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Lie-admissible coalgebras

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Abstract

After introducing the concept of Lie-admissible coalgebras, we study a remarkable class corresponding to coalgebras whose coassociator satisfies invariance conditions with respect to the symmetric group Σ_3. We then study the convolution and tensor products.

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1 Definitions and examples

In this work $\mathbb{K}$ indicates a field of characteristic zero. Let M be a $\mathbb{K}$-vector space and Δ a $\mathbb{K}$-linear comultiplication map $\Delta : M \to M \otimes M$. The coassociator of Δ is denoted by

$$\tilde{A}(\Delta) = (\Delta \otimes Id) \circ \Delta - (Id \otimes \Delta) \circ \Delta \tag{1.1}$$

and the flip $\tau : M^{\otimes 2} \to M^{\otimes 2}$ is the linear map defined by $\tau(x \otimes y) = y \otimes x$.

Let Σ_3 be the symmetric group of degree 3. We denote by c_1 and c_2 the 3-cycles of Σ_3 and τ_{ij} the transposition echanging i and j. For every $\sigma \in \Sigma_3$, we define a linear map $\Phi^M_\sigma : M^{\otimes 3} \to M^{\otimes 3}$ by

$$\Phi^M_\sigma(x_1 \otimes x_2 \otimes x_3) = x_{\sigma^{-1}(1)} \otimes x_{\sigma^{-1}(2)} \otimes x_{\sigma^{-1}(3)}$$

Definition 1. The pair (M, Δ) is a Lie-admissible coalgebra if the linear map $\Delta_L : M \to M \otimes M$ defined by $\Delta_L = \Delta - \tau \circ \Delta$ is a Lie coalgebra comultiplication, that is, if Δ_L satisfies

$$\begin{cases} \tau \circ \Delta_L = -\Delta_L \\ \tilde{A}(\Delta_L) + \Phi^M_{c_1} \circ \tilde{A}(\Delta_L) + \Phi^M_{c_2} \circ \tilde{A}(\Delta_L) = 0 \end{cases}$$

Recall that a multiplication μ of a $\mathbb{K}$-algebra $(\mathcal{A}, \mu)$ is Lie-admissible if its associator

$$A(\mu) = \mu \circ (\mu \otimes Id) - \mu \circ (Id \otimes \mu)$$

satisfies

$$\Sigma_{\sigma \in \Sigma_3} \, (-1)^{\epsilon(\sigma)} A(\mu) \circ \Phi^{\mathcal{A}}_\sigma = 0$$

where $\epsilon(\sigma)$ is the sign of the permutation σ. This means that the algebra $(\mathcal{A}, [,])$ whose product is given by the bracket $[x, y] = \mu(x, y) - \mu(y, x)$ is a Lie algebra. We have a similar characterization of a Lie-admissible comultiplication.

Proposition 1. *A comultiplication Δ on M is a Lie-admissible comultiplication if and only if Δ satisfies*

$$\sum_{\sigma \in \Sigma_3} (-1)^{\epsilon(\sigma)} \Phi^M_\sigma \circ \tilde{A}(\Delta) = 0 \tag{1.2}$$

where $\epsilon(\sigma)$ denotes the sign of the permutation σ.

Proof. It is a direct consequence of Equation (1.1) because

$$\begin{aligned} \tilde{A}(\tau \circ \Delta) &= ((\tau \circ \Delta) \otimes Id) \circ \tau \circ \Delta - (Id \otimes (\tau \circ \Delta)) \circ (\tau \circ \Delta) \\ &= -(\Phi^M_{\tau_{13}} \circ \tilde{A})(\Delta) \end{aligned}$$

This proves the proposition. □

Examples

- Every coassociative coalgebra is a Lie-admissible coalgebra.
- The comultiplication of a pre-Lie coalgebra (M, Δ) satisfies

$$\tilde{A}(\Delta) - \Phi^M_{\tau_{23}} \circ \tilde{A}(\Delta) = 0 \tag{1.3}$$

Since the composition of (1.3) by $\Phi^M_{c_1}$ and $\Phi^M_{c_2}$ gives respectively

$$\Phi^M_{c_1} \circ \tilde{A}(\Delta) - \Phi^M_{\tau_{13}} \circ \tilde{A}(\Delta) = 0$$

and

$$\Phi^M_{c_2} \circ \tilde{A}(\Delta) - \Phi^M_{\tau_{12}} \circ \tilde{A}(\Delta) = 0$$

we obtain Identity (1.2) by summation of (1.3) with these two equations and every pre-Lie coalgebra is Lie-admissible.

In the following sections we generalize these examples.

2 G_i-coalgebras

An interesting class of Lie-admissible coalgebras is obtained by dualizing the G_i-associative algebras. These Lie-admissible algebras has been introduced in [9] and developed in [3]. Let us point out these initially notations.

2.1 Σ_3-associative algebras

Let $\mathbb{K}[\Sigma_3]$ be the group algebra associated to Σ_3, where $\mathbb{K}$ is a field of characteristic zero. Every $v \in \mathbb{K}[\Sigma_3]$ decomposes as follows:

$$v = a_1 id + a_2\tau_{12} + a_3\tau_{13} + a_4\tau_{23} + a_5 c_1 + a_6 c_2$$

or simply

$$v = \sum_{\sigma \in \Sigma_3} a_\sigma \sigma$$

where $a_\sigma \in \mathbb{K}$. If $\mathcal{A}$ is a $\mathbb{K}$-vector space, then we define from such a vector v the endomorphism $\Phi^{\mathcal{A}}_v$ of $\mathcal{A}^{\otimes 3}$ by

$$\Phi^{\mathcal{A}}_v = \sum_{\sigma \in \Sigma_3} a_\sigma \Phi^{\mathcal{A}}_\sigma$$

Consider the natural right action of Σ_3 on $\mathbb{K}[\Sigma_3]$:

$$\Sigma_3 \times \mathbb{K}[\Sigma_3] \to \mathbb{K}[\Sigma_3], \qquad (\sigma, \sum_i a_i\sigma_i) \mapsto \sum_i a_i \sigma^{-1} \circ \sigma_i$$

The corresponding orbit of a vector $v \in \mathbb{K}[\Sigma_3]$ is denoted by $\mathcal{O}(v)$ and generates a linear subspace $F_v = \mathbb{K}(\mathcal{O}(v))$ of $\mathbb{K}[\Sigma_3]$. It is an invariant subspace of $\mathbb{K}[\Sigma_3]$. Therefore, using Mashke's theorem, it is a direct product of irreducible invariant subspaces.

Let $(\mathcal{A}, \mu)$ be a $\mathbb{K}$-algebra with multiplication μ and $A(\mu)$ its associator.

Definition 2. An algebra $(\mathcal{A},\mu)$ is a Σ_3-associative algebra if there exists $v \in \mathbb{K}[\Sigma_3]$, $v \neq 0$ such that $A(\mu)\circ\Phi_v^{\mathcal{A}} = 0$.

Proposition 2. *Let v be in $\mathbb{K}[\Sigma_3]$ such that dim $F_v = 1$. Then $v = \alpha V$ or $v = \alpha W$ with $\alpha \in \mathbb{K}$ and the vectors V and W are the following vectors:*

$$V = Id - \tau_{12} - \tau_{13} - \tau_{23} + c_1 + c_2 \tag{2.1}$$
$$W = Id + \tau_{12} + \tau_{13} + \tau_{23} + c_1 + c_2 \tag{2.2}$$

The first case corresponds to the character of Σ_3 given by the sign, the second corresponds to the trivial case.

Every algebra $(\mathcal{A},\mu)$ whose associator satisfies

$$A(\mu)\circ\Phi_V^{\mathcal{A}} = 0$$

is a Lie-admissible algebra. Likewise an algebra $(\mathcal{A},\mu)$ whose associator satisfies

$$A(\mu)\circ\Phi_W^{\mathcal{A}} = 0$$

is 3-power associative, that is, it satisfies $A(\mu)(x,x,x) = 0$ for every $x \in A$.

2.2 G_i-associative algebras

The class of Σ_3-associative Lie-admissible algebras contains interesting subclasses associated to the subgroups G_i of Σ_3 that we naturally call G_i-associative algebras. Let us introduce some notations. Consider the subgroups of Σ_3:

$$\begin{array}{lll} G_1 = \{Id\}, & G_2 = \{Id,\tau_{12}\}, & G_3 = \{Id,\tau_{23}\} \\ G_4 = \{Id,\tau_{13}\}, & G_5 = \{Id,c_1,c_2\} = A_3, & G_6 = \Sigma_3 \end{array}$$

Definition 3. Let G_i be a subgroup of Σ_3. The algebra $(\mathcal{A},\mu)$ is G_i-associative if

$$\sum_{\sigma\in G_i} (-1)^{\varepsilon(\sigma)} A(\mu)\circ\Phi_\sigma^{\mathcal{A}} = 0 \tag{2.3}$$

Proposition 3. *Every G_i-associative algebra is a Σ_3-associative algebra.*

Proof. Every subgroup G_i of Σ_3 corresponds to an invariant linear space $F(v_i)$ generated by a single vector $v_i \in \mathbb{K}[\Sigma_3]$. More precisely we consider $v_1 = Id$, $v_2 = Id - \tau_{12}$, $v_3 = Id - \tau_{23}$, $v_4 = Id - \tau_{13}$, $v_5 = Id + c_1 + c_2$ and $v_6 = V$ that we have defined in (2.1). □

Proposition 4. *Every G_i-associative algebra is a Lie-admissible algebra.*

Proof. The vector V belongs to the orbits $\mathcal{O}(v_i)$ for every v_i. Thus, if μ is a G_i-associative product, it also satisfies

$$A(\mu)\circ\Phi_V^{\mathcal{A}} = 0$$

and μ is a Lie-admissible multiplication. □

We deduce the following type of Lie-admissible algebras:

1. A G_1-associative algebra is an associative algebra.
2. A G_2-associative algebra is a Vinberg algebra. If A is finite-dimensional, the associated Lie algebra is provided with an affine structure.

3. A G_3-associative algebra is a pre-Lie algebra.
4. If $(\mathcal{A},\mu)$ is G_4-associative then μ satisfies

$$(X\cdot Y)\cdot Z - X\cdot(Y\cdot Z) = (Z\cdot Y)\cdot X - Z\cdot(Y\cdot X)$$

with $X\cdot Y=\mu(X,Y)$.

5. If $(\mathcal{A},\mu)$ is G_5-associative then μ satisfies the generalized Jacobi condition :

$$(X\cdot Y)\cdot Z + (Y\cdot Z)\cdot X + (Z\cdot X)\cdot Y = X\cdot(Y\cdot Z) + Y\cdot(Z\cdot X) + Z\cdot(X\cdot Y)$$

with $X\cdot Y=\mu(X,Y)$. Moreover if the product is skew-symmetric, then it is a Lie algebra bracket.

6. A G_6-associative algebra is a Lie-admissible algebra.

2.3 G_i-coalgebras

Dualizing the formula (2.3) we obtain the notion of G_i-coalgebra.

Definition 4. A G_i-coalgebra is a $\mathbb{K}$-vector space M provided with a comultiplication Δ satisfying

$$\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}\Phi_\sigma^M\circ\tilde{A}(\Delta)=0$$

Remark. We can present an equivalent and axiomatic definition of the notion of G_i-associative algebra. A G_i-associative algebra is $(\mathcal{A},\mu,\eta,G_i)$ where $\mathcal{A}$ is a vector space, G_i a subgroup of Σ_3, $\mu:\mathcal{A}\otimes\mathcal{A}\longrightarrow\mathcal{A}$ and $\eta:\mathbb{K}\rightarrow\mathcal{A}$ are linear maps satisfying the following axioms:

1. (G_i-ass): The square

$$\begin{tikzcd}
\mathcal{A}\otimes\mathcal{A}\otimes\mathcal{A} \arrow[r, "(\mu\otimes Id)_{G_i}"] \arrow[d, "(Id\otimes\mu)_{G_i}"'] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\mu"'] \\
\mathcal{A}\otimes\mathcal{A} \arrow[r, "\mu"] & \mathcal{A}
\end{tikzcd}$$

commutes, where $(Id\otimes\mu)_{G_i}$ is the linear mapping defined by:

$$(Id\otimes\mu)_{G_i}=\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}(Id\otimes\mu)\circ\Phi_\sigma^{\mathcal{A}}$$

If we impose that the algebra is unitary we have to add the following axiom:

2. (Un) The following diagram is commutative:

$$\begin{tikzcd}
\mathbb{K}\otimes\mathcal{A} \arrow[r, "\eta\otimes id"] \arrow[dr] & \mathcal{A}\otimes\mathcal{A} \arrow[d, "\mu"] & \mathcal{A}\otimes\mathbb{K} \arrow[l, "id\otimes\eta"'] \arrow[dl] \\
& \mathcal{A} &
\end{tikzcd}$$

The axiom (G_i-ass) expresses that the multiplication μ is G_i-associative whereas the axiom (Un) means that the element $\eta(1)$ of $\mathcal{A}$ is a left and right unit for μ. We want to dualize the previous diagrams to obtain the notions of corresponding coalgebras. Let Δ be a comultiplication $\Delta:M\longrightarrow M\otimes M$. We define the bilinear map

$$G_i\circ(\Delta\otimes Id):M^{\otimes 3}\longrightarrow M^{\otimes 3}$$

by

$$G_i \circ (\Delta \otimes Id) = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \Phi_\sigma^{\mathcal{A}} \circ (\Delta \otimes Id)$$

A G_i-coalgebra is a vector space M provided with a comultiplication $\Delta : M \longrightarrow M \otimes M$ and a counit $\epsilon : M \to \mathbb{K}$ such that

1. (G_i-ass co) The following square is commutative:

$$\begin{tikzcd} M \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & M \otimes M \arrow[d, "G_i \circ (Id \otimes \Delta)"] \\ M \otimes M \arrow[r, "G_i \circ (\Delta \otimes Id)"] & M \otimes M \otimes M \end{tikzcd}$$

   If we suppose moreover that the coalgebra is counitary we have to add the following axiom:

2. (Coun) The following diagram is commutative:

$$\begin{tikzcd} \mathbb{K} \otimes M & M \otimes M \arrow[l, "\epsilon \otimes id"'] \arrow[r, "id \otimes \epsilon"] & M \otimes \mathbb{K} \\ & M \arrow[ul] \arrow[u, "\Delta"] \arrow[ur] & \end{tikzcd}$$

A morphism of G_i-coalgebras

$$f : (M, \Delta, \epsilon, G_i) \to (M', \Delta', \epsilon', G_i)$$

is a linear map from M to M' such that

$$(f \otimes f) \circ \Delta = \Delta' \circ f \quad \text{and} \quad \epsilon = \epsilon' \circ f$$

Proposition 5. *Every G_i-coalgebra is a Lie-admissible coalgebra.*

Proof. The Lie-admissible coalgebras are given by the relation

$$\sum_{\sigma \in \Sigma_3} (-1)^{\epsilon(\sigma)} \Phi_\sigma^M \circ \tilde{A}(\Delta) = \Phi_V^M \circ \tilde{A}(\Delta) = 0$$

Since for every $v_i = \sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} \sigma$ we have $V \in F_{v_i}$, we deduce the proposition. □

2.4 The dual space of a G_i-coalgebra

For any natural number n and any $\mathbb{K}$-vector spaces E and F, we denote by

$$\lambda_n : Hom(E, F)^{\otimes n} \longrightarrow Hom(E^{\otimes n}, F^{\otimes n})$$

the natural embedding

$$\lambda_n(f_1 \otimes \ldots \otimes f_n)(x_1 \otimes \ldots \otimes x_n) = f_1(x_1) \otimes \ldots \otimes f_n(x_n)$$

Proposition 6. *The dual space of a G_i-coalgebra is provided with a structure of G_i-associative algebra.*

Proof. Let (M,Δ) be a G_i-coalgebra. We consider the multiplication on the dual vector space M^* of M defined by $\mu = \Delta^* \circ \lambda_2$. It provides M^* with a G_i-associative algebra structure. In fact we have

$$\mu(f_1 \otimes f_2) = \mu_{\mathbb{K}} \circ \lambda_2(f_1 \otimes f_2) \circ \Delta \tag{2.4}$$

for all $f_1, f_2 \in M^*$ where $\mu_{\mathbb{K}}$ is the multiplication in $\mathbb{K}$. Equation (2.4) becomes:

$$\begin{aligned}\mu \circ (\mu \otimes Id)(f_1 \otimes f_2 \otimes f_3) &= \mu_{\mathbb{K}} \circ (\lambda_2(\mu(f_1 \otimes f_2) \otimes f_3)) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ \lambda_2((\mu_{\mathbb{K}} \circ \lambda_2(f_1 \otimes f_2) \circ \Delta) \otimes f_3) \circ \Delta \\ &= \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta\end{aligned}$$

The associator $A(\mu)$ satisfies

$$\begin{aligned}A(\mu) = &+ \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta \\ &- \mu_{\mathbb{K}} \circ (Id \otimes \mu_{\mathbb{K}}) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (Id \otimes \Delta) \circ \Delta\end{aligned}$$

and using associativity and commutativity of the multiplication in $\mathbb{K}$, we obtain

$$A(\mu) = \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ ((\Delta \otimes Id) \circ \Delta - (Id \otimes \Delta) \circ \Delta)$$

Thus

$$\begin{aligned}\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} A(\mu) \circ \Phi_\sigma^{M^*} &= \mu_{\mathbb{K}} \circ (\mu_{\mathbb{K}} \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (G_i \circ (\Delta \otimes Id) \circ \Delta \\ &\qquad - G_i \circ (Id \otimes \Delta) \circ \Delta) \\ &= 0\end{aligned}$$

and (M^*,μ) is a G_i-associative algebra. □

Proposition 7. *The dual vector space of a finite dimensional G_i-associative algebra has a G_i-coalgebra structure.*

Proof. Let $\mathcal{A}$ be a finite dimensional G_i-associative algebra and let $\{e_i, i = 1, ..., n\}$ be a basis of $\mathcal{A}$. If $\{f_i\}$ is the dual basis then $\{f_i \otimes f_j\}$ is a basis of $\mathcal{A}^* \otimes \mathcal{A}^*$. The coproduct Δ on $\mathcal{A}^*$ is defined by

$$\Delta(f) = \sum_{i,j} f(\mu(e_i \otimes e_j)) f_i \otimes f_j$$

In particular

$$\Delta(f_k) = \sum_{i,j} C_{ij}^k f_i \otimes f_j$$

where C_{ij}^k are the structure constants of μ related to the basis $\{e_i\}$. Then Δ is the comultiplication of a G_i-associative coalgebra. □

3 The convolution product

Let us recall that if $(\mathcal{A},\mu)$ is associative $\mathbb{K}$-algebra and (M,Δ) a coassociative $\mathbb{K}$-coalgebra (i.e. a G_1-coalgebra) then the convolution product

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

provides $Hom(M,\mathcal{A})$ with an associative algebra structure. This result can be extended to the G_i-associative algebras and coalgebras. But we have to introduce the notion of $G_i^!$-coalgebras defined by the Koszul duality in the operads theory [2] and [3].

3.1 The $G_i^!$-algebras and coalgebras

Let $G_i - \mathcal{A}ss$ be the quadratic operad associated to the G_i-associative algebras. In [3] and [8], we show that these operads satisfy the Koszul duality as soon as $i = 1, 2, 3, 6$. Let $G_i - \mathcal{A}ss^!$ be the dual operad. We will call a $G_i^!$-algebra any algebra on $G_i - \mathcal{A}ss^!$. These algebras are defined as follows:

Definition 5. For $i \geq 2$, a $G_i^!$-algebra is an associative algebra satisfying

- for $i = 2$: $x_1 \cdot x_2 \cdot x_3 = x_2 \cdot x_1 \cdot x_3$,
- for $i = 3$: $x_1 \cdot x_2 \cdot x_3 = x_1 \cdot x_3 \cdot x_2$,
- for $i = 4$: $x_1 \cdot x_2 \cdot x_3 = x_3 \cdot x_2 \cdot x_1$,
- for $i = 5$: $x_1 \cdot x_2 \cdot x_3 = x_2 \cdot x_3 \cdot x_1 = x_3 \cdot x_1 \cdot x_2$,
- for $i = 6$: $x_1 \cdot x_2 \cdot x_3 = x_{\sigma(1)} \cdot x_{\sigma(2)} \cdot x_{\sigma(3)}$ for all x_1, x_2, x_3 and $\sigma \in \Sigma_3$.

Definition 6. For $i \geq 2$, a $G_i^!$-coalgebra is a coassociative coalgebra satisfying

$$\Phi_\sigma^M \circ (Id \otimes \Delta) \circ \Delta = (Id \otimes \Delta) \circ \Delta \quad \text{for every} \quad \sigma \in G_i$$

We will provide $Hom(M, \mathcal{A})$ with a structure of G_i-associative algebra.

Proposition 8. *Let $(\mathcal{A}, \mu)$ be a G_i-associative algebra and (M, Δ) a $G_i^!$-coalgebra. Then the algebra $(Hom(M, \mathcal{A}), \star)$ is a G_i-associative algebra where $\star$ is the convolution product*

$$f \star g = \mu \circ \lambda_2(f \otimes g) \circ \Delta$$

Proof. Let us compute the associator $A(\star)$ of the convolution product. Since

$$\begin{aligned}(f_1 \star f_2) \star f_3 &= \mu \circ \lambda_2((f_1 \star f_2) \otimes f_3) \circ \Delta \\ &= \mu \circ \lambda_2((\mu \circ \lambda_2(f_1 \otimes f_2) \circ \Delta) \otimes f_3) \circ \Delta \\ &= \mu \circ (\mu \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta\end{aligned}$$

we have

$$\begin{aligned}A(\star)(f_1 \otimes f_2 \otimes f_3) &= \mu \circ (\mu \otimes Id) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta \\ &\quad - \mu \circ (Id \otimes \mu) \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (Id \otimes \Delta) \circ \Delta\end{aligned}$$

Therefore

$$\begin{aligned}&\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} A(\star) \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\ &\qquad = \mu \circ (\mu \otimes Id) \circ (\sum_{\sigma \in G_i} \lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3))) \circ (\Delta \otimes Id) \circ \Delta \\ &\qquad - \mu \circ (Id \otimes \mu) \circ (\sum_{\sigma \in G_i} \lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3))) \circ (Id \otimes \Delta) \circ \Delta\end{aligned}$$

But

$$\lambda_3(\Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3)) = \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ \Phi_{\sigma^{-1}}^M$$

This gives

$$\begin{aligned}&\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} A(\star) \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\ &\qquad = \mu \circ (\mu \otimes Id) \circ (\sum_{\sigma \in G_i} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^M \circ (\Delta \otimes Id) \circ \Delta \\ &\qquad - \mu \circ (Id \otimes \mu) \circ (\sum_{\sigma \in G_i} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ \Phi_{\sigma^{-1}}^M \circ (Id \otimes \Delta) \circ \Delta\end{aligned}$$

Since Δ is coassociative,

$$(\Delta \otimes Id) \circ \Delta = (Id \otimes \Delta) \circ \Delta$$

the $G_i^!$-coalgebra structure implies

$$\Phi_\sigma^M \circ (Id \otimes \Delta) \circ \Delta = \Phi_\sigma^M \circ (\Delta \otimes Id) \circ \Delta = (\Delta \otimes Id) \circ \Delta$$

Then

$$\begin{aligned}
&\sum_{\sigma \in G_i} (-1)^{\epsilon(\sigma)} A(\star) \circ \Phi_\sigma^{Hom(M,A)}(f_1 \otimes f_2 \otimes f_3) \\
&\qquad = \mu \circ (\mu \otimes Id) \circ (\sum_{\sigma \in G_i} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes Id) \circ \Delta \\
&\qquad - \mu \circ (Id \otimes \mu) \circ (\sum_{\sigma \in G_i} \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3)) \circ (\Delta \otimes Id) \circ \Delta \\
&\qquad = \sum_{\sigma \in G_i} A(\mu) \circ \Phi_\sigma^A \circ \lambda_3(f_1 \otimes f_2 \otimes f_3) \circ (\Delta \otimes Id) \circ \Delta \\
&\qquad = 0
\end{aligned}$$

This proves the proposition. □

3.2 Lie-admissible bialgebras.

Definition 7. A Lie-admissible bialgebra is a triple $(\mathcal{A}, \mu, \Delta)$ where $(\mathcal{A}, \mu)$ is a Lie-admissible algebra and $(\mathcal{A}, \Delta)$ a Lie-admissible coalgebra with a compatibility condition between Δ and μ:

$$\Delta \circ A(\mu) \circ \Phi_{G_6}^{\mathcal{A}} = 0$$

Here we do not assume that the algebra and coalgebra are unitary and counitary. Among Lie-admissible bialgebras, we shall have the class of G_i-bialgebras. As example, a compatibility condition for pre-Lie bialgebras (that is G_3-bialgebras) is given by

$$\Delta \circ \mu = (Id \otimes \mu) \circ (\Delta \otimes Id) + (\mu \otimes Id) \circ \Phi_{\tau_{23}}^{\mathcal{A}} \circ (\Delta \otimes Id)$$

4 Tensor product of Lie-admissible algebras and coalgebras

4.1 Tensor product of G_i and $G_i^!$-algebras

We know that the tensor product of associative algebras can be provided with an associative algebra structure. In other words, the category of associative algebras is monoidal and closed for the tensor product. This is not true in general for other categories of Σ_3-associative algebras.

Proposition 9. *Let $(\mathcal{A}, \mu_A)$ and $(\mathcal{B}, \mu_B)$ be two Σ_3-associative algebras respectively defined by the relations $A(\mu_A) \circ \Phi_v^{\mathcal{A}} = 0$ and $A(\mu_B) \circ \Phi_w^{\mathcal{B}} = 0$. Then $(\mathcal{A} \otimes_{\mathbb{K}} \mathcal{B}, \mu_A \otimes \mu_B)$ is a Σ_3-associative algebra if and only if $\mathcal{A}$ and $\mathcal{B}$ are associative algebras (i.e G_1-associative algebras).*

Proof. See [4]. □

But we have:

Theorem 10. *If $\mathcal{A}$ is a G_i-associative algebra and $\mathcal{B}$ a $G_i^!$-algebra (with the same index) then $\mathcal{A} \otimes \mathcal{B}$ can be provided with a G_i-algebra structure for $i = 1, \ldots, 6$.*

Proof. Let us consider on $\mathcal{A}\otimes\mathcal{B}$ the classical tensor product

$$\mu_A\otimes\mu_B((a_1\otimes b_1)\otimes(a_2\otimes b_2))=\mu_A(a_1\otimes a_2)\otimes\mu_B(b_1\otimes b_2)$$

To simplify, we denote by μ the product $\mu_A\otimes\mu_B$. As $\mathcal{B}$ is an associative algebra, the associator $A(\mu)$ satisfies

$$A(\mu)((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3))=A(\mu_A)(a_1\otimes a_2\otimes a_3)\otimes\mu_B\circ(\mu_B\otimes Id)(b_1\otimes b_2\otimes b_3)$$

Therefore

$$\begin{aligned}&\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}A(\mu)\circ\Phi_\sigma^{\mathcal{A}\otimes\mathcal{B}}((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3))\\&\quad=\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}A(\mu_A)\circ\Phi_\sigma^{\mathcal{A}}(a_1\otimes a_2\otimes a_3)\otimes\mu_B\circ(\mu_B\otimes Id)\circ\Phi_\sigma^{\mathcal{B}}(b_1\otimes b_2\otimes b_3)\end{aligned}$$

But $\mathcal{B}$ a $G_i^!$-algebra. Then

$$\mu_B\circ(\mu_B\otimes Id)\circ\Phi_\sigma^{\mathcal{B}}(b_1\otimes b_2\otimes b_3)=\mu_B\circ(\mu_B\otimes Id)(b_1\otimes b_2\otimes b_3)$$

for any $\sigma\in G_i$. So we obtain

$$\begin{aligned}&\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}A(\mu)\circ\Phi_\sigma^{\mathcal{A}\otimes\mathcal{B}}((a_1\otimes b_1)\otimes(a_2\otimes b_2)\otimes(a_3\otimes b_3))\\&\quad=(\sum_{\sigma\in G_i}(-1)^{\epsilon(\sigma)}A(\mu_A)\circ\Phi_\sigma^{\mathcal{A}}(a_1\otimes a_2\otimes a_3))\otimes\mu_B\circ(\mu_B\otimes Id)(b_1\otimes b_2\otimes b_3)\\&\quad=0\end{aligned}$$

This proves the proposition. ☐

4.2 Tensor product of G_i-coalgebras

Let (M_1,Δ_1) and (M_2,Δ_2) be two Lie-admissible coalgebras and Δ the composite

$$M_1\otimes M_2\xrightarrow{\Delta_1\otimes\Delta_2}(M_1\otimes M_1)\otimes(M_2\otimes M_2)\xrightarrow{id_{M_1}\otimes\tau\otimes id_{M_2}}(M_1\otimes M_2)\otimes(M_1\otimes M_2)$$

If Δ_1 is a comultiplication of G_i-coalgebra, what should be the structure of (M_2,Δ_2) such that Δ is a comultiplication of G_i-coalgebra too?

Proposition 11. *Let (M_1,Δ_1) be a G_i-coalgebra and (M_2,Δ_2) a $G_i^!$-coalgebra. Then $(M_1\otimes M_2,\Delta)$ is provided with a G_i-coalgebra structure.*

Proof. Using classical notations we have

$$\tilde{A}(\Delta)(v\otimes w)=v_1^1\otimes w_1^1\otimes v_1^2\otimes w_1^2\otimes v_2\otimes w_2-v_1\otimes w_1\otimes v_2^1\otimes w_2^1\otimes v_2^2\otimes w_2^2$$

Let $\chi:(M_1\otimes M_2)^{\otimes 3}\rightarrow M_1^{\otimes 3}\otimes M_2^{\otimes 3}$ be the isomorphism given by

$$\chi(v_1\otimes w_1\otimes v_2\otimes w_2\otimes v_3\otimes w_3)=v_1\otimes v_2\otimes v_3\otimes w_1\otimes w_2\otimes w_3$$

Thus we obtain, from the hypothesis on Δ_2

$$\chi\circ\Phi_{G_i}^{M_1\otimes M_2}\circ\tilde{A}(\Delta)=\Phi_{G_i}^{M_1}\circ\tilde{A}(\Delta_1)\otimes(\Delta_2\otimes Id)\circ\Delta_2$$

which is zero because Δ_1 is a G_i-comultiplication. As χ is an isomorphism, we deduce the proposition. ☐

Remark. In [5] we have generalized this study and defined for any quadratic operad $\mathcal{P}$ a quadratic operad $\tilde{\mathcal{P}}$ so that the tensor product of a $\mathcal{P}$-algebra with a $\tilde{\mathcal{P}}$-algebra is provided with a $\mathcal{P}$-algebra structure. In the previous case we have always $\tilde{\mathcal{P}}=\mathcal{P}^!$.

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