Journal of
Applied & Computational Mathematics

Arafa, J Appl Computat Math 2016, 5:6
DOI: 10.4172/2168-9679.1000336

Research Article OMICS International

Laplace Homotopy Perturbation Method (LHAM) to Fractional Oscillation Equations

Arafa AAM*

Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt

Abstract

In this paper, Laplace Homotopy perturbation method (LHAM) is used to find the approximate solution of the fractional Oscillation equations. The fractional derivatives are described in the Caputo sense. We compare the exact solutions with our results without fractional derivatives. The resulting solutions spread faster than the classical solutions and may exhibit asymmetry, depending on the fractional derivative used. Numerical results are demonstrated the accuracy, efficiency and high rate of convergence of this method.

Keywords: Laplace homotopy perturbation; Oscillation equations; Fractional calculus

Introduction

In recent years, fractional derivatives have received considerable interest in recent years. In many applications, they provide more accurate models of systems under consideration. For example, they have been used successfully to model frequency dependent damping behavior of many viscoelastic materials. Other authors have demonstrated applications of fractional derivatives in the areas of electrochemical processes [1,2], dielectric polarization [3], colored noise [4], viscoelastic materials [5-8] and chaos [9]. Mainardi [10] and Rossikhin and Shitikova [11] presented survey of the application of fractional derivatives, in general to solid mechanics, and in particular to modeling of viscoelastic damping. Magin [12-14] presented a three part critical review of applications of fractional calculus in bioengineering. Applications of fractional derivatives in other fields and related mathematical tools and techniques could be found in [12-20]. In fact, real world processes generally or most likely are fractional order systems

This paper considers two examples of more general Oscillator fractional differential equations of the form:

$$D^2x(t)+f(t,x(t),D^\alpha x(t))=0 \quad (1.1)$$

Subject to the initial conditions:

$$x(0)=a, \qquad \frac{d}{dt}x(0)=b \quad (1.2)$$

The first attempt to describe, qualitatively, the oscillatory behavior of the heart, was made by Van der Pol, in 1926. He observed for the first time the relaxation oscillations, by studying an electrical circuit that presents self-entertained oscillations, with amplitude which does not depend on the initial conditions [21]. Oscillators, both linear and nonlinear, are often used to represent different biological systems. Various chemical activities, human diseases, biological rhythms, and even neural activity have all been modeled by nonlinear equations.

The concept of fractional or non-integer order derivation and integration can be traced back to the genesis of integer order calculus itself [22]. Almost most of the mathematical theory applicable to the study of non-integer order calculus was developed through the end of 19th century. However it is in the past hundred years that the most intriguing leaps in engineering and scientific application have been found. The calculation technique has in some cases had to change to meet the requirement of physical reality. The use of fractional differentiation for the mathematical modeling of real world physical problems has been widespread in recent years, e.g. the modeling of earthquake, the fluid dynamic traffic model with fractional derivatives, measurement of viscoelastic material properties, etc.

The reason of using fractional order differential equations (FOD) is that FOD are naturally related to systems with memory which exists in most biological systems. Also they are closely related to fractals which are abundant in biological systems. The results derived of the fractional system (1.1) are of a more general nature. Respectively, solutions to the fractional diffusion equation spread at a faster rate than the classical diffusion equation, and may exhibit asymmetry. However, the fundamental solutions of these equations still exhibit useful scaling properties that make them attractive for applications.

The derivatives are understood in the Caputo sense. The general response expression contains a parameter describing the order of the fractional derivative that can be varied to obtain various responses.

The LHAM will be applied for computing solutions to the systems of fractional partial differential equations considered in this paper. This method has been used to obtain approximate solutions of a large class of linear or nonlinear differential equations. It is also quite straightforward to write computer codes in any symbolic languages. The method provides solutions in the form of power series with easily computed terms. It has many advantages over the classical techniques mainly; it provides efficient numerical solutions with high accuracy, minimal calculations.

Fractional Calculus

There are several approaches to the generalization of the notion of differentiation to fractional orders e.g. Riemann- Liouville, GruÖnwald-Letnikow, Caputo and Generalized Functions approach [23]. Riemann-

***Corresponding author:** Arafa AAM, Department of Mathematics, Faculty of Science, Port Said University, Port Said, Egypt, Tel: +20 66 3402344; E-mail: anaszi2@yahoo.com

Received December 03, 2016; **Accepted** December 23, 2016; **Published** December 30, 2016

Citation: Arafa AAM (2016) Laplace Homotopy Perturbation Method (LHAM) to Fractional Oscillation Equations. J Appl Computat Math 5: 336. doi: 10.4172/2168-9679.1000336

Liouville fractional derivative is mostly used by mathematicians but this approach is not suitable for real world physical problems since it requires the definition of fractional order initial conditions, which have no physically meaningful explanation yet. Caputo introduced an alternative definition, which has the advantage of defining integer order initial conditions for fractional order differential equations [23]. Unlike the Riemann-Liouville approach, which derives its definition from repeated integration, the GruÖnwald-Letnikow formulation approaches the problem from the derivative side. This approach is mostly used in numerical algorithms.

Here, we mention the basic definitions of the Caputo fractional-order integration and differentiation, which are used in the upcoming paper and play the most important role in the theory of differential and integral equation of fractional order.

The main advantages of Caputo's approach are the initial conditions for fractional differential equations with Caputo derivatives take on the same form as for integer order differential equations.

Definition 1

The fractional derivative of $f(x)$ in the Caputo sense is defined as:

$$D^{\alpha} f(x) = I^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-t)^{m-\alpha+1} f^{(m)}(t)dt$$

for $m-1<\alpha\le m,\ m\in N,\ x>0.$

For the Caputo derivative we have

$D^{\alpha}C=0$, C is constant

$$D^{\alpha} t^{n} = \begin{cases} 0, & (n\le\alpha-1) \\ \dfrac{\Gamma(n+1)}{\Gamma(n-\alpha+1)} t^{n-\alpha}, & (n>\alpha-1) \end{cases} \quad (2.1)$$

Definition 2

For m to be the smallest integer that exceeds α, the Caputo fractional derivatives of order α>0 is defined as

$$D^{\alpha}u(x,t) = \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)}\displaystyle\int_0^t (t-\tau)^{m-\alpha+1}\frac{\partial^{m}u(x,\tau)}{\partial\tau^{m}}d\tau, & for\ \text{m-1}<\alpha<\text{m} \\ \dfrac{\partial^{m}u(x,t)}{\partial t^{m}}, & for\ \alpha=m\in N \end{cases}$$

Analysis of the Method

In this section, in order to verify numerically whether the proposed methodology leads to higher accuracy, we evaluate two examples to get the numerical solution of the problem (1.1). To show the efficiency of the present method for our problem, we compare the results with the exact solution.

Example 1: Consider the fractional composite Oscillation:

$$\frac{d^2x}{dt^2} + \frac{d^{\alpha}x}{dt^{\alpha}} + x = 8 \quad (3.1)$$

Subject to the initial condition

$$x(0)=0, \qquad \frac{d}{dt}x(0)=0 \quad (3.2)$$

The exact solution when $\alpha\to 1$ is:

$$x(t) = 8(1-e^{-t/2}(\cos\frac{\sqrt{3}}{2}t + \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t)) \quad (3.3)$$

We will apply LHAM to nonlinear fractional Van der Pol Oscillation (3.1). The technique consists first for applying Laplace transform to both side of equation (3.1), we find

$$L\{\frac{d^2x}{dt^2}\} + L\{\frac{d^{\alpha}x}{dt^{\alpha}}\} + L\{x\} = 8L\{1\} \quad (3.4)$$

We can write this equation in the forms

$$L\{\frac{d^2x}{dt^2}\} = 8L\{1\} - L\{x\} - L\{\frac{d^{\alpha}x}{dt^{\alpha}}\} \quad (3.5)$$

Applying the formula for Laplace transform, we obtain

$$s^2L\{x(t)\} - s\,x(0) - \frac{d}{dt}x(0) = \frac{8}{s} - L\{x(t)\} - L\{\frac{d^{\alpha}x}{dt^{\alpha}}\} \quad (3.6)$$

Using the initial condition (3.2), we have

$$s^2L\{x(t)\} = \frac{8}{s} - L\{x(t)\} - L\{\frac{d^{\alpha}x}{dt^{\alpha}}\} \quad (3.7)$$

Or

$$L\{x(t)\} = \frac{8}{s^3} - \frac{1}{s^2}L\{x(t)\} - \frac{1}{s^2}L\{\frac{d^{\alpha}x}{dt^{\alpha}}\} \quad (3.8)$$

By the homotopy technique [24,25], defines the homotopy $H(x,p)$:$\Omega\times[0,1]\to R$ which satisfies,

$$H(x,p) = (1-p)[L\{x(t)\} - \frac{8}{s^3}] + p[L\{x(t)\} - \frac{8}{s^3} + \frac{1}{s^2}L\{x(t)\} + \frac{1}{s^2}L\{\frac{d^{\alpha}x}{dt^{\alpha}}\}] = 0 \quad (3.9)$$

Where $p\in[0,1]$ is an imbedding parameter, $x(0)$ is an initial approximation.

The basic assumption is that the solution of equations (3.9) can be expressed as a powers series in p

$$x = x_0 + px_1 + p^2x_2 + \ldots \quad (3.10)$$

Substitute Equation (3.10) into Equation (3.9) and rearranging the results based on p-terms, we find:

$$p^0 = L\{x_0(t)\} = \frac{8}{s^3} \quad (3.11)$$

Applying the inverse Laplace transform for equation (3.11) we get

$$x_0(t) = 4t^2 \quad (3.12)$$

$$p^1:\ L\{x_1(t)\} = -\frac{1}{s^2}L\{x_0(t)\} - \frac{1}{s^2}L\{\frac{d^{\alpha}x_0}{dt^{\alpha}}\} \quad (3.13)$$

Substitute this value of $x_0(t)$ in equation (3.12) into equation (3.13) gives

$$p^1:\ L\{x_1(t)\} = -8\frac{1}{s^5} - 8\frac{1}{s^{5-\alpha}} \quad (3.14)$$

Applying the inverse Laplace transform for equation (3.14) we get

$$x_1(t) = -\frac{1}{3}t^4 - \frac{8}{\Gamma(5-\alpha)}t^{4-\alpha} \quad (3.15)$$

$$p^2:\ L\{x_2(t)\} = -\frac{1}{s^2}L\{x_1(t)\} - \frac{1}{s^2}L\{\frac{d^{\alpha}x_1}{dt^{\alpha}}\} \quad (3.16)$$

$$x_2(t) = \frac{1}{90}t^6 + \frac{16}{\Gamma(7-\alpha)}t^{6-\alpha} + \frac{8}{\Gamma(7-2\alpha)}t^{6-2\alpha} \quad (3.17)$$

$$p^3:\ L\{x_3(t)\} = -\frac{1}{s^2}L\{x_2(t)\} - \frac{1}{s^2}L\{\frac{d^{\alpha}x_2}{dt^{\alpha}}\} \quad (3.18)$$

$$x_3(t) = -\frac{1}{45360}t^8 - \frac{24}{\Gamma(9-\alpha)}t^{8-\alpha} - \frac{24}{\Gamma(9-2\alpha)}t^{8-2\alpha} - \frac{8}{\Gamma(9-3\alpha)}t^{8-3\alpha} \quad (3.19)$$

Citation: Arafa AAM (2016) Laplace Homotopy Perturbation Method (LHAM) to Fractional Oscillation Equations. J Appl Computat Math 5: 336. doi: 10.4172/2168-9679.1000336

The approximate solution of equation (3.1) can be readily obtained by equations (3.12), (3.15), (3.17), and (3.19) as follow (Figure 1 and Table 1):

$$x(t) = \lim_{p \to 1} x = x_0 + x_1 + x_2 + \ldots \qquad (3.20)$$

Example 2: Consider the time fractional Van der Pol Oscillation:

$$\frac{d\ x}{dt} + (1 + x^2)\frac{d\ x}{dt} + x = 2\cos t - \cos^3 t \qquad (3.21)$$

Subject to the initial condition

$$x(0) = 0, \qquad \frac{d}{dt}x(0) = 1 \qquad (3.22)$$

The exact solution when α → 1 is:

$$x(t) = \sin t \qquad (3.23)$$

Applying Laplace transform to both side of equation (3.21), we find

$$L\{\frac{d^2x}{dt^2}\} + L\{(1 + x^2)\frac{d^\alpha x}{dt^\alpha}\} + L\{x\} = 2L\{\cos t\} - L\{\cos^3 t\} \qquad (3.24)$$

We can write this equation in the forms

$$L\{\frac{d^2x}{dt^2}\} = \frac{5}{4}L\{\cos t\} - \frac{1}{4}L\{\cos 3t\} - L\{x\} - L\{\frac{d^\alpha x}{dt^\alpha}\} - L\{x^2\frac{d^\alpha x}{dt^\alpha}\} \qquad (3.25)$$

Applying the formula for Laplace transform, we obtain

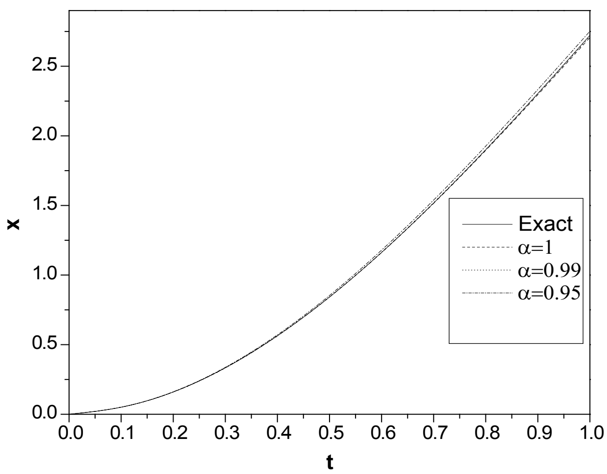

Figure 1: Approximate solutions for example 1 comparison with the exact solutions.

t	Exact Solution	*LAHM* (α=1)	*LAHM* (α=0.99)	*LAHM* (α=0.95)
0.0	0.0000000	0.0000000	0.0000000	0.0000000
0.1	0.09983	0.09983	0.09999	0.10058
0.2	0.19867	0.1986	0.1991	0.201
0.3	0.29552	0.29514	0.2961	0.29974
0.4	0.38942	0.38816	0.38964	0.39533
0.5	0.47943	0.47617	0.47821	0.48611
0.6	0.56464	0.55744	0.56006	0.57027
0.7	0.64422	0.62997	0.63316	0.64566
0.8	0.71736	0.69139	0.69512	0.70981
0.9	0.78333	0.73892	0.74314	0.75981
1.0	0.84147	0.76935	0.77396	0.7923

Table 1: The numerical results for Example 1.

$$s^2L\{x(t)\} - s\,x(0) - \frac{d}{dt}x(0) = \frac{5}{4}\frac{s}{s^2+1} - \frac{1}{4}\frac{s}{s^2+9} - L\{x(t)\} - L\{\frac{d^\alpha x}{dt^\alpha}\} - L\{x^2\frac{d^\alpha x}{dt^\alpha}\} \qquad (3.26)$$

Using the initial condition (3.22), we have

$$s^2L\{x(t)\} = 1 + \frac{5}{4}\frac{s}{s^2+1} - \frac{1}{4}\frac{s}{s^2+9} - L\{x(t)\} - L\{\frac{d^\alpha x}{dt^\alpha}\} - L\{x^2\frac{d^\alpha x}{dt^\alpha}\} \qquad (3.27)$$

Or

$$L\{x(t)\} = \frac{1}{s^2} + \frac{5}{4}\frac{1}{s(s^2+1)} - \frac{1}{4}\frac{1}{s(s^2+9)} - \frac{1}{s^2}L\{x(t)\} - \frac{1}{s^2}L\{\frac{d^\alpha x}{dt^\alpha}\} - \frac{1}{s^2}L\{x^2\frac{d^\alpha x}{dt^\alpha}\} \qquad (3.28)$$

By using homotopy technique, we find:

$$H(x,p) = (1-p)[L\{x(t)\} - \frac{1}{s^2}\}] + p[L\{x(t)\} - \frac{1}{s^2} - \frac{5}{4}\frac{1}{s(s^2+1)} + \frac{1}{4}\frac{1}{s(s^2+9)} + \frac{1}{s^2}L\{x(t)\} + \frac{1}{s^2}L\{\frac{d^\alpha x}{dt^\alpha}\} + \frac{1}{s^2}L\{N(x)\}] = 0 \qquad (3.29)$$

Where $p\in[0,1]$ is an imbedding parameter, $x(0)$ is an initial approximation.

The basic assumption is that the solution of equations (3.29) can be expressed as a powers series in p

$$x = x_0 + px_1 + p^2x_2 + \ldots \qquad (3.30)$$

Substitute equation (3.30) into equation (3.29) and rearranging the results based on p-terms, we find:

$$p^0 = L\{x_0(t)\} = \frac{1}{s^2} \qquad (3.31)$$

$$p^1:\ L\{x_1(t)\} = \frac{5}{4}\frac{1}{s(s^2+1)} - \frac{1}{4}\frac{1}{s(s^2+9)} - \frac{1}{s^2}L\{x_0(t)\} - \frac{1}{s^2}L\{\frac{d^\alpha x_0}{dt^\alpha}\} - \frac{1}{s^2}L\{x_0^2\frac{d^\alpha x_0}{dt^\alpha}\} \qquad (3.32)$$

Applying the inverse Laplace transform for equation (3.31) we get

$$x_0(t) = t \qquad (3.33)$$

Substitute these value of $x_0(t)$ into equation (3.32) gives

$$p^1:\ L\{x_1(t)\} = \frac{5}{4}\frac{1}{s(s^2+1)} - \frac{1}{4}\frac{1}{s(s^2+9)} - \frac{1}{s^4} - \frac{1}{s^{4-\alpha}} - \frac{\Gamma(4-\alpha)}{\Gamma(2-\alpha)}\frac{1}{s^{6-\alpha}} \qquad (3.34)$$

We can write the last equation in the form:

$$p^1:\ L\{x_1(t)\} = \frac{44}{36}\frac{1}{s} - \frac{5}{4}\frac{s}{s^2+1} + \frac{1}{36}\frac{s}{s^2+9} - \frac{1}{s^4} - \frac{1}{s^{4-\alpha}} - \frac{\Gamma(4-\alpha)}{\Gamma(2-\alpha)}\frac{1}{s^{6-\alpha}} \qquad (3.35)$$

Applying the inverse Laplace transform for equation (3.35) we get

$$x_1(t) = \frac{44}{36} - \frac{5}{4}\cos t + \frac{1}{36}\cos 3t - \frac{1}{6}t^3 - \frac{1}{\Gamma(4-\alpha)}t^{3-\alpha} - \frac{(3-\alpha)(2-\alpha)}{\Gamma(6-\alpha)}t^{5-\alpha} \qquad (3.36)$$

The approximate solution of equation (3.21), therefore, can be readily obtained

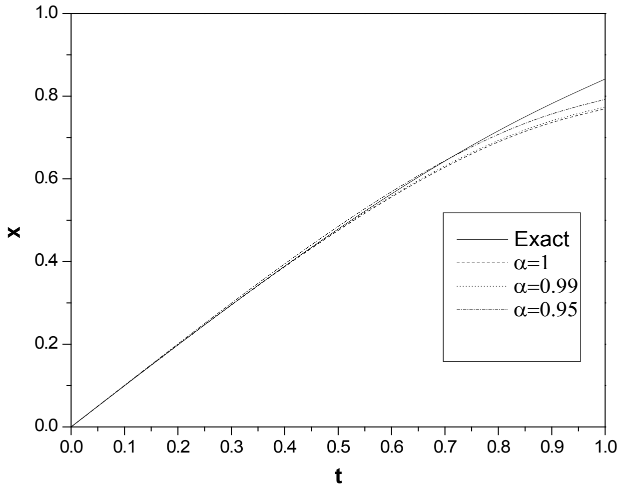

Figure 2: Approximate solutions for example 2 comparison with the exact solutions.

t	Exact Solution	*LAHM* (α=1)	*LAHM* (α=0.99)	*LAHM* (α=0.95)
0.0	0	0	0	0
0.1	0.099833	0.099829	0.099988	0.100577
0.2	0.198669	0.198598	0.199101	0.200996
0.3	0.29552	0.295144	0.2961	0.299744
0.4	0.389418	0.388161	0.389641	0.395325
0.5	0.479426	0.476167	0.47821	0.486114
0.6	0.564642	0.557442	0.560064	0.570269
0.7	0.644218	0.629971	0.633164	0.645661
0.8	0.717356	0.691389	0.695122	0.709806
0.9	0.783327	0.738922	0.743137	0.759805
1.0	0.841471	0.769345	0.773958	0.792299

Table 2: The numerical results for Example 2.

$$x(t) = \lim_{p \to 1} x = x_0 + x_1 + x_2 + \ldots \quad (3.37)$$

Therefore substitute the value of $x_0(t)$ and $x_1(t)$ from equation (3.33) and equation (3.36) yields (Figure 2 and Table 2):

$$x(t) = \frac{44}{36} - \frac{5}{4}\cos t + \frac{1}{36}\cos 3t + t - \frac{1}{6}t^3 - \frac{1}{\Gamma(4-\alpha)}t^{3-\alpha} - \frac{(3-\alpha)(2-\alpha)}{\Gamma(6-\alpha)}t^{5-\alpha} \quad (3.38)$$

Conclusion

We employ the LHPM to obtain approximate solutions for fractional nonlinear oscillatory equations. Excellent agreement between approximate and exact solutions. The LHPM has great potential and can be applied to other strongly nonlinear oscillators with non-polynomial terms. The results revealed that the LHPM is a powerful mathematical tool for the exact and numerical solutions of nonlinear equations in terms of accuracy and efficiency. The corresponding numerical solutions are obtained according to the recurrence relation using Mathematica.

References

1. Ichise M, Nagayanagi Y, Kojima Y (1971) An analog simulation of non-integer order transfer functions for analysis of electrode processes. J Electronical Chem Interfacial Electrochem 33: 253-265.
2. Sun HH, Onaral B, Tsao Y (1984) Application of positive reality principle to metal electrode linear polarization phenomena. IEEE Trans Biomed Eng BME 31: 664-674.
3. Sun HH, Abdelwahab AA, Onaral B (1984) Linear approximation of transfer function with a pole of fractional order. IEEE Trans Automat Control AC 29: 441-444.
4. Mandelbrot B (1967) Some noises with 1/f spectrum, a bridge between direct current and white noise. IEEE Trans Inform Theory 13: 289-298.
5. Bagley RL, Calico RA (1991) Fractional order state equations for the control of viscoelastic structures. J Guid Control Dynam 14: 304-311.
6. Koeller RC (1984) Application of fractional calculus to the theory of viscoelasticity. J Appl Mech 51: 299-307.
7. Koeller RC (1986) Polynomial operators. Stieltjes convolution and fractional calculus in hereditary mechanics. Acta Mech 58: 251-264.
8. Skaar SB, Michel AN, Miller RK (1988) Stability of viscoelastic control systems. IEEE Trans Automat Control AC 33: 348-357.
9. Hartley TT, Lorenzo CF, Qammar HK (1995) Chaos in a fractional order chua system. IEEE Trans. Circuits Systems I 42: 485-490.
10. Mainardi F (1997) Fractional calculus: some basic problem in continuum and statistical mechanics. In: Mainardi F (ed.) Fractals and Fractional Calculus in Continuum Mechanics, Springer, Wein, New York pp. 291-348.
11. Rossikhin YA, Shitikova MV (1997) Applications of fractional calculus to dynamic problems of linear and nonlinear hereditary mechanics of solids. Appl Mech Rev 50: 15-67.
12. Magin RL (2004) Fractional calculus in bioengineering. Crit Rev Biomed Eng 32: 1-104.
13. Magin RL (2004) Fractional calculus in bioengineering-part 2. Crit Rev Biomed Eng 32: 105-193.
14. Magin RL (2004) Fractional calculus in bioengineering-part 3. Crit Rev Biomed Eng 32: 194-377.
15. Oldham KB, Spanier J (1974) The Fractional Calculus. Academic Press, New York.
16. Samko SG, Kilbas AA, Marichev OI (1993) Fractional Integrals and Derivatives-Theory and Applications. Gordon and Breach Science Publishers, Longhorne, PA.
17. Podlubny I (1999) Fractional Differential Equations. Academic Press, New York.
18. El-Sayed AMA, Rida SZ, Arafa AMA (2009) On the Solutions of Time-fractional Bacterial Chemotaxis in a Diffusion Gradient Chamber. Int J of Nonlinear Science 7: 485-492.
19. El-Sayed AMA, Rida SZ, Arafa AMA (2010) On the Solutions of the generalized reaction-diffusion model for bacteria growth. Acta Appl Math 110: 1501.
20. El-Sayed AMA, Rida SZ, Arafa AAM (2009) Exact solutions of fractional-order biological population model. Communications in Theoretical Physics 52: 992.
21. Rocşoreanu C, Georgescu A, Giurgiţanu N (2000) The FitzHugh-Nagumo model. Mathematical Modelling: Theory and Applications p. 10.
22. Munkhammar JD (2005) Fractional calculus and the Taylor-Riemann series. Undergrad J Math 6: 1-19.
23. Podlubny I (1999) Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications. Academic Press, New York.
24. He JH (2003) Homotopy perturbation method: A new nonlinear analytical technique. Appl Math Comput 135: 73-79.
25. He JH (2003) A coupling method of a homotopy technique and a perturbation technique for non-linear problems. Int J Non-linear Mech 35: 37-43.

Citation: Arafa AAM (2016) Laplace Homotopy Perturbation Method (LHAM) to Fractional Oscillation Equations. J Appl Computat Math 5: 336. doi: 10.4172/2168-9679.1000336