\Delta \otimes id)\Delta(1) = (id \otimes \Delta)\Delta(1)$ since

$$\begin{aligned}(\Delta \otimes id)\Delta(1) &= \Delta(1) \otimes 1 - \Delta(1) \otimes e - \Delta(e) \otimes 1 + 2\Delta(e) \otimes e \\ &= 1\otimes1\otimes1 - 1\otimes e\otimes1 - e\otimes1\otimes1 + 2e\otimes e\otimes1 - 1\otimes1\otimes e + 1\otimes e\otimes e + \\ &\quad e\otimes1\otimes e - 2e\otimes e\otimes e - e\otimes e\otimes1 + 2e\otimes e\otimes e \\ &= 1\otimes1\otimes1 - 1\otimes e\otimes1 - e\otimes1\otimes1 + e\otimes e\otimes1 - 1\otimes1\otimes e + 1\otimes e\otimes e + e\otimes1\otimes e \\ &= (1-e)\otimes(1-e)\otimes(1-e) + e\otimes e\otimes e.\end{aligned}$$

and on the other hand

$$\begin{aligned}(id \otimes \Delta)\Delta(1) &= 1 \otimes \Delta(1) - 1 \otimes \Delta(e) - e \otimes \Delta(1) + 2e \otimes \Delta(e) \\ &= 1\otimes1\otimes1 - 1\otimes e\otimes1 - e\otimes1\otimes1 + e\otimes e\otimes1 - 1\otimes1\otimes e + 1\otimes e\otimes e + e\otimes1\otimes e \\ &= (1-e)\otimes(1-e)\otimes(1-e) + e\otimes e\otimes e.\end{aligned}$$

The identity (1.5) is also satisfied. Indeed, let $a \in \mathcal{B}\setminus\{1\}$

$$\begin{aligned}&(\varepsilon \otimes id)\Delta(a) = \varepsilon(a)a = a = id(a), \\ &(id \otimes \varepsilon)\Delta(a) = \varepsilon(a)a = a = id(a) \\ &(\varepsilon \otimes id)\Delta(1) = \varepsilon(1)1 - \varepsilon(1)e - \varepsilon(e)1 + 2\varepsilon(e)e = id(1), \\ &(id \otimes \varepsilon)\Delta(1) = \varepsilon(1)1 - \varepsilon(e)1 - \varepsilon(1)e + 2\varepsilon(e)e = id(1).\end{aligned}$$

The comultiplication Δ is in fact an algebra homomorphism (1.6). Indeed, let $a_1, a_2 \in \mathcal{B}\setminus\{1\}$, since $a_1.a_2 \in \mathcal{A}$ we have

$$\Delta(a_1) \bullet \Delta(a_2) = (a_1 \otimes a_1) \bullet (a_2 \otimes a_2) = a_1 \cdot a_2 \otimes a_1 \cdot a_2 = \Delta(a_1 \cdot a_2).$$

Also, for $a \in \mathcal{B}\setminus\{1\}$ we have

$$\begin{aligned}\Delta(a) \bullet \Delta(1) &= (a \otimes a) \bullet (1\otimes1 - 1\otimes e - e\otimes1 + 2\times1\otimes1) \\ &= a\otimes a - a\otimes a - a\otimes a + 2a\otimes a = a\otimes a = \Delta(a).\end{aligned}$$

We check the compatibility condition (1.7). Indeed, since e and $1-e$ are orthogonal idempotent elements, that is $e \cdot e = e, (1-e)\cdot(1-e) = 1-e$ and $(1-e)\cdot e = e\cdot(1-e) = 0$, we have

$$(1 \otimes \Delta(1)) \bullet (\Delta(1) \otimes 1) = (1-e)\otimes(1-e)\otimes(1-e) + e\otimes e\otimes e = (\Delta \otimes id)\Delta(1).$$

And similarly

$$(\Delta(1) \otimes 1) \bullet (1 \otimes \Delta(1)) = (1-e)\otimes(1-e)\otimes(1-e) + e\otimes e\otimes e = (\Delta \otimes id)\Delta(1).$$

Finally, we check that the identity (1.8) is satisfied for any element in $\mathcal{B}$ Indeed, let $a_1, a_2, a_3 \in \mathcal{B}$ with $a_2 \in \mathcal{B}\setminus\{1\}$. Since $a_1 \cdot a_2 \in \mathcal{B}\setminus\{1\}$ (resp. $a_2 \cdot a_3 \in \mathcal{B}\setminus\{1\}$), then $(a_1 \cdot a_2)\cdot a_3 \in \mathcal{B}\setminus\{1\}$ (resp. $a_1\cdot(a_2\cdot a_3) \in \mathcal{B}\setminus\{1\}$). Therefore $\varepsilon(a_1\cdot a_2\cdot a_3) = 1$ and $\varepsilon(a_1\cdot a_2)\varepsilon(a_2\cdot a_3) = 1$.

Assume now that a_2 = 1, then the left hand side becomes $\varepsilon(a_1\cdot1\cdot a_3) = \varepsilon(a_1\cdot a_3)$ and the right hand side writes $\varepsilon(a_1\cdot1_{(1)})\varepsilon(1_{(2)}\cdot a_3) = \varepsilon(a_1\cdot(1-e))\varepsilon((1-e)\cdot a_3) + \varepsilon(a_1\cdot e)\varepsilon(e\cdot a_3)$. We consider the following particular cases:

(1) $a_1 = 1$ and $a_3 = 1$

$$\varepsilon(1\cdot1_{(1)})\varepsilon(1_{(2)}\cdot1) = \varepsilon(1-e)\varepsilon(1-e) + \varepsilon(e)\varepsilon(e) = 2 = \varepsilon(1).$$

(2) $a_1 = 1$ and $a_3 \neq 1$

$$\varepsilon(1\cdot1_{(1)})\varepsilon(1_{(2)}\cdot a_3) = \varepsilon(1-e)\varepsilon((1-e)\cdot a_3) + \varepsilon(e)\varepsilon(e\cdot a_3) = \varepsilon(a_3) = 1.$$

(3) $a_1 \neq 1$ and $a_3 = 1$

$$\varepsilon(a_1\cdot1_{(1)})\varepsilon(1_{(2)}\cdot1) = \varepsilon(a_1\cdot(1-e))\varepsilon(1-e) + \varepsilon(a_1\cdot e)\varepsilon(e\cdot1) = \varepsilon(a_1) = 1.$$

(4) $a_1 \neq 1$ and $a_3 \neq 1$

$$\varepsilon(a_1\cdot1_{(1)})\varepsilon(1_{(2)}\cdot a_3) = \varepsilon(a_1\cdot(1-e))\varepsilon((1-e)\cdot a_3) + \varepsilon(a_1\cdot e)\varepsilon(e\cdot a_3) = \varepsilon(a_1)\varepsilon(a_3) = 1,$$

which is equal to $\varepsilon(a_1\cdot a_3)$ because $a_1\cdot a_3 \in \mathcal{B}\setminus\{1\}$

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This ends the proof that $\mathcal{B}$ is endowed with a weak bialgebra structure.

Remark 2.2: *The weak bialgebra obtained above is not a regular bialgebra since for a $a \in \mathcal{B}\backslash\{1\}$, we have*

$\varepsilon(a.1) = 1$ and $\varepsilon(a)\,\varepsilon(1) = 2$.

Corollary 2.3: *Let $\mathcal{A}$ be an associative algebra with a unit 1. If $\mathcal{A}\backslash\{1\}$, is a subalgebra and there exists an element e in $\mathcal{A}\backslash\{1\}$such that $e\,.\,e = e$ and $e\,.\,a = a.\,e = a$ for all $a \in \mathcal{A}\backslash\{1\}$. Then there exists a weak bialgebra structure on $\mathcal{A}$ given by:*

$$\Delta(1) = (1-e)\otimes(1-e) + e\otimes e,$$
$$\Delta(a) = a\otimes a,\ \ \forall a \in A\setminus\{1\},$$
$$\varepsilon(1) = 2,$$
$$\varepsilon(a) = 1,\ \ \forall a \in A\setminus\{1\}.$$

Remark 2.4: *The weak bialgebra $\mathcal{B}$ of theorem 2.1 is not always a weak Hopf algebra with the antipode S = id. The identities (1.9),(1.10),(1.11) are fulfilled for x = 1. But for x∈ $\mathcal{B}\backslash\{1\}$,, the identities (1.9),(1.10) lead to the condition $x\,.\,x = e$ while the identity (1.10) leads to$(x\,.\,x).\,x = x$.*

Example 2.5: *Let $\mathcal{A}$ be a 2-dimensional associative algebra with unit 1. We assume that e is an idem-potent element ($e\,.\,e = e$)different from 1 in $\mathcal{A}$. Then there exists a weak bialgebra structure on $\mathcal{A}$ given by*

$$\Delta(1) = (1-e)\otimes(1-e) + e\otimes e,$$
$$\Delta(e) = e\otimes e,$$
$$\varepsilon(1) = 2,\quad \varepsilon(e) = 1.$$

Moreover, with S = id, the bialgebra $\mathcal{A}$ becomes a weak Hopf algebra.

Example 2.6: *Let $\mathcal{A} = \mathbb{K}\,\mathrm{x}\ldots\mathbb{K}$ be an n-dimensional unital associative algebra. Assume that $\mathfrak{b} = \{e_i\}_{1\le i\le n}$ is a basis of $\mathcal{A}$ such that $e_1 = 1$ and $\{e_i\}_{2\le i\le n}$ are orthogonal idempotent elements. Then there exist weak bialgebra structures on $\mathcal{A}$ given by setting, for a fixed integer k∈{2,…n},*

$$\Delta(1) = (1-e_k)\otimes(1-e_k) + e_k\otimes e_k,$$
$$\Delta(e_i) = e_i\otimes e_i,\quad i \in \{2,\cdots n\},$$
$$\varepsilon(1) = 2,$$
$$\varepsilon(e_i) = 1,\ \ i \in \{2,\cdots n\}.$$

In the following, we have the following more general result.

Theorem 2.7: *Let $\mathcal{A}$ be a finite-dimensional unital associative algebra with unit $e_1 = 1$. Let $\mathfrak{b} = \mathfrak{b}_1 \cup \mathfrak{b}_2$ be a basis of $\mathcal{A}$ with $\mathfrak{b}_1 = \{e_i\}_{i=1,\cdots p}$. Assume that $span(\mathfrak{b}_2)$ is a subalgebra of $\mathcal{A}$ and*

$$e_i\cdot e_j = e_{max(i,j)},\ \forall i,j = 1,\cdots,p,\ e_i\cdot f = f\cdot e_i = f,\ \ \forall f \in \mathfrak{b}_2.$$

Then the comultiplication Δ and the counit ε defined by

$$\Delta(e_p) = e_p\otimes e_p$$
$$\Delta(e_i) = (e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \Delta(e_{i+1}),\ \ \forall i, i = 1,\cdots,p-1,$$
$$\Delta(f) = f\otimes f,\ \ \forall f \in \mathfrak{b}_2,$$
$$\varepsilon(e_i) = p - i + 1,\ \ \forall i, i = 1,\cdots,p,$$
$$\varepsilon(f) = 1, \forall f \in \mathfrak{b}_2,$$

endow $\mathcal{A}$ with a weak bialgebra structure.

Proof: The image by Δ of an element $e_{p-i} \in \mathfrak{b}_1, i = 1,\cdots,p-1$, can written in the form:

$$\Delta(e_i) = (e_i - e_{i+1})\otimes(e_i - e_{i+1}) + (e_{i+1} - e_{i+2})\otimes(e_{i+1} - e_{i+2}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p.$$

It follows for $i = 1,\cdots,p-1$ that $\Delta(e_i - e_{i+1}) = (e_i - e_{i+1})\otimes(e_i - e_{i+1})$ and $\varepsilon(e_i - e_{i+1}) = 1$. Let us show that Δ is coassociative. For i = 1,…, p - 1, we have

$$(id\otimes\Delta)\Delta(e_i) = (id\otimes\Delta)[(e_i - e_{i+1})\otimes(e_i - e_{i+1})\cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p] =$$
$$(e_i - e_{i+1})\otimes(\Delta(e_i) - \Delta(e_{i+1})) + \cdots + (e_{p-1} - e_p)\otimes(\Delta(e_{p-1}) - \Delta(e_p)) + e_p\otimes\Delta(e_p) = (e_i - e_{i+1})\otimes(e_i - e_{i+1})$$
$$\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p\otimes e_p,$$

$$(\Delta\otimes id)\Delta(e_i) = (\Delta\otimes id)[(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p]$$
$$= (\Delta(e_i) - \Delta(e_{i+1}))\otimes(e_i - e_{i+1}) + \cdots + (\Delta(e_{p-1}) - \Delta(e_p))\otimes(e_{p-1} - e_p) + \Delta(e_p)\otimes e_p$$
$$= (e_i - e_{i+1})\otimes(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p\otimes e_p.$$

Then, $(id\otimes\Delta)\Delta(e_i) = (\Delta\otimes id)\Delta(e_i)$. Obviously, one gets the coassociativity for e_p and any $f \in \mathfrak{b}_2$. We show that ε is a counit. For i = 1,…, p - 1 we have

$$(id\otimes\varepsilon)\Delta(e_i) = (id\otimes\varepsilon)[(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p]$$
$$= (e_i - e_{i+1})(\varepsilon(e_i) - \varepsilon(e_{i+1})) + \cdots + (e_{p-1} - e_p)(\varepsilon(e_{p-1}) - \varepsilon(e_p)) + e_p\varepsilon(e_p)$$
$$= e_i - e_{i+1} + e_{i+1} - e_{i+2} + e_{i+2} + \cdots - e_{p-1} + e_{p-1} - e_p + e_p = id(e_i),$$

$$(\varepsilon\otimes id)\Delta(e_i) = (\varepsilon\otimes id)[(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p]$$
$$= (\varepsilon(e_i) - \varepsilon(e_{i+1}))(e_i - e_{i+1}) + \cdots + (\varepsilon(e_{p-1}) - \varepsilon(e_p))(e_{p-1} - e_p) + \varepsilon(e_p)e_p$$
$$= e_i - e_{i+1} + e_{i+1} - e_{i+2} + e_{i+2} + \cdots - e_{p-1} + e_{p-1} - e_p + e_p = id(e_i).$$

Then $(id\otimes\varepsilon)\Delta(e_i) = (\varepsilon\otimes id)\Delta(e_i) = id(e_i)$. The coassociativity is obviously satisfied for the group like elements.

The comultiplication Δ is compatible with the multiplication. Indeed, let $e_i, e_k \in \mathfrak{b}_1$, i, k = 1,…,p - 1 and $f \in \mathfrak{b}_2$.

$$\Delta(e_i)\bullet\Delta(f) = [(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p].(f\otimes f)$$
$$= e_p\cdot f\otimes e_p\cdot f = f\otimes f = \Delta(e_i\cdot f).$$

Similarly we have $\Delta(f)\bullet\Delta(e_i) = \Delta(f\cdot e_i)$.

Assume $i \ge k$, by a direct calculation we have

$$\Delta(e_i)\bullet\Delta(e_k) = [(e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p]$$
$$\bullet[(e_k - e_{k+1})\otimes(e_k - e_{k+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p]$$
$$= (e_i - e_{i+1})\otimes(e_i - e_{i+1}) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p = \Delta(e_i) = \Delta(e_i\cdot e_k).$$

Also for any $f_1, f_2 \in \mathfrak{b}_2$, we have

$$\Delta(f_1)\bullet\Delta(f_2) = (f_1\otimes f_1)\bullet(f_2\otimes f_2) = f_1\cdot f_2\otimes f_1\cdot f_2 = \Delta(f_1\cdot f_2).$$

In the following we check the identities (1.7). We have

$$[\Delta(e_1)\otimes e_1]\cdot[e_1\otimes\Delta(e_1)] =$$
$$[(e_1 - e_2)\otimes(e_1 - e_2)\otimes e_1 + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes e_1 + e_p\otimes e_p\otimes e_1]\cdot[e_1\otimes(e_1 - e_2)\otimes$$
$$(e_1 - e_2) + \cdots + e_1\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_1\otimes e_p\otimes e_p]$$
$$= (e_1 - e_2)\otimes(e_1 - e_2)\otimes(e_1 - e_2) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p\otimes e_p,$$

and

$$[e_1\otimes\Delta(e_1)]\cdot[\Delta(e_1)\otimes e_1] =$$
$$[e_1\otimes(e_1 - e_2)\otimes(e_1 - e_2) + \cdots + e_1\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_1\otimes e_p\otimes e_p].[(e_1 - e_2)\otimes(e_1 - e_2)\otimes$$
$$e_1 + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes e_1 + e_p\otimes e_p\otimes e_1]$$
$$= (e_1 - e_2)\otimes(e_1 - e_2)\otimes(e_1 - e_2) + \cdots + (e_{p-1} - e_p)\otimes(e_{p-1} - e_p)\otimes(e_{p-1} - e_p) + e_p\otimes e_p\otimes e_p.$$

Then $[e_1\otimes\Delta(e_1)]\bullet[\Delta(e_1)\otimes e_1] = [\Delta(e_1)\otimes e_1]\bullet[e_1\otimes\Delta(e_1)] = (\Delta\otimes id)\Delta(e_1)$.

Now we check the identity (1.8). We consider first a triple (e_i, e_j, e_k). Assume that $j \le k$, in this case $\varepsilon(e_i\cdot e_j\cdot e_k) = \varepsilon(e_i\cdot e_k) = p - \max(i,k) + 1$.

On the other hand we have

$$\varepsilon(e_i\cdot(e_j)_{(1)})\varepsilon((e_j)_{(2)}\cdot e_k) = \varepsilon(e_i\cdot(e_j - e_{j+1}))\varepsilon((e_j - e_{j+1})\cdot e_k)) + \cdots + \varepsilon(e_i\cdot(e_{p-1} - e_p))\varepsilon((e_{p-1} - e_p)\cdot e_k)$$
$$+\varepsilon(e_i\cdot e_p)\varepsilon((e_p\cdot e_k) = \varepsilon(e_i\cdot(e_k - e_{k+1}))\varepsilon(e_k - e_{k+1}) + \cdots + \varepsilon(e_i\cdot(e_{p-1} - e_p))\varepsilon(e_{p-1} - e_p) + \varepsilon(e_i\cdot e_p)\varepsilon(e_p)$$
$$= \varepsilon(e_i\cdot(e_k - e_{k+1})) + \varepsilon(e_i\cdot(e_{k+1} - e_{k+2})) + \cdots + \varepsilon(e_i\cdot(e_{p-1} - e_p)) + \varepsilon(e_i\cdot e_p) = \varepsilon(e_i\cdot e_k) - \varepsilon(e_i\cdot e_{k+1}) + \varepsilon(e_i\cdot e_{k+1})$$
$$-\varepsilon(e_i\cdot e_{k+2}) + \varepsilon(e_i\cdot e_{k+2}) + \cdots - \varepsilon(e_i\cdot e_{p-1}) + \varepsilon(e_i\cdot e_{p-1}) - \varepsilon(e_i\cdot e_p) + \varepsilon(e_i\cdot e_p) = \varepsilon(e_i\cdot e_k) = p - \max(i,k) + 1.$$

If $j > k$, then $\varepsilon(e_i\cdot e_j\cdot e_k) = \varepsilon(e_i\cdot e_j) = p - \max(i,j) + 1$.

Also

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$$\varepsilon(e_i \cdot (e_j)_{(1)})\varepsilon((e_j)_{(2)} \cdot e_k) = \varepsilon(e_i \cdot (e_j - e_{j+1}))\varepsilon((e_j - e_{j+1}) \cdot e_k) + \cdots + \varepsilon(e_i \cdot (e_{p-1} - e_p)\varepsilon((e_{p-1} - e_p) \cdot e_k)$$
$$+\varepsilon(e_i \cdot e_p)\varepsilon(e_p \cdot e_k) = \varepsilon(e_i \cdot (e_j - e_{j+1}))\varepsilon(e_j - e_{j+1}) + \varepsilon(e_i \cdot (e_{j+1} - e_{j+2}))\varepsilon(e_{j+1} - e_{j+2}) + \cdots + \varepsilon(e_i \cdot (e_{p-1} - e_p))$$
$$\varepsilon(e_{p-1} - e_p) + \varepsilon(e_i \cdot e_p)\varepsilon(e_p) = \varepsilon(e_i \cdot (e_j - e_{j+1})) + \cdots + \varepsilon(e_i \cdot (e_{p-1} - e_p)) + \varepsilon(e_i \cdot e_p) = \varepsilon(e_i \cdot e_j) - \varepsilon(e_i \cdot e_{j+1})$$
$$+\varepsilon(e_i \cdot e_{j+1}) + \cdots - \varepsilon(e_i \cdot e_{p-1}) + \varepsilon(e_i \cdot e_{p-1}) - \varepsilon(e_i \cdot e_p) + \varepsilon(e_i \cdot e_p) = \varepsilon(e_i \cdot e_j) = p - \max(i, j) + 1.$$

For a triple (f, e_i, e_j), we obtain $\varepsilon(f \cdot e_i \cdot e_j) = \varepsilon(f) = 1$ and on the other hand

$$\varepsilon(f \cdot (e_i)_{(1)})\varepsilon((e_i)_{(2)} \cdot e_j) = \varepsilon(f \cdot (e_i - e_{i+1}))\varepsilon((e_i - e_{i+1}) \cdot e_j)) + \cdots + \varepsilon(f \cdot (e_{p-1} - e_p))\varepsilon((e_{p-1} - e_p) \cdot e_j)$$
$$+\varepsilon(f \cdot e_p)\varepsilon((e_p \cdot e_j) = \varepsilon(f \cdot e_p)\varepsilon(e_p \cdot e_j) = \varepsilon(f)\varepsilon(e_p) = 1.$$

For a triple (a_1, f, a_2) where $a_1, a_1 \in \mathcal{A}$ we obtain $\varepsilon(a_1 . f . a_2) = \varepsilon(f) = 1$ because $a_1 . f$ and $f . a_2$ belong to $span(b_2)$ which is in fact an ideal. On the other hand we have $\varepsilon(a_1 \cdot (f)_{(1)})\varepsilon((f)_{(2)} \cdot a_2) = \varepsilon(f)\varepsilon(f) = 1.$

We show in the following that the commutative algebra $\mathcal{A} = \mathbb{K} \times \ldots \times \mathbb{K}$ carries a structure of a weak Hopf algebra. To this end we write the algebra in a suitable basis.

Proposition 2.8: *Let $\mathcal{A}$ be a unital algebra with unit e_2 such that on a basis $\{e_i\}_{i=2,\cdots,n}$ of $\mathcal{A}$ the multiplication is given by $m(e_i \otimes e_j) = e_{max(i,j)}$, $i, j = 2, \cdots, n$. Let $\mathcal{B}$ be the result of adjoining a second unit $e_1 = 1$ to $\mathcal{A}$. Set for a comultiplication Δ, a counit ε and an antipode S*

$$\Delta(e_n) = e_n \otimes e_n,$$
$$\Delta(e_i) = (e_i - e_{i+1}) \otimes (e_i - e_{i+1}) + \Delta(e_{i+1}), \quad \forall i, i = 1 \cdots n-1,$$
$$\varepsilon(e_i) = n - i + 1, \quad \forall i, i = 1 \cdots n,$$
$$S = id$$

Then $\mathcal{B}$ becomes a weak Hopf algebra.

Proof: The structure of a weak bialgebra follows from the previous theorem. It remains to verify the antipode's identities (1.9)(1.10)(1.11). We have for $i = 1, \ldots, n - 1$

$$m(id \otimes S)\Delta(e_i) = m(\Delta(e_i)) = m((e_i - e_{i+1}) \otimes (e_i - e_{i+1}) + \cdots + (e_{n-1} - e_n) \otimes (e_{n-1} - e_n) + e_n \otimes e_n)$$
$$= e_i - e_{i+1} + e_{i+1} + \cdots - e_{n-1} + e_{n-1} - e_n + e_n = e_i,$$

$$(\varepsilon \otimes id)[\Delta(e_1) \bullet (e_i \otimes e_1)] = (\varepsilon \otimes id)([(e_1 - e_2) \otimes (e_1 - e_2) + \cdots + (e_{n-1} - e_n) \otimes (e_{n-1} - e_n)$$
$$+e_n \otimes e_n] \cdot (e_i \otimes e_1)) = \varepsilon(e_i - e_{i+1})(e_i - e_{i+1}) + \cdots + \varepsilon(e_{n-1} - e_n)(e_{n-1} - e_n) + \varepsilon(e_n)e_n = e_i - e_{i+1}$$
$$+e_{i+1} + \cdots - e_{n-1} + e_{n-1} - e_n + e_n = e_i,$$

and similarly $(id \otimes \varepsilon)[(e_1 \otimes e_i) \bullet \Delta(e_1)] = e_i.$

Thus (1.9)(1.10) hold.

The identity (1.11) is also satisfied, we use a previous calculation of $(\Delta \otimes id)(\Delta(e_i))$ and $m((e_i - e_{i+1}) \otimes (e_i - e_{i+1})) = e_i - e_{i+1}$, then

$$m(m \otimes id)(S \otimes id \otimes S)(\Delta \otimes id)(\Delta(e_i)) = m(m \otimes id)(id \otimes id \otimes id)(\Delta \otimes id)(\Delta(e_i))$$
$$= m(m \otimes id)(\Delta \otimes id)(\Delta(e_i)) = m(m \otimes id)((e_i - e_{i+1}) \otimes (e_i - e_{i+1}) + \cdots + (e_{n-1} - e_n) \otimes (e_{n-1} - e_n) + e_n \otimes e_n)$$
$$= m((e_i - e_{i+1}) \otimes (e_i - e_{i+1}) + \cdots + (e_{n-1} - e_n) \otimes (e_{n-1} - e_n) + e_n \otimes e_n) = e_i = S(e_i).$$

The proof for en follows from easy direct calculations.

Remark 2.9: *Similarly we may endow the algebra generated by n orthogonal idempotent elements with the structure of weak Hopf algebra. The algebra structure is isomorphic to the algebra structure considered in Proposition 2.8, one may consider the same comultiplication and counit as in this Proposition.*

Now, we provide constructions of weak bialgebras starting from any bialgebra. We show that any n-dimensional bialgebra can be extended to (n + 1)-dimensional weak bialgebra.

Theorem 2.10: *Let $\mathcal{B}$ be a bialgebra and e_2 its unit. We consider the set $\mathcal{B}'$ as a result of adjoining a unit e_1 to $\mathcal{B}$ with respect to the multiplication. We extend the comultiplication Δ and the counit ε in the following way*

$$\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2,$$
$$\varepsilon(e_1) = 2.$$

Then $\mathcal{B}'$ becomes a weak bialgebra.

Proof: The identities (1.2)-(1.7) follow from the proof of Theorem 2.1 and the fact that $\Delta(e_2) = e_2 \otimes e_2$ and $\varepsilon(e_2) = 1.$ since $\mathcal{B}$ is a bialgebra with unit e_2. It remains to check the compatibility of the counit with the comultiplication. The identity (1.8) is satisfied when it deals with 3 elements of $\mathcal{B}$.

For a triple (e_1, a, b), we have

$$\varepsilon(e_1 \cdot a_{(1)})\varepsilon(a_{(2)} \cdot b) = \varepsilon(a_{(1)})\varepsilon(a_{(2)} \cdot b) = \varepsilon(\varepsilon(a_{(1)})a_{(2)} \cdot b) = \varepsilon(a \cdot b) = \varepsilon(e_1 \cdot a \cdot b).$$

The case of triples (a, b, e_1), is similar. Let us consider now triples of the form (a, e_1, b). The left hand side of (1.8) becomes $\varepsilon(a \cdot e_1 \cdot b) = \varepsilon(a \cdot b)$ and the right hand side writes

$$\varepsilon(a \cdot e_{1(1)})\varepsilon(e_{1(2)} \cdot b) = \varepsilon(a \cdot (e_1 - e_2))\varepsilon((e_1 - e_2) \cdot b) + \varepsilon(a \cdot e_2)\varepsilon(e_2 \cdot b).$$

We consider the following particular cases:

(1) $a = e_1$ and $b = e_1$

$$\varepsilon(e_1 \cdot e_{1(1)})\varepsilon(e_{1(2)} \cdot e_1) = \varepsilon(e_1 - e_2)\varepsilon(e_1 - e_2) + \varepsilon(e_2)\varepsilon(e_2) = 2 = \varepsilon(e_1).$$

(2) $a = 1$ and $b \neq e_1$

$$\varepsilon(e_1 \cdot e_{1(1)})\varepsilon(e_{1(2)} \cdot b) = \varepsilon(e_1 - e_2)\varepsilon((e_1 - e_2) \cdot b) + \varepsilon(e_2)\varepsilon(e_2 \cdot b) = \varepsilon(b).$$

(3) $a \neq e_1$ and $b = e_1$

$$\varepsilon(a \cdot e_{1(1)})\varepsilon(e_{1(2)} \cdot e_1) = \varepsilon(a \cdot (e_1 - e_2))\varepsilon(e_1 - e_2) + \varepsilon(a \cdot e_2)\varepsilon(e_2 \cdot e_1) = \varepsilon(a).$$

(4) $a \neq 1$ and $b \neq 1$

$$\varepsilon(a \cdot e_{1(1)})\varepsilon(e_{1(2)} \cdot b) = \varepsilon(a \cdot (e_1 - e_2))\varepsilon((e_1 - e_2) \cdot b) + \varepsilon(a \cdot e_2)\varepsilon(e_2 \cdot b) = \varepsilon(a)\varepsilon(b) = \varepsilon(a \cdot b)$$

Because $a, b \in \mathcal{B}$ and $\mathcal{B}$ is a bialgebra.

The dimension of $\mathcal{B}'$ is $dim\, \mathcal{B}' = dim\, \mathcal{B} + 1$.

Remark 2.11: The counit of $\mathcal{B}$ is not an algebra homomorphism, indeed $\varepsilon(e_1 \cdot e_2) = \varepsilon(e_2) = 1$ while $\varepsilon(e_1)\varepsilon(e_2) = 2.$

The following theorem provides a way for extending an n-dimensional Hopf algebra to an $(n + 1)$-dimensional weak Hopf algebra.

Theorem 2.12: *Let $\mathcal{H}$ be a Hopf algebra and e_2 its unit. We consider the set $\mathcal{H}'$ as a result of adjoining a unit e_1 to $\mathcal{H}$ with respect to the multiplication. We extend the comultiplication Δ, the counit ε and the antipode S in the following way*

$$\Delta(e_1) = (e_1 - e_2) \otimes (e_1 - e_2) + e_2 \otimes e_2, \varepsilon(e_1) = 2$$
$$S(e_1) = e_1.$$

Then $\mathcal{H}'$ becomes a weak Hopf algebra.

Proof: The structure of a weak bialgebra follows from Theorem 2.10. The remaining identities are given for e_1 by straightforward calculations.

Example 2.13 (Sweedler's 5-dimensional weak Hopf algebra): *Assume that char($\mathbb{K}$) ≠ 2. Let $\mathcal{H}$ be the Sweedler 4-dimensional Hopf algebra given by generators and relations as follows: $\mathcal{H}$ is generated as a $\mathbb{K}$ -algebra by c and x satisfying the relations:*

$$c^2 = e, x^2 = 0, x \cdot c = -c \cdot x, \tag{2.5}$$

where e is the unit:

Let $\mathcal{H}'$ be the algebra obtained by adjoining a new unit 1 to $\mathcal{H}$. Then $\mathcal{H}'$ is a 5-dimensional weak bialgebra defined as a $\mathbb{K}$ -algebra, with basis {1,e,x,c,c.x} and relations (2.5) and $e\cdot c=c\cdot e=c, e\cdot x=x\cdot e=x, e\cdot e=e.$ The coalgebra structure is defined by :

$$\Delta(1)=(1-e)\otimes(1-e)+e\otimes e,$$
$$\Delta(c)=c\otimes c, \Delta(e)=e\otimes e, \Delta(x)=c\otimes x+x\otimes e, \varepsilon(1)=2, \varepsilon(e)=1, \varepsilon(c)=1, \varepsilon(x)=0,$$

The antipode is given by:

$$S(1)=1, S(e)=e, S(c)=c, S(x)=-c\cdot x. \tag{2.6}$$

This weak Hopf algebra is non-commutative and non-cocommutative.

Example 2.14 (Taft's weak Hopf algebras): *Let $n\geq 2$ be an integer and λ be a primitive n-th root of unity. Consider Taft's algebras $\mathcal{H}_{n^2}(\lambda)$, generalizing Sweedler's Hopf algebra, defined by the generators c and x and where e is the unit, with the relations:*

$$c^n=e, x^n=0, x\cdot c=\lambda\, c\cdot x. \tag{2.7}$$

Let $\mathcal{H}'$ be the algebra obtained by adjoining a new unit 1 to $\mathcal{H}_{n^2}(\lambda)$.

We set a coalgebra structure defined by:

$$\Delta(1)=(1-e)\otimes(1-e)+e\otimes e,$$
$$\Delta(e)=e\otimes e, \Delta(c)=c\otimes c, \Delta(x)=c\otimes x+x\otimes e,$$
$$\varepsilon(1)=2, \varepsilon(e)=1, \varepsilon(c)=1, \varepsilon(x)=0$$

Then $\mathcal{H}'$ becomes an (n^2+1)-dimensional weak bialgebra, having a basis $\{1, c^i x^j, 0\leq i,j\leq n-1\}$. It carries a structure of weak Hopf algebra with an antipode defined by:

$$S(1)=1, S(e)=e, S(c)=c^{-1}, S(x)=-c^{-1}\cdot x. \tag{2.8}$$

Next two propositions give other constructions of weak bialgebras starting from bialgebras, the proofs are similar to previous ones.

Proposition 2.15: *Let $\mathcal{B}$ be a bialgebra and u be its unit. Let $\mathcal{B}'$ be a result of adjoining to $\mathcal{B}$ successive unit elements e and 1 with respect to the multiplication. We extend the comultiplication Δ and the counit ε in the following way*

$$\Delta(1)=1\otimes(e-u)+u\otimes(1-2e+2u),$$
$$\Delta(e)=e\otimes(e-u)+u\otimes(2u-e),$$
$$\varepsilon(1)=2, \varepsilon(e)=2.$$

Then $\mathcal{B}'$ is a weak bialgebra.

Proposition 2.16: *Let $\mathcal{B}$ be a bialgebra and u its unit. Let $\mathcal{B}'$ be a result of adjoining to $\mathcal{B}$ two successive unit elements e and 1 with respect to the multiplication and such that the comultiplication Δ and the counit ε are extended in the following way*

$$\Delta(1)=(1-e)\otimes(1-e)+(e-u)\otimes(e-u)+u\otimes u,$$
$$\Delta(e)=(e-u)\otimes(e-u)+u\otimes u,$$
$$\varepsilon(1)=3, \varepsilon(e)=2.$$

Then $\mathcal{B}'$ is a weak bialgebra.

Remark 2.17: *In the previous propositions, if the bialgebra $\mathcal{B}$ is a Hopf algebra then $\mathcal{B}'$ becomes a weak Hopf algebra by setting S(1) = 1 and S(e) = e.*

Algebraic Varieties of Weak Bialgebras

Let V be an n-dimensional $\mathbb{K}$-vector space and $\mathfrak{b}=\{e_1,\cdots,e_n\}$, be a basis of V. Let $\mathcal{H}=(V,m,\eta,\Delta,\varepsilon)$ (resp. $\mathcal{H}=(V,m,\eta,\Delta,\varepsilon,S)$) be a weak bialgebra (resp. weak Hopf algebra). Set $\eta(1)=e_1$ for the unit. Let us write multiplication m, comultiplication Δ, counit ε and antipode S write, with respect to the basis b,

$$m(e_i,e_j)=\sum_{k=1}^n C_{i,j}^k e_k, \Delta(e_k)=\sum_{i,j=1}^n D_k^{i,j} e_i\otimes e_j, \varepsilon(e_k)=f_k, S(e_i)=\sum_{i,j=1}^n s_{i,j}e_j.$$

The collection $\{C_{i,j}^k, D_k^{i,j}, f_k : i,j,k=1,\cdots,n\}$ is the set of structure constants of the weak bialgebra H,with respect to the basis b. Any n-dimensional weak bialgebra is identified to a point of $\mathbb{K}^{2n^3+n}$, determined by a collection $\{C_{i,j}^k, D_k^{i,j}, f_k : i,j,k=1,\cdots,n\}\in\mathbb{K}^{2n^3+n}$, satisfying for $i,j,k,s\in\{1,..,n\}$ the following equations:

$$\sum_{\ell=1}^n C_{i,j}^\ell C_{\ell,k}^s - C_{i,\ell}^s C_{j,k}^l=0, \tag{3.1}$$

$C_{1,i}^j=C_{i,1}^j=\delta_{ij}$ where δ_{ij} is the Kronecker symbol;(3.2)

$$\sum_{\ell=1}^n D_s^{\ell,k} D_\ell^{i,j} - D_s^{i,\ell} D_\ell^{j,k}=0, \tag{3.3}$$

$$D_i^{k,j} f_k = D_i^{j,k} f_k=\delta_{i,j}, \tag{3.4}$$

$$\sum_{\ell=1}^n (C_{i,j}^\ell D_\ell^{s,k} - \sum_{p,q,m=1}^n D_i^{\ell m} D_j^{p,q} C_{\ell,p}^s C_{m,q}^k)=0, \tag{3.5}$$

$$\sum_{\ell=1}^n (D_1^{\ell,j} D_\ell^{s,k} - \sum_{m=1}^n D_1^{\ell,j} D_1^{s,m} C_{\ell,m}^k)=0, \tag{3.6}$$

$$\sum_{\ell,m=1}^n (C_{i,j}^\ell C_{\ell,k}^m f_m - \sum_{p,q=1}^n D_j^{\ell,m} C_{i,\ell}^p C_{m,k}^q f_p f_q)=0. \tag{3.7}$$

We denote by $\mathcal{BF}_n$ the set of n-dimensional weak bialgebras. The previous system of equations endows $\mathcal{BF}_n$ with a structure of affine algebraic variety imbedded in $\mathbb{K}^{2n^3+n}$.

Similarly, an n-dimensional weak Hopf algebra $\mathcal{H}=(V,m,\eta,\Delta,\varepsilon,S)$ is determined, with respect to the basis b of V, by a collection of structure constants $\{C_{i,j}^k, D_k^{i,j}, f_k, s_{i,j} : i,j,k=1,\cdots,n\}\in\mathbb{K}^{2n^3+n^2+n}$ satisfying the equations (3.1)-(3.7), and in addition the following equations :

for $i,j\in\{1,..,n\}$

$$\sum_{t,r,k=1}^n D_i^{t,k} s_{k,r} C_{t,r}^j - \sum_{t,k=1}^n D_1^{t,j} C_{t,i}^k f_k=0, \tag{3.8}$$

$$\sum_{t,r,k=1}^n D_i^{k,t} s_{k,r} C_{r,t}^j - \sum_{t,k=1}^n D_1^{j,t} C_{i,t}^k f_k=0, \tag{3.9}$$

$$\sum_{p,q,k,r,m,\ell,t=1}^n D_i^{p,q} D_p^{k,r} s_{r,m} s_{q,\ell} C_{m,r}^t C_{t,\ell}^j - s_{i,j}=0. \tag{3.10}$$

We denote by $\mathcal{HF}_n$ the set of n-dimensional weak Hopf algebras.

We define the action of linear groups on the algebraic varieties of weak bialgebras $\mathcal{BF}_n$ and similarly on the algebraic varieties of weak Hopf algebras $\mathcal{HF}_n$.

$$GL_n(\mathbb{K})\times\mathcal{BF}_n\to\mathcal{BF}_n,$$
$$(g,\mathcal{H})\mapsto g.\mathcal{H}.$$

This action is defined for all *x, y* in *V* by

$$(g\cdot m)(x\otimes y)=g^{-1}(m(g(x)\otimes g(y))),$$
$$(g\cdot\Delta)(x)=g^{-1}\otimes g^{-1}(\Delta g(x)),$$
$$(g\cdot\varepsilon)(x)=\varepsilon(g(x)).$$

The action on the antipode is given by

$$g\cdot S=g^{-1}\circ S\circ g.$$

The orbit of a weak bialgebra (resp. weak Hopf algebra) $\mathcal{H}$ describes the isomorphisms class, it is characterized by:

$$\vartheta(\mathcal{H})=\{g\cdot\mathcal{H}:g\in GL_n(\mathbb{K})\}.$$

The stabilizer of $\mathcal{H}$ is

$$stab(\mathcal{H})=\{g\in GL_n(\mathbb{K}):g\cdot\mathcal{H}=\mathcal{H}\},$$

which corresponds to the automorphisms group of $\mathcal{H}$. We have $dim\ \vartheta(\mathcal{H})=n^2-dim\ Aut(\mathcal{H})$.

Classifications and Homomorphism Groups

In this section, we establish a classification, up to isomorphism, of weak bialgebras and weak Hopf algebras of dimension 2 and 3.

Classification of associative algebras

The classification of n-dimensional associative algebras is known for $n\le 5$, [21,22]. We recall the results in dimensions 2 and 3. Let $\{e_1,\ldots, e_n\}$ be a basis of the underlaying vector space.

Proposition 4.1: *Every 2-dimensional associative algebra is isomorphic to one of the following algebras:*

$$m_1^2(e_1,e_1)=e_1, m_1^2(e_1,e_2)=e_2, m_1^2(e_2,e_1)=e_2, m_1^2(e_2,e_2)=0$$
$$m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2.$$

Proposition 4.2: *Every 3-dimensional associative algebra is isomorphic to one of the following algebras:*

$$m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,$$
$$m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,$$

$$m_2^3(e_1,e_1)=e_1, m_2^3(e_1,e_2)=e_2, m_2^3(e_2,e_1)=e_2, m_2^3(e_2,e_2)=e_2, m_2^3(e_1,e_3)=e_3,$$
$$m_2^3(e_3,e_1)=e_3, m_2^3(e_2,e_3)=e_3, m_2^3(e_3,e_2)=e_3, m_2^3(e_3,e_3)=0,$$

$$m_3^3(e_1,e_1)=e_1, m_3^3(e_1,e_2)=e_2, m_3^3(e_2,e_1)=e_2, m_3^3(e_2,e_2)=e_2, m_3^3(e_1,e_3)=e_3,$$
$$m_3^3(e_3,e_1)=e_3, m_3^3(e_2,e_3)=0, m_3^3(e_3,e_2)=0, m_3^3(e_3,e_3)=0,$$

$$m_4^3(e_1,e_1)=e_1, m_4^3(e_1,e_2)=e_2, m_4^3(e_2,e_1)=e_2, m_4^3(e_2,e_2)=0, m_4^3(e_1,e_3)=e_3,$$
$$m_4^3(e_3,e_1)=e_3, m_4^3(e_2,e_3)=0, m_4^3(e_3,e_2)=0, m_4^3(e_3,e_3)=0,$$

$$m_5^3(e_1,e_1)=e_1, m_5^3(e_1,e_2)=e_2, m_5^3(e_2,e_1)=e_2, m_5^3(e_2,e_2)=e_2, m_5^3(e_1,e_3)=e_3,$$
$$m_5^3(e_3,e_1)=e_3, m_5^3(e_2,e_3)=e_3, m_5^3(e_3,e_2)=0, m_5^3(e_3,e_3)=0.$$

Next, we will build 2 and 3-dimensional weak bialgebras and weak Hopf algebras using the previous associative algebras. All the calculations are done using a computer algebra system.

Classification of 2-dimensional weak bialgebras and weak Hopf algebras

Let $\{e_1, e_2\}$ be a basis of $V=\mathbb{C}^2$.

Proposition 4.3: *Every 2-dimensional weak bialgebra is isomorphic to one of the following weak bialgebras*:

$$(1)\quad \begin{array}{l} m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=1.\end{array}$$

$$(2)\quad \begin{array}{l} m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=(e_1-e_2)\otimes(e_1-e_2)\otimes+e_2\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=1.\end{array}$$

$$(3)\quad \begin{array}{l} m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_2)=e_2\otimes e_2,\\ \varepsilon(e_1)=2, \varepsilon(e_2)=1.\end{array}$$

From the precedent classification, we derive the 2-dimensional weak Hopf algebras.

Proposition 4.4: *There exist, up to isomorphism, two 2-dimensional weak Hopf algebras, which are given by:*

$$(1)\quad \begin{array}{l} m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=1,\\ S(e_1)=e_1, S(e_2)=e_2.\end{array}$$

$$(2)\quad \begin{array}{l} m_2^2(e_1,e_1)=e_1, m_2^2(e_1,e_2)=e_2, m_2^2(e_2,e_1)=e_2, m_2^2(e_2,e_2)=e_2,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_2)=e_2\otimes e_2,\\ \varepsilon(e_1)=2, \varepsilon(e_2)=1,\\ S(e_1)=e_1, S(e_2)=e_2.\end{array}$$

Classification of 3-dimensional weak bialgebras and weak Hopf algebras

Let $V=\mathcal{C}^3$ be a 3-dimensional vector space with a basis $\{e_1, e_2, e_3\}$. We provide all 3-dimensional weak bialgebras. Then we specify which of them correspond to weak Hopf algebras.

Proposition 4.5: *Every 3-dimensional weak bialgebra is isomorphic to one of the following weak bialgebras.*

$$(1)\quad \begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes(e_1-e_3)+e_2\otimes(2e_3-e_2)+e_3\otimes(2e_2-e_3-e_1),\\ \Delta(e_3)=e_1\otimes(e_2-e_3)+e_2\otimes(e_1-2e_2+e_3)+e_3\otimes(e_2+e_3-e_1),\\ \varepsilon(e_1)=\varepsilon(e_2)=\varepsilon(e_3)=1.\end{array}$$

$$(2)\quad \begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=\varepsilon(e_2)=\varepsilon(e_3)=1.\end{array}$$

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$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=(e_2-e_3)\otimes e_3+e_3\otimes(e_2-e_3),\\ \varepsilon(e_1)=\varepsilon(e_2)=1, \varepsilon(e_3)=0. \end{array} \tag{3}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=(e_2-e_3)\otimes e_3+e_3\otimes e_2,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=\varepsilon(e_2)=\varepsilon(e_3)=1. \end{array} \tag{4}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2+(e_1-e_2)\otimes e_3,\\ \Delta(e_3)=(e_1-e_3)\otimes e_3+e_3\otimes e_2,\\ \varepsilon(e_1)=\varepsilon(e_2)=1, \varepsilon(e_3)=0. \end{array} \tag{5}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2+e_3\otimes(e_1-e_2),\\ \Delta(e_3)=e_2\otimes e_3+e_3\otimes e_1-e_3\otimes e_3,\\ \varepsilon(e_1)=\varepsilon(e_2)=\varepsilon(e_3)=1. \end{array} \tag{6}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=\varepsilon(e_2)=\varepsilon(e_3)=1. \end{array} \tag{7}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=2, \varepsilon(e_2)=1, \varepsilon(e_3)=1. \end{array} \tag{8}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=(e_2-e_3)\otimes e_3+e_3\otimes(e_2-e_3),\\ \varepsilon(e_1)=2, \varepsilon(e_2)=1, \varepsilon(e_3)=0. \end{array} \tag{9}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+(e_2-e_3)\otimes(e_2-e_3)+e_3\otimes e_3,\\ \Delta(e_2)=(e_2-e_3)\otimes(e_2-e_3)+e_3\otimes e_3,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=3, \varepsilon(e_2)=2, \varepsilon(e_3)=1. \end{array} \tag{10}$$

$$\begin{array}{l} m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\ m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\ \Delta(e_1)=e_1\otimes(e_2-e_3)+e_3\otimes(e_1-2e_2+2e_3),\\ \Delta(e_2)=(e_2-e_3)\otimes(e_2-e_3)+e_3\otimes e_3,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=2, \varepsilon(e_2)=2, \varepsilon(e_3)=1. \end{array} \tag{11}$$

$$\begin{array}{l} m_2^3(e_1,e_1)=e_1, m_2^3(e_1,e_2)=e_2, m_2^3(e_2,e_1)=e_2, m_2^3(e_2,e_2)=e_2, m_2^3(e_1,e_3)=e_3,\\ m_2^3(e_3,e_1)=e_3, m_2^3(e_2,e_3)=e_3, m_2^3(e_3,e_2)=e_3, m_2^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2,\\ \Delta(e_3)=e_1\otimes e_3+e_3\otimes e_1-e_3\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{12}$$

$$\begin{array}{l} m_2^3(e_1,e_1)=e_1, m_2^3(e_1,e_2)=e_2, m_2^3(e_2,e_1)=e_2, m_2^3(e_2,e_2)=e_2, m_2^3(e_1,e_3)=e_3,\\ m_2^3(e_3,e_1)=e_3, m_2^3(e_2,e_3)=e_3, m_2^3(e_3,e_2)=e_3, m_2^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2,\\ \Delta(e_3)=e_1\otimes e_3-e_2\otimes e_3+e_3\otimes e_1-e_3\otimes e_2+e_3\otimes e_3,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{13}$$

$$\begin{array}{l} m_2^3(e_1,e_1)=e_1, m_2^3(e_1,e_2)=e_2, m_2^3(e_2,e_1)=e_2, m_2^3(e_2,e_2)=e_2, m_2^3(e_1,e_3)=e_3,\\ m_2^3(e_3,e_1)=e_3, m_2^3(e_2,e_3)=e_3, m_2^3(e_3,e_2)=e_3, m_2^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2,\\ \Delta(e_3)=e_1\otimes e_3-e_2\otimes e_3+e_3\otimes e_1,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{14}$$

$$\begin{array}{l} m_2^3(e_1,e_1)=e_1, m_2^3(e_1,e_2)=e_2, m_2^3(e_2,e_1)=e_2, m_2^3(e_2,e_2)=e_2, m_2^3(e_1,e_3)=e_3,\\ m_2^3(e_3,e_1)=e_3, m_2^3(e_2,e_3)=e_3, m_2^3(e_3,e_2)=e_3, m_2^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2,\\ \Delta(e_3)=e_1\otimes e_3-e_3\otimes e_2+e_3\otimes e_1-e_2\otimes e_3,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{15}$$

$$\begin{array}{l} m_3^3(e_1,e_1)=e_1, m_3^3(e_1,e_2)=e_2, m_3^3(e_2,e_1)=e_2, m_3^3(e_2,e_2)=e_2, m_3^3(e_1,e_3)=e_3,\\ m_3^3(e_3,e_1)=e_3, m_3^3(e_2,e_3)=e_3, m_3^3(e_3,e_2)=0, m_3^3(e_3,e_3)=0,\\ \Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=e_3\otimes e_3,\\ \varepsilon(e_1)=2, \varepsilon(e_2)=\varepsilon(e_3)=1. \end{array} \tag{16}$$

$$\begin{array}{l} m_3^3(e_1,e_1)=e_1, m_3^3(e_1,e_2)=e_2, m_3^3(e_2,e_1)=e_2, m_3^3(e_2,e_2)=e_2, m_3^3(e_1,e_3)=e_3,\\ m_3^3(e_3,e_1)=e_3, m_3^3(e_2,e_3)=e_3, m_3^3(e_3,e_2)=0, m_3^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2,\\ \Delta(e_3)=e_1\otimes e_3+e_3\otimes e_1-e_2\otimes e_3-e_3\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{17}$$

$$\begin{array}{l} m_3^3(e_1,e_1)=e_1, m_3^3(e_1,e_2)=e_2, m_3^3(e_2,e_1)=e_2, m_3^3(e_2,e_2)=e_2, m_3^3(e_1,e_3)=e_3,\\ m_3^3(e_3,e_1)=e_3, m_3^3(e_2,e_3)=e_3, m_3^3(e_3,e_2)=0, m_3^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2-e_3\otimes e_3,\\ \Delta(e_3)=e_1\otimes e_3-e_2\otimes e_3+e_3\otimes e_1-e_3\otimes e_2,\\ \varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0. \end{array} \tag{18}$$

$$\begin{array}{l} m_3^3(e_1,e_1)=e_1, m_3^3(e_1,e_2)=e_2, m_3^3(e_2,e_1)=e_2, m_3^3(e_2,e_2)=e_2, m_3^3(e_1,e_3)=e_3,\\ m_3^3(e_3,e_1)=e_3, m_3^3(e_2,e_3)=e_3, m_3^3(e_3,e_2)=0, m_3^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2,\\ \Delta(e_3)=e_2\otimes e_3+e_3\otimes e_2,\\ \varepsilon(e_1)=\varepsilon(e_2)=1, \varepsilon(e_3)=0. \end{array} \tag{19}$$

$$\begin{array}{l} m_5^3(e_1,e_1)=e_1, m_5^3(e_1,e_2)=e_2, m_5^3(e_2,e_1)=e_2, m_5^3(e_2,e_2)=e_2, m_5^3(e_1,e_3)=e_3,\\ m_5^3(e_3,e_1)=e_3, m_5^3(e_2,e_3)=e_3, m_5^3(e_3,e_2)=0, m_5^3(e_3,e_3)=0,\\ \Delta(e_1)=e_1\otimes e_1,\\ \Delta(e_2)=e_2\otimes e_2+e_3\otimes e_3,\\ \Delta(e_3)=e_2\otimes e_3+e_3\otimes e_2,\\ \varepsilon(e_1)=\varepsilon(e_2)=1, \varepsilon(e_3)=0. \end{array} \tag{20}$$

The 3-dimensional weak Hopf algebras are given by the following proposition:

Proposition 4.6: *Every 3-dimensional weak Hopf algebra is isomorphic to one of the following weak Hopf algebras.*

(1)
$$\begin{aligned}
&m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\
&m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\
&\Delta(e_1)=e_1\otimes e_1,\\
&\Delta(e_2)=e_1\otimes e_2+e_2\otimes e_1-e_2\otimes e_2-e_2\otimes e_3-e_3\otimes e_2+2e_3\otimes e_3,\\
&\Delta(e_3)=e_1\otimes e_3+e_2\otimes e_2-2e_2\otimes e_3+e_3\otimes e_1-2e_3\otimes e_2+e_3\otimes e_3,\\
&\varepsilon(e_1)=1, \varepsilon(e_2)=\varepsilon(e_3)=0,\\
&S(e_1)=e_1, S(e_2)=e_2, S(e_3)=e_2-e_3.
\end{aligned}$$

(2)
$$\begin{aligned}
&m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\
&m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\
&\Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+e_2\otimes e_2,\\
&\Delta(e_2)=e_2\otimes e_2,\\
&\Delta(e_3)=(e_2-e_3)\otimes e_3+e_3\otimes(e_2-e_3),\\
&\varepsilon(e_1)=2, \varepsilon(e_2)=1, \varepsilon(e_3)=0,\\
&S(e_1)=e_1, S(e_2)=e_2, S(e_3)=e_3.
\end{aligned}$$

(3)
$$\begin{aligned}
&m_1^3(e_1,e_1)=e_1, m_1^3(e_1,e_2)=e_2, m_1^3(e_2,e_1)=e_2, m_1^3(e_2,e_2)=e_2, m_1^3(e_1,e_3)=e_3,\\
&m_1^3(e_3,e_1)=e_3, m_1^3(e_2,e_3)=e_3, m_1^3(e_3,e_2)=e_3, m_1^3(e_3,e_3)=e_3,\\
&\Delta(e_1)=(e_1-e_2)\otimes(e_1-e_2)+(e_2-e_3)\otimes(e_2-e_3)+e_3\otimes e_3,\\
&\Delta(e_2)=(e_2-e_3)\otimes(e_2-e_3)+e_3\otimes e_3,\\
&\Delta(e_3)=e_3\otimes e_3,\\
&\varepsilon(e_1)=3, \varepsilon(e_2)=2, \varepsilon(e_3)=1,\\
&S(e_1)=e_1, S(e_2)=e_2, S(e_3)=e_3.
\end{aligned}$$

Automorphisms group

In this Section, we compute the automorphisms groups of 2-dimensional and 3-dimensional weak bialgebras and weak Hopf algebras obtained above. First, we write down the conditions which should be satisfied in order that two weak bialgebras lie in the same orbit.

Let $\mathcal{H}_1=(V,m,\Delta_1,\varepsilon_1)$ and $\mathcal{H}_2=(V,m,\Delta_2,\varepsilon_2)$ be two weak bialgebras with the same orbit, then there exists a linear bijective map g: $V\to V$ which ensure the transport of the structure. We set, with respect to a basis $\{e_i\}_{i=1,\cdots,n}$,

$$g(e_i)=\sum_{j=1}^{n}T_{i,j}e_j,\quad m(e_i,e_j)=\sum_{k=1}^{n}C_{i,j}^{k}e_k,$$

$$\Delta_1(e_i)=\sum_{j,k=1}^{n}D_{1,i}^{j,k}e_j\otimes e_k,\quad \Delta_2(e_i)=\sum_{j,k=1}^{n}D_{2,i}^{j,k}e_j\otimes e_k,$$

$$\varepsilon_1(e_i)=f_{1,i},\quad \varepsilon_2(e_i)=f_{2,i}.$$

These two weak bialgebras $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic if the following conditions are satisfied:

$$\sum_{p=1}^{n}T_{i,p}D_{1,p}^{s,r}-\sum_{p,q=1}^{n}D_{2,i}^{p,q}T_{p,s}T_{q,r}=0\quad i,s,r=1,\cdots,n, \tag{4.1}$$

$$\sum_{j=1}^{n}T_{i,j}f_{1,j}-f_{2,i}=0\quad i=1,\cdots,n, \tag{4.2}$$

$$\sum_{t=1}^{n}T_{t,k}C_{i,j}^{t}-\sum_{s,r=1}^{n}T_{i,s}T_{j,r}C_{s,r}^{k}=0\quad i,j,k=1,\cdots,n. \tag{4.3}$$

Automorphisms group of 2-dimensional weak bialgebras: The automorphisms groups of all 2-dimensional weak bialgebras are groups of order 2 given by:

$$G=<\{\begin{pmatrix}1&1\\0&-1\end{pmatrix}\}>.$$

We obtain a similar group for 2-dimensional weak Hopf algebras.

Automorphisms group of 3-dimensional weak bialgebras: The automorphisms group of the weak bialgebras (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), is the group of order 6 given by:

$$G=<\{\begin{pmatrix}1&0&0\\0&1&1\\0&0&-1\end{pmatrix},\begin{pmatrix}1&1&0\\0&0&1\\0&-1&-1\end{pmatrix}\}>.$$

The automorphisms group of the weak bialgebras (12), (13), (14), (15), (16), (17), (19), is the group:

$$G=\{\begin{pmatrix}1&0&0\\0&1&0\\0&0&\alpha^{\theta}\end{pmatrix},\theta\in\mathbb{Z},\alpha\in\mathbb{C}^*\}.$$

The automorphisms group of the weak bialgebra (18) is the group:

$$G=<\{\begin{pmatrix}1&0&0\\0&1&0\\0&-r/2&\pm 1/2\sqrt[2]{(4e-r^2)}\end{pmatrix},4e-r^2\neq 0,r,e\in\mathbb{C}\}>.$$

The automorphisms group of the weak bialgebra (20) is the group:

$$G=<\{\begin{pmatrix}1&0&0\\0&1&0\\0&r/2&\pm 1/2\sqrt[2]{(4e+r^2)}\end{pmatrix},4e+r^2\neq 0,r,e\in\mathbb{C}\}>.$$

The automorphisms group of the 3-dimensional weak Hopf algebras is the group of order 6 given by:

$$G=<\{\begin{pmatrix}1&0&0\\0&1&1\\0&0&-1\end{pmatrix},\begin{pmatrix}1&1&0\\0&0&1\\0&-1&-1\end{pmatrix}\}>.$$

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