

Liouvilie Type Results for a Class of Quasilinear Parabolic Problem

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Abstract

In this paper, we investigate the parabolic logistic equation with blow-up initial and boundary values on a smooth bounded domain

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = a(x,t) f(u) on \Omega \times (0,T), \\ u = \infty on \partial \Omega \times (0,T) \cup \overline{\Omega} \times \{0\} \end{cases}$$

where T>0. Under suitable assumptions on a(x, t) and f, we show that such solution stays bounded in any compact subset of Ω as t increase to T. Other asymptotic estimates will given in this work.

Keywords: Parabolic Cauchy problem; Blow-up solutions; Liouville theorems; Comparison principle; Super and subsolution

Introduction

Let Ω a bounded domain in \mathbb{R}^N , $(N \ge 3)$ with boundary $\partial \Omega$. In this work we consider the boundary blow-up parabolic problem:

$$\begin{aligned} \Delta u &- \frac{\partial u}{\partial t} = a(x,t) f(u) on \Omega \times (0,T), \\ u &= \infty on \partial \Omega \times (0,T) \cup \overline{\Omega} \times \{0\} \end{aligned}$$
(1.1)

The nonlinearity f is assumed to fulfil either

(F)
$$f \in C^{1}(R), f \ge 0$$
 and $f^{j} \ge 0$

and the Keller-Osserman condition

(KO)
$$\int_{a}^{\infty} \left[f(t) \right]^{-\frac{1}{2}} \le \infty, \forall a \ge 0 \text{ where } F(t) = \int_{a}^{s} (s) ds$$

Throughout this work, we assume that the function a(x, t) is continuous on $\Omega \times [0, T]$ and there exist a constant $\gamma >$ -2and positive continuous functions $\alpha_1(t)$ and $\alpha_2(t)$ on [0, T] such that, for $(x, t) \in [0, T]$, we have

$$\alpha_1(t) (d(x,\partial\Omega))^{\gamma} \le \alpha(x,t) \le \alpha_2(t) (d(x,\partial\Omega))^{\gamma}$$
(1.2)
And $a(x,T) = 0$ on $\Omega \times \{T\}$

Remark that $\alpha_1(T)=0$.

The main purpose of this work is to find proprieties of large solutions (blow-up solutions) of (1.1), that is solutions u satisfying

$$\lim_{d(x,\partial\Omega)\to 0} u(x,t) = 0 \tag{1.3}$$

uniformly for any t \in (0,T). We denote that u= ∞ on $\overline{\Omega} \times \{0\}$ means that

$$\lim_{t\to 0} u(x,t) = \infty$$

uniformly for $x \in \overline{\Omega}$

The existence of such solution is associated to the existence of large solutions to the stationary equation

$$\Delta v - a(x,t) f(v) = 0 \text{, in } R^n \times (o,t)$$
(1.4)

satisfying

$$\lim_{d(x,\partial\Omega)\to 0} \nu(x) = \infty$$
(1.5)

and solutions of the ODE

$$\varphi' - a(x,t) f(\varphi) = 0, \quad (0,\infty)$$
 (1.6)

Subject to the initial blow-up condition:

$$\lim_{t \to 0} \varphi(t) = \infty \tag{1.7}$$

The mathematical theory of Blow-up began in the sixties of the latest century, with the works of Fujita and Friedman [1-3]. The two main models considered in those works were the semilinear heat equation with $f(u)=u^p$ and $f(u)=e^u$. From those days, an increasing interest on blow-up problems has attracted a great number of researchers [4-6]. Equations of the type (1.1) arise especially in population dynamics and ecological models, where the nonnegative quantity u typically stands for the concentration of a species [7-9]. Stationary problems associated to (1.1)1 arises in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings [10-12] and in mathematical modeling if non- Newtonian fluids, see [13-15] for a discussion of the physical background. Our study in the present work was partly motivated first by Belhaj et al. [16], in which the authors treated the problem (1.1) in the stationary case and the papers [17,18] concerning the long time asymptotic behavior of a parabolic logistic equation with a degenerate spatial-temporal coefficient, of the form

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = a(x,t)u - b(x,t)u^{p} \text{ on } \Omega \times (0,T), \\ u = \infty \text{ on } \partial \Omega \times (0,T) \cup \overline{\Omega} \times \{0\} \end{cases}$$
(1.8)

In the author studies [6] the existence and the existence of blow-up parabolic problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = f(u) on \Omega \times (0, T), \\ u = \infty on \quad \partial \Omega \times (0, T) \cup \overline{\Omega} \times \{0\} \end{cases}$$
(1.9)

where satisfies the assumption in the following form

$$\forall (x, y) \in \mathbb{R}^2, x \ge y \ge m \Longrightarrow f(x, y) \ge f(x) + f(y) - L$$

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In this note, we will also study the qualitative properties of the global equation

$$\Delta u - \frac{\partial u}{\partial t} = a(x,t) f(u) on R^n \times (o,T)$$
(1.10)

Throughout this study, we will make reference to the assumptions:

(F1)
$$\exists c_0 \ge 0, y_o \ge 0$$
 and $m \ge 1f(y) \ge c_o y^m$, $\forall y \ge y_o$

Under hypothesis (F1), our first result deals with the existence of function depending only on the variable t and $d(x,\partial\Omega)$ which is an upper bound of the solution of the problem (1.1). Precisely, we have

Theorem 1.1

Assume that f satisfies condition (F_2) and there exists c_2>0 and $\gamma{<}2$ such that

$$\frac{c_2}{\left(d\left(x,\partial\Omega\right)\right)\gamma} \le a\left(x,t\right) \tag{1.11}$$

Then there exist a positive constant C such that any solution u of (1.1) satisfies

$$u(x,t) \le C \left(1 + \left[(q-2)bt \right]^{\frac{1}{2-q}} \right) \left[1 + \frac{1}{(d(x,\partial\Omega))} \right]^{\frac{1}{m-1}}$$
(1.12)

where q>1 and the real b is a positif.

Under the hypothesis of Theorem 1.1, by extending $d(x, \partial \Omega)$ to the infinity, we obtain our second result.

Theorem 1.2

Assume that f satisfies condition (F₂) and there exists c₂>0 and <2 such that

$$\frac{c_2}{\left(d\left(x,\partial\Omega\right)\right)\gamma} \le a\left(x,t\right) \tag{1.13}$$

Then there exist a positive constant C such that any solution u of (1.10) satisfies

$$u(x,t) \le C \left(1 + \left[\left(q-2\right)bt \right]^{\frac{1}{2-q}} \right)$$
(1.14)

where q>1 and the real b is a positif.

Note that the proof Theorem 1.1 and Theorem 1.2 and other preliminary result are the aim of the second section. The third paragraph is devoted to the study of explosive solution (1.1). If we set

$$V(x_o, r, a, b) = \{(x, t) \in \mathbb{R}^N, ||x - x_o||\} \le r \text{ and } a \le t \le b$$

And $\Xi = \{|V(x_o, r, a, b), x_o \in \mathbb{R}^N, r \ge 0, a, b \in \mathbb{R}\}$

It's well known that Ξ is a basis of open sets in X. For $V \in \Xi$ we denote by $\partial_{\mu} V$ the heat boundary of V i.e. the following set

 $\partial_h V = \left\{ (x, t) \in \partial V \le t \ b \in R \right\}$

whenever $V = V(x_o, \mathbf{r}, a, b)$

The result interesting existence of explosive solutions is the following

Theorem 1.3

Suppose that there exists m>1, satisfying

$$a(x,t)f(y) \ge y^m, (x,t) \in \mathbb{R}^N \times (0,T)$$
 and $y \ge 0$

Then for every $V \in \Xi$; there exists a unique positive function u such that

$$\Delta u - \frac{\partial u}{\partial t} - a(x,t) f(u) = 0 \text{ in } R^N \times (0,T)$$
(1.15)
And

 $\lim_{y \in V} u(y) = \infty \text{ for every } z \in \partial_h V$ (1.16)

Theorem 1.4

Under condition (1.2),(F1) and (KO), problem (1.1) has a maximal solution \overline{u} and the minimal positive solution \underline{u} in the sense that any positive solution u of (1.1) satisfies $\underline{u} \le u \le \overline{u}$ furthermore, for any $t_o \in (0,T)$ there exist positive constants c_1 and c_2 depending on t_o , such that, for $(x,t) \in \Omega \times [0,T]$

$$\overline{u}(x,t) \le c_1 \left(t^{\frac{1}{1-m}} + \left(d(x,\partial\Omega) \right)^{\frac{2+\gamma}{m-1}} \right)$$
(1.17)

And

$$\underline{u}(x,t) \ge c_2 \left(t^{\frac{1}{1-m}} + \left(d\left(x,\partial\Omega\right) \right)^{-\frac{2+\gamma}{m-1}} \right)$$
(1.18)

Our result of uniqueness of solution of (1.1) is the following

Theorem 1.5

Let f be a concave function on $[0,\infty)$. Then under assumptions of Theorem 1.4, the problem (1.1) has a unique positive solution.

Preliminaries

We start this section by some preliminary results. For the reader's convenience, we recall the defination of supersolution and the subsolution of (1.1). To this end, let Ω denotes a bounded domain of \mathbb{R}^{N} .

Definition 2.1

Let Ω be a bounded domain of \mathbb{R}^n . A function $u \in L^2_{loc} 2l \ u \in l$ $((0;\infty);0H_{loc}^{-1}(\Omega))\cup H_{loc}^{-1x}(0;\infty; L^2(\Omega))$ such that $f(u)\in L_{loc}^{-2}(0,\infty;L_{loc}^{-2}(\Omega))$ is a weak supersolution (resp. subsolution) of (1.1) if,

$$\iint_{\Omega} \left[\left(\frac{\partial u}{\partial t} \right) \varphi + \nabla u \nabla \varphi - a(x,t) f(u) \varphi \right] (,t) dx \ge 0$$
(2.1)

for all t>0 and $\phi \in H_0^{-1}(\Omega)$, $\phi \ge 0$. A function u is a solution of (1.1) if and only if u is a supersolution and a subsolution of (1.1).

In the sequel, we consider a real number R>0 and $x_0 \in R^N$: The first preliminary result we need is a standard comparison principle which will be used frequently in this paper.

Proposition 2.2

Let u be a subsolution of (1.1) and v be a supersolution of (1.1) such that

$$\limsup_{x \to z} \left[v(x,t) - u(x,t) \right] \ge 0 \text{ for all } z \in \partial \Omega \text{ and } t \in (0,T)$$
(2.2)

And

$$\lim_{t \to 0^+} \left\| (u - v)^+ (., t) \right\|^2 = 0$$

Suppose that $f(v(x,t)) \ge f(u(x,t))$ on the set $u(x,t) \le v(x,t)$, then $u(x, t) \le v(x, t)$; for all $x \in \Omega$ and t (0, T).

Proof: The function $\varphi = (u - v)^+(.; t) \in W_0^{1,p}(\Omega)$; for all $t \in (0, t)$.

Using the fact that u is a subsolution and v is a supersolution of (1.1) and testing in (1.1) by (ϕ =u-v)⁺(.; t); we obtain:

$$\int_{\Omega} \frac{\partial u}{\partial t} \varphi + \int_{u \ge v+\epsilon} \nabla u \nabla \varphi dx - \int_{u \ge v+\epsilon} a(x,t) f(u) \varphi dx \le 0$$
(2.3)

And

$$\int_{\Omega} \frac{\partial u}{\partial t} \varphi + \int_{u \ge v + \epsilon} \nabla v \nabla \varphi dx - \int_{u \ge v + \epsilon} a(x, t) f(v) \varphi dx \le 0$$
(2.4)

Denote by

$$I_{1} = \int_{u \ge v+\epsilon} (\nabla v - \nabla u) \nabla (u - v) dx$$
and
$$(2.5)$$

$$I_{2} = \int_{u \ge v+\epsilon} a(x,t) (f(u) - f(v)) (u - v - \epsilon) dx$$
(2.6)

Combining eqns. (2.3) and (2.4), we get $f_{\Omega} \frac{\partial \varphi}{\partial t} \varphi \leq I_1 + I_2$. On the other hand, we have,

$$I_1 \leq 0$$
 and $I_2 \leq 0$

Hence, we get,

$$\int_{\Omega} \frac{\partial \varphi}{\partial t} \varphi \leq 0$$

That is $\frac{\partial}{\partial t} \int_{\Omega} \varphi^2(x, t) dx \leq 0$.

Then the function $t \mapsto \int_{\Omega} \varphi^2(x,t) dx$ in non-increasing on the set (0,T) and we get

 $\int_{\Omega} \varphi^2(x,t) dx \le \lim_{t \to 0} \left\| \varphi^+(x,t) \right\|^2 = 0$

Since u and v are continuous, therefore $\phi^+=(u-v)^+(., t)=0$ and $u(x, t) \le v(x, t)$, for all $x \in \Omega$ and $t \in (0; T)$: as required.

In the sequel, we introduce the function h on (0,1) which is very useful for the proof of Theorem 1.1.

$$h(t) = 1 + \left[(q-2)bt \right]^{\frac{1}{2-q}}, q \ge 2 \ b \ge 0$$
Note the function h satisfies
$$(2.7)$$

Note the function h satisfies

on
$$h'(t) + bh^{q-1}(t) \ge 0$$
 (0, ∞) (2.8)

And
$$\lim_{t \to 0^+} h(t) = +\infty$$
 and $\lim_{t \to \infty} h(t) = 1$ (2.9)

Remarks 2.3: Proposition 2.2 is well known [3] for the first half of this Proposition. It holds on a more general setting allowing f to depend on x; t and u, and for weak super and sub-solutions of (1.1) which are unbounded but satisfy certain growth conditions near x=1. We refer the interested readers [19] in which Lemma 2.3 is proved as a special case by using a maximum principle in a study [1] (Chapter2, Theorem 9) and the monotone iteration method in Theorem 3.1 of a study [19]. To obtain information about the solution, we will need the following result, which plays an important role in the proof of our main result. To this end, let denote a bounded domain of \mathbb{R}^n and we consider a real $\mathbb{R} \ge 1$ and $x_o \in \mathbb{R}^n$.

Lemma 2.4

There exists $\lambda{>}0$ and a nondecreasing function h on (0,1) such that the function $v_{_\lambda}$ defined by, Let

$$v_{\lambda}(x,t) = \frac{\lambda h(t)}{\left(R^2 - |x - x_o|^2\right)^{\frac{2}{m-1}}} on \ B(x_o, R) \times (0,T)$$

Satisfies where is a positive constant and h as in (2.7). We have

$$\lim_{\infty \to z} v\lambda(x,t) = +\infty \text{ for all } z \in \partial B(x_o, R), t \in (0,T)$$
(2.10)

And
$$-\frac{\partial u\lambda}{\partial t} + \Delta v_{\lambda} - \beta v_{\lambda} \le 0$$
 in $B(x_o, R) \times (0, T)$ (2.11)

Proof: For each $x \in B(x_o, R)$, we denote $r = |x-x_o|$ and we consider $\lambda h(t)$

the function
$$v_{\lambda}$$
 defined by $v_{\lambda} = \frac{1}{\left(R^2 - r^2\right)^{\frac{2}{m-1}}}$ for some $\lambda > 1$ [20-25]. So

it's an easy task to show that v_{λ} fulfil the first part of the Lemma. By mean of a straightforward calculation we verify that (2.11) is equivalent to (2.12)

$$h'(t)(R^{2}-r^{2})^{2}+\beta h^{m}\lambda^{m-1} \ge h(t)\left[\frac{4n}{m-1}(R^{2}-r^{2})+8\frac{m+1}{(m-1)^{2}}r^{2}\right] \quad (2.12)$$

Using assumption on the function h, precisely (2.8), eqn. (2.12) holds true and so (2.11) if

$$-bh^{q-1}(t)(R^2-r^2)^2+\beta h^m\lambda^{m-1}\geq h(t)\left[\frac{4n}{m-1}(R^2-r^2)+8\frac{m+1}{(m-1)^2}r^2\right]$$

Since h>1 them, (2.12) implies (2.11), if

$$-b(R^{2}-r^{2})^{2}+\beta\lambda^{m-1} \ge \left[\frac{4n}{m-1}(R^{2}-r^{2})+8\frac{m+1}{(m-1)^{2}}r^{2}\right]$$
(2.13)

$$\lambda \ge \left[\frac{b}{\beta}R^4 + \left[\frac{4n}{\beta(m-1)} + 8\frac{m+1}{\beta(m-1)^2}\right]R^2\right]^{m-1}$$
(2.14)

set

$$\lambda_{o} \ge \left[\frac{b}{\beta}R^{4} + \left[\frac{4n}{\beta(m-1)} + 8\frac{m+1}{\beta(m-1)^{2}}\right]R^{2}\right]^{\frac{1}{m-1}}$$
(2.15)

Proposition 2.5

Assume that f satisfes condition (F2) and there exist $c_2{>}0$ and $\gamma{<}2$ such that

$$\frac{c_2}{\left(d\left(x,\partial\Omega\right)\right)^{\gamma}} \le a(x,t) \tag{2.16}$$

Then there exist a positive constant C such that any solution u of (1.1) satisfies

$$u(x,t) \le C \left(1 + \left[(q-2)bt \right]^{\frac{1}{2-q}} \right) \left[1 + \frac{1}{\left(d(x,\partial\Omega) \right)^2} \right]^{\frac{1}{m-1}}$$
(2.17)

Proof: Let u a continuous solution of (1.1) and choose $\eta > 0$ such that $u(x)>y_0$ whenever $d(x,\partial\Omega) \leq$ wherever y_0 is defined as in (F1). Set

$$E = \left\{ x \in D, d\left(x, \partial \Omega\right) \le \frac{\eta}{2} \right\}$$

For $x \in E$, we set $B=B(x,R=d(x, \partial \Omega))=2$). Hence, if $y \in B$, we get from (F1) and (2.19)

$$-u_{t}(y,t) + \Delta u(y,t) = a(x,t) f(u)(y,t)$$

$$\geq \frac{C_{2}C_{O}}{(\varsigma(y))^{\gamma^{2}}} u^{m}(y,t)$$

$$\geq \frac{C_{2}C_{O}}{(3R)^{\gamma^{2}}} u^{m}(y,t)$$

Therefore, the solution u of (1.1) is subsolution of the problem

$$\frac{\partial v\lambda}{\partial t} + \Delta v_{\lambda} - \beta v_{\lambda}^{m} = 0$$
(2.18)

On the set in $B(x_0,R)\times(0,T)$, where rom (F1) andwhere 2:

Let
$$\lambda_{o} = \left[\frac{b}{\beta}R^{4} + \left(\frac{4n}{\beta(m-1)} + 8\frac{m+1}{\beta(m-1)^{2}}\right)R^{2}\right]^{\frac{1}{m-1}}$$
 given in (2.15), then the

function $v_{\lambda 0}$ is a sursolution of (2.18) on $B(x_0,R)$ satisfying $v_{\lambda 0}(x_0,0) = \lim_{t \to 0} v_{\lambda 0}(x,t) = +\infty \ge u(x,0) \quad \forall x \in B(x_0,R)$ in addition we have

$$\limsup \left(v_{\lambda o} \left(x, t \right) - u \left(x, t \right) \right) \ge 0, \forall t \ge 0$$

and for all $z \in \partial \Omega$. Since the function $r \to r^m$ is decreasing on $[0,\infty)$; and $v_{\lambda_0} \ge 0$ on the set $\{u \ge v_{\lambda_0}\}$, it follows from Proposition 2.2 that $u(x_0; t) \ge v_{\lambda_0}(x_0; t)$ and we get

$$u(x_{o},t) \leq \left(1 + \left[(q-2)bt\right]^{\frac{1}{2-q}}\right) \left[\frac{b}{\beta} + \left(\frac{4n}{\beta(m-1)} + \frac{m+1}{\beta(m-1)}\right) \frac{1}{R^{2}}\right]$$

Since $R=d(x,\partial\Omega)/2$ then we get,

$$u(x,t) \leq \left(1 + \left[(q-2)bt\right]^{\frac{1}{2-q}}\right) \left[\frac{b}{\beta} + \frac{\delta}{d(x,\partial\Omega)^2}\right]^{\frac{1}{m-1}}$$
(2.19)

Where $\delta = \frac{16n}{\beta(m-1)} + 32 \frac{m+1}{\beta(m-1)^2}$ then if we denote $C = \sup\left(\frac{b}{\beta}, \delta\right)$, we

can write the following identity

$$u(x,t) \le C \left(1 + \left[(q-2)bt \right]^{\frac{1}{2-q}} \right) \left[\frac{b}{\beta} + \frac{\delta}{d(x,\partial\Omega)^2} \right]^{\frac{1}{m-1}}$$
(2.20)

Proof of Theorem 1.3

In what follows, we consider X=R^N and $L = \Delta - \frac{\partial}{\partial t}$, the heat operator on X. For every $x_o \in R^N$, r>0 and a, $b \in R$ with a
b we denote by

$$V(x_o, r, a, b) = \left\{ (x, t) \in X, \|x - x_o\| \le r \text{ and } a \le t \le b \right\}$$

And $\Xi = \{V(x_o, r, a, b), x_o \in \mathbb{R}^N, r \ge 0, a, b \in \mathbb{R}\}$

It's well known that Ξ is a basis of open sets in X. For $V\in\Xi$ we denote by $\partial_h V$ the heat boundary of V i.e. the following set

$$\partial_h V = \{ (x,t) \in \partial V, a \le t \le b \in R \}$$

whenever $V=V(x_0, r, a, b)$. By a study [2], we have the following Lemma.

Lemma 3.1

For every V $\in \Xi$ there exists a positive function $u \in V$ with the following properties:

1) u and all derivatives $\frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_i^2}, \frac{\partial u}{\partial t}$ for i $\in \{1, 2, ..., N\}$ are uniformly holder continuous in V.

2) $\Delta - \frac{\partial}{\partial t} \le 0 \text{ on } V$

3) $\lim_{y\to z \in V} u(y)=0$ for every $z \in \partial h_y$.

Remarks 3.2: The function u of the previous Lemma satisfies the following properties [26,27]: There exists a constant M which is an upper bound to u (0<u<M);

$$\left\|\nabla u\right\|^{2} = \sum_{i=1}^{n} \left(\frac{\partial u}{\partial x_{i}}\right)^{2} \le M^{2}$$
(3.1)

$$0 \le \frac{\partial u}{\partial t} - \Delta u \le M \tag{3.2}$$

Proposition 3.3

Let m>1. For every $V \in \Xi$, there exists a function $w=w(x, t) \in C^{+}(V)$

such that

$$Lw = \Delta w - \frac{\partial w}{\partial t} \le a(x,t)f(v)$$
(3.3)

And
$$\lim_{(x,t)\to(z,s)} w(x,t) = \infty \text{ for all } (z,s) \in \partial_h V$$
 (3.4)

Proof: Let u be a function as in Lemma 3.1 and $w = \frac{1}{(cu)^{\beta}}$. We obtain the statement if we can choose c and β such that

$$Lw = \Delta w - \frac{\partial w}{\partial t} \le a(x,t) f(v)$$

We have $\Delta w = c^2 \beta(\beta + 1)(cu)^{-\beta - 2} |\nabla u|^2 - c\beta(cu)^{-\beta - 1} \Delta u$
And hence $\Delta w - \frac{\partial w}{\partial t} = c\beta(cu)^{-\beta - 1} \left(\frac{\partial u}{\partial t} - \Delta u\right) + c^2 \beta(\beta + 1)(cu)^{-\beta - 2} |\nabla u|^2$
Using remark (40), we get $\Delta w - \frac{\partial w}{\partial t} \le (cu)^{-\beta - 2} [c\beta M cu + c^2 \beta(\beta + 1)M^2]$
Then we get
 $\Delta w - \frac{\partial w}{\partial t} \le (cu)^{-\beta - 2} [c\beta M cu + c^2 \beta(\beta + 1)M^2]$

$$\Delta w - \frac{\partial d}{\partial t} \leq (cu)^{m} \left[c\beta Mcu + c^{2}\beta(\beta + 1)M^{2} \right]$$

for $\beta = \frac{2}{m-1}$ and $c = \frac{1}{M(\beta + 2)}$, we obtain
$$\Delta w - \frac{\partial w}{\partial t} \leq w^{m}$$

And using the fact that $w^m \le a(x,t)f(w)$, we get

$$\Delta w - \frac{\partial w}{\partial t} \le a(x,t) f(w)$$

The property 3 in Remark (3.2) yields the statement about the limit of w at the heat boundary.

Proof of Theorem 1.3 Completed

By a study [1], there exists for every $n \in N$ a unique function $u_{_n}$ on V such that

$$\begin{cases} \Delta u_n - \frac{\partial u_n}{\partial t} = a(x,t) f(u_n) \\ u_n = n \end{cases}$$
(3.5)

moreover, (u_n) is increasing.

Let v be a function on V with the properties of the Proposition 3.3. By the maximum principle, we have $u_n \le v$ for every n and we get $u=\supu_n$ satisfies on V the required equation. Let $v\in C^1(V)$ satisfying $\Delta v - \frac{\partial u}{\partial t} = a(x,t)f(v)$ with infinite limit at the heat boundary, by minimum principle, we have $u_n \le v$ for every $n \in N$. By passing to the limit as $n \rightarrow \infty$, we get $u \le v$.

Let $(x_0, t_0) \in V$ and since V is convex, it's a star domain at (x_0, t_0) and $(x_0+r(x-x_0); t_0+r2(t-t_0))$ is in V for every $(x, t) \ge V$ and $r \in [0, 1]$. Let us consider,

$$w(x) = \beta v \left(x_o + r \left(x - x_o \right) t_o + r^2 \left(t - t_o \right) \right)$$

We can verify that

$$\Delta w - \frac{\partial u}{\partial t} = w^m \le a(x,t) f(w)$$

For $\beta = r^2$, we obtain

$$\Delta w - \frac{\partial u}{\partial t} = w^m \le a(x, t) f(w)$$

Using minimum principle we get,

Page 4 of 7

$$(r^{2})^{\frac{1}{m-1}}v(x_{o}+r(x-x_{o}),t_{o}+r^{2}(t-t_{o})) \leq u(x,t) \text{ for every } r \in [0,1]$$

By passing to the limit as $r \uparrow 1$, we get

 $v(x, t) \leq u(x, t).$

Thus u=v, which is the required uniqueness.

Proof of Theorem 1.4

The proof of Theorem 1.4 is based on various comparison arguments [28-31]. We begin this section by some intermediary results which play an important role to the proof of the existence of minimal and maximal positive solution of the problem (1.1).

Lemma 4.1

The unique solution z* of the problem

$$-\Delta z = -d\left(x,\partial\Omega\right)^{\gamma} z^{m}, z = \infty on\partial\Omega$$

$$\tag{4.1}$$

Satisfies
$$\lim_{x \to i \le 2} \frac{z^*(x)}{(d(x))^{\frac{2+y}{m-1}}} = \left[\frac{(2+\gamma)(1+\gamma+m)}{(m-1)^2}\right]^{\frac{1}{m-1}}$$
(4.2)

Proof: It suffices to apply Theorem 6.15 in a study [8].

The second intermediary result is the following, note that it's proof is immediately.

Lemma 4.2

The auxiliary problem

$$v + v = 0 \qquad v(0) = \infty \tag{4.3}$$

has a unique solution v* given by the following formula

$$v^{*}(t) = \frac{1}{\left\lceil (1-m)t \right\rceil^{\frac{1}{m-1}}}, t \ge 0$$
(4.4)

Remarks 4.3: 1) There exist two positive constants c_1 and c_2 such that the unique solution z^* of (4.1) satisfies

$$c_{1}(d(x))^{\frac{2+\gamma}{m-1}} \le x^{+}(x) \le c_{2}(d(x))^{\frac{2+\gamma}{m-1}}$$
(4.5)

2) Due to the fact that the fact that the function α_1 (defined in eqn. (1.2)) is continuous and positive on [0,T], we may suppose that

$$\alpha_1 \ge m_{\epsilon} \text{ in } [0, T - \epsilon] \text{ for } \epsilon << 1 \text{ small}$$

$$(4.6)$$

3) If $\gamma \in (-2, 0)$, then we can and $c_3 > 0$ such that $c_3(d(x)) \ge 1$, in this case we obtain

$$\left(v^{+}\right)^{\prime} \geq -c_{a}\left(d\left(x\right)\right)^{\gamma}\left(v^{+}\right)^{m}$$

$$\tag{4.7}$$

For $n \in N^*$ and $\epsilon > 0$, we consider the problem

$$\begin{cases} \Delta u_n - \frac{\partial u_n}{\partial t} = a(x,t) f(u_n) & \Omega \times [0, T - \epsilon] \\ u_n = n \, on \, \partial \Omega \times [0, T - \epsilon] \cup \overline{\Omega} \times \{0\} \end{cases}$$

$$(4.8)$$

It's clear that (4.8) has a unique positive solution u_n . Using the classical comparison principle, we get $u_n < u_n + 1$. That is the sequence u_n is strictly increasing on the set $\overline{\Omega} \times [0, T - \epsilon]$. Now, we try to find a supersolution of (4.8) which is independent of the integer n.

Lemma 4.4

The problem (4.8) possess a super-solution U*.

Proof: We begin by the case $\gamma \in (-2,0)$. Using eqns. (4.5) and (4.7),

we can find a constant M>1 sufficiently large such that U*:= $M(v^* + z^*)$ satisfies,

$$U_{t}^{+} - \Delta U^{+} \ge -m_{\varepsilon} \left(d(x) \right)^{\gamma} \left(U^{*} \right)^{m}, x \in \Omega, t \ge 0$$

Using assumption (F1) on the function f, (4.6) and (1.2), we obtain

$$U_{t}^{*} - \Delta U^{*} \ge -a(x,t)f(U^{*}), \text{ for } x \in \Omega, t \in [0, T-\varepsilon]$$

$$(4.9)$$

Therefore U* is a super-solution of (4.8) and U*is independent on n. Now we prove the existence of super- solution for the case $\gamma \ge 0$ Set $U_0 = M_o (v^* + z^*)$ and $m_\gamma = inf\left(m, \frac{2m + \gamma}{2 + \gamma}\right)$, then there exists a constant $c_5 > 0$ such that

$$U_o^{\underline{m}\gamma} \le c_5 d(x)^{\gamma \frac{m-1}{2+\gamma}}$$

Indeed $U_o \ge z^* \ge c_1 d(x)^{\frac{2+\gamma}{m-1}}$. Due to left term of inequality (4.5) there exist $c_a > 0$, such that

$$d(x)^{\gamma} \leq c4z^{+}(x)^{-\gamma \frac{m-1}{2+\gamma}}$$

And thus
$$-\Delta z^* \ge -c_4 \left(z^*\right)^{m\gamma}, m_{\gamma} = \inf\left(m, \frac{2m+\gamma}{2+\gamma}\right)$$

Consequently, there exists $M_0 >> 1$ large enough such that -M ($v_{\pm} + \tau_{\pm}$) satisfying (U_1) $\downarrow_{\pm} \rightarrow U_2 \stackrel{-C_0 m_{\pm}}{\longrightarrow} U_2 \stackrel{m_{\pm}}{\longrightarrow} U_2$

$$J_0 = M_0 (v^* + z^*) \text{ satisfying } (U_o) t - \Delta U_o \frac{\partial}{\partial c_o} U_o^T$$

where
$$c_0$$
 is the constant defined in (F1) and m_e as in (4.6). Then,
 $(U_o)t - \Delta U_o \frac{-c_o m_e}{c_5} d(x)^{\gamma} U_o^m$

By assumptions (F1), (1.2) and (4.6) we have

$$(U_o)_t - \Delta U_o \ge -a(x,t)f(U_o)$$

Therefore, $U^*=U_0$ satisfies (4.9), as required.

Now we are able to prove Theorem 1.4. The next Propositions deal the existence of minimal and maximal solutions of the problem (1.1).

Proposition 4.5

Assume the condition (1.2),(F1) and (KO) are satisfied then the problem (1.1) has a minimal positive solution u, in the sens that any positive solution u of (1.1) satisfies $\underline{u} \leq u$.

Proof: First, it must be mentioned that we conserve notations used in Lemma 4.1, 4.2 and 4.10.

Let n be an integer number. Then U^{*} a super-solution and u_n a solution of (4.8), we have U^{*}(x; t)>u_n(X; t) on $\partial\Omega \times$ (0; T- ε) and on $\overline{\Omega}$ for all small ε >0. Using the comparison principle, we obtain

$$u_n(X, \mathbf{t}) \leq U^+(x, t) \text{ on } \overline{\Omega} \times (0, T - \epsilon), \epsilon >> 1$$

It should be noticed that, for any compact K of and for $\epsilon \ll 1$, the function U^{*} is bounded on the set K × (ϵ , T - ϵ). By standard regularity arguments, $u_n(x; t) \rightarrow u(x; t)$ as $n \rightarrow \infty$ uniformly on any compact subset of $\Omega \times (0; T)$, where u satisfies (1.1). Thus \underline{u} , is a solution to (1.1) [32,33]. Note that u is the minimal positive solution of (1.1). Indeed, let u a positive solution of (1.1) and apply the comparison principle we ge:

By passing to the limit as $n \rightarrow \infty$, we obtain

 $\underline{u} \leq u \quad on \ \Omega \times \left(0, T\right)$

Since ϵ is arbitrary small, then

 $\underline{u} \le u \quad on \ \Omega \times (0,T)$

Thus u is a the minimal positive solution of the problem (1.1).

Proposition 4.6

Assume that conditions (1.2), (F1) and (KO) are satisfied. Then problem (1.1) has a maximal positive solution u, in the sens that any positive solution u of (1.1) satisfies $u \le \overline{u}$

Proof: Now we will prove the existence of the maximal solution of the problem (1.1). For this aim, we set

 $\Omega_{\epsilon} := \left\{ x \in \Omega; d(x, \partial \Omega) \ge \epsilon \right\}$

where ϵ is a positive small real and we consider the following problem

$$\begin{cases} \Delta u - \frac{\partial u}{\partial t} = a(x,t) f(u) on \Omega_{\epsilon} \times (\epsilon,T), \\ u = \infty on \ \partial \Omega \times (\epsilon,T) \cup \Omega_{\epsilon} \times \{\epsilon\} \end{cases}$$
(4.10)

let us denote u^{ϵ} the minimal positive solution of (4.10). If we apply the parabolic comparison principle, we get

 $\underline{u}^{\epsilon_1} \geq \underline{u}^{\epsilon_2} \geq \underline{u} \text{ if } \epsilon_1 \geq \epsilon_2 \geq 0$

Moreover, we can extract a decreasing sequence $\epsilon_n \rightarrow 0$ satisfying $\underline{u}^{\epsilon_n} \rightarrow \overline{u}$ as $\epsilon_n \rightarrow 0$ and \overline{u} is a solution of (1.1). Indeed, for any positive solution u of (1.1) and using the parabolic comparison principle we obtain

 $\underline{u}^{\in n} \ge u$ in $\Omega_n \times (\in_n, T)$

By passing to the limit as $n \rightarrow \infty$, we get $\overline{u} \ge u$, which proves that u is the maximal solution of (1.1).

Proof of Theorem 1.4 Completed

First step: Proof of the inequality (1.18)

Using the previous arguments, we have

$$\underline{u} \ge \underline{u} \le M\left(v_{\delta} + z_{\delta}^{+}\right) \text{ in } \Omega_{\delta} \times \left(\delta, T - \epsilon\right)$$

$$(4.11)$$

Where M≥1 is a constant independent of $\delta, v^*_{\delta}(t)=v^*(t-\delta)$, and the function z^*_{δ} is the unique positive solution of the problem

$$-\Delta z = -d\left(x,\partial\Omega\right)^{\gamma} z^{m}, z = \infty \quad on \ \partial\Omega_{\delta} \tag{4.12}$$

In the case $\gamma \in (-2, 0)$. In the other case i.e. $\gamma \ge 0$, the function $z*_{\delta}$ will be considered as the unique positive solution of

$$-\Delta z = -d\left(x,\partial\Omega\right)^{\gamma} z^{m}, z = \infty \quad on \ \partial\Omega_{\delta}$$

$$(4.13)$$

Note that the existence and uniqueness of $z*_{\delta}$ was treated in a study [8] precisely Theorem 6.15.

Passing to the limit as $\delta \rightarrow 0$ in (4.11), we get $z_{\delta}^* \rightarrow z_a$ and

$$\underline{u} \le \overline{u} \le M \left(v^+ + z^+ \right) \text{ in } \Omega \times (0, T - \epsilon) \tag{4.14}$$

Combining eqns. (4.17), (4.5) and (4.14), we get immediately the inequality (1.17) in Theorem 1.4.

Second step: Proof of the inequality (17)

To reach this aim, we pursue the same method used in the proof of

(1.17) and we need the following two Lemma [34,35].

Lemma 4.7

The auxiliary problem

 $-\Delta z = -d\left(x,\partial\Omega\right)^{\gamma} z^{m}, z = n \quad on \ \partial\Omega \tag{4.15}$

has a unique solution z_n , which converges uniformly on any compact set of Ω to the solution z^* of (4.1).

Lemma 4.8

The auxiliary problem

$$v' + v^m = 0, t \ge \mu \text{ and } v(-\mu) = \infty$$
 (4.16)

has a unique solution v* given by the following formula

$$v_{\mu}^{*}(t) = \frac{1}{\left[(1-m)(t+\mu)\right]^{\frac{1}{m-1}}}, t \ge 0$$
(4.17)

Remarks 4.9: 1) Remark that $v_{\mu}^* \rightarrow v^*$ as $\mu \rightarrow 0$ where v^* is the solution of (4.3). For the proof, we refer to Theorem 6.15 in a study [8].

2) Due to the fact that the fact that the function α_2 (defined in (1.2)) is continuous and positive on [0, T], we may suppose that there exists a constant $K_c>0$ such that

$$\alpha_2 \le K_{\epsilon} \text{ in } [0, T-\epsilon] \text{ for } \epsilon \le 1$$

$$(4.18)$$

3) Due to eqn. (4.5), there exists a constant $c_{2}>0$ such that

$$z_{7}^{-1}(d(x))^{\frac{2+\gamma}{m-1}} \le z^{+}(x) \le c_{7}(d(x))^{\frac{2+\gamma}{m-1}}$$
(4.19)

4) If $\gamma > 0,$ then we can find $c_{_8} > 0$ such that $c_{_8}(d(x)^{\gamma} \le 1$ in In this case we obtain

$$(v^{+}\mu) \leq -c_{8} (d(x))^{\gamma} (v^{+}\mu)^{m}$$
 (4.20)

Lemma 4.10

The problem (4.8) possess a sub-solution U_*

Proof: We begin by the case $\gamma \ge 0$. For $k \ge 0$, small enough, we set

$$U_{+} = c_{o}k_{e}(v^{+}\mu + z^{+})$$

where c_0 as in (F1).

Using equations (4.19) and (4.20), we get

 $(U_*)_t - \Delta U_* \leq -c_o k_{\epsilon} (d(x))^{\gamma} U_*^m$, for $x \in \Omega$ and $t \geq -\mu$

Shrinking the constant
$$k_{e}$$
 and using assumption (F1), we can write
 $(U_{e}) = \Delta U_{e} \leq a(x, t) f(U_{e})$

$$\left(U_{*}\right)_{t} - \Delta U_{*} \leq a(x,t) f\left(U_{*}\right)$$

Therefore the function fufill the proposition in this case.

If $\gamma \in (-2, 0)$. Due to eqn. (4.5), there exists $c_q > 0$ such that

$$(d(x))^{\gamma} \ge c_{9} \left(z^{+}\right)^{\overline{m}_{\gamma}-m} \text{ where } \overline{m}_{\gamma} = \sup\left(m, \frac{2m+1}{m+2}\right)$$

For k₁>0 small, we get
$$U_{+}^{1} = c_{o}k_{1}\left((v_{\mu}^{+} + z^{+}\right)$$
(4.21)

$$(\Delta U_*^1)_t - \Delta U_*^1 \le -k_1 \left(d(x)\right)^{\gamma} \left(U_*^1\right)^m, \text{ for } x \in \Omega \text{ and } t \ge -v$$

Choose k₁, such that $k_1(d(x))^{\gamma} \le a(x; t)$, then we get by (F1)

 $(\Delta U_*^1)_{t} - \Delta U_*^1 \le k_1 (d(x))^{\gamma} (U_*^1)^{m}$

Now we are ready to prove desisable inequality (1.17). Let \overline{u} is a minimal solution of (1.1) and z_n is solution of (4.16). By shrinking the real k1 (defined as (4.21), we can write,

 $\underline{u}(x,t) \ge 2k_1 z_n(x)$ for $x \in \Omega$ and $t \in (0,t-\epsilon)$

Due to Lemma 4.7 we have also

$$\underline{u}(x,t) \ge 2k_1 z^*(x) \quad for \quad x \in \Omega \quad and \quad t \in (0,t-\epsilon)$$

Therefore if we set $[\varepsilon = \partial \Omega \times (0, t - \varepsilon] \cup \Omega \times \{0\}$, we obtain

$$\lim_{(\infty,t)\to\Gamma\in}\inf\left(u(x,t)-U_+(x,t)\right)\geq 0$$

Page 6 of 7

(5.1)

If we apply maximum principle, we get

 $\underline{u}(x,t) \ge k_1 \left(v_{\mu}^{+}(t) + z^{+}(x) \right), \text{ for } x \in \Omega \text{ and } t \in (0,t-\epsilon)$ (4.22)

Combining eqns. (4.19), (4.20) and (4.22), we obtain inequality (1.17)

Proof of Theorem 1.5

Proof: It suffices to show that $\underline{u} = \overline{u}$ in $\Omega \times (0,T,-\varepsilon)$ for $0 < \varepsilon < T$. Using inequalities (1.17) and (1.18), there exists a positive constant $\mu > 1$ such that

 $\underline{u} \le \overline{u} \le \mu u \quad in \, \Omega \times (0, T - \epsilon)$

Suppose that $\underline{u} = \overline{u}$ in the set $\Omega \times (0, T-\varepsilon)$ and set

$$V = \underline{u} - \frac{1}{2\mu} \left(\overline{u} - \underline{u} \right)$$

It's easy to verify that

 $\frac{2\mu}{2\mu+1}V + \frac{1}{2\mu+1}\overline{u} = \overline{u}$

And $\underline{u} \ge V$ in $\Omega \times (0, T - \epsilon)$

Using the fact that -f is convex we get,

 $f(\underline{u}) \ge \frac{2\mu}{2\mu+1} f(V) + \frac{1}{2\mu+1} f(\overline{u})$

Consequently,

$$V_{t} - \Delta V = a\left(x,t\right) \left[\frac{2\mu + 1}{2\mu} f\left(V\right) + \frac{1}{2\mu} f(\overline{u})\right] \ge a\left(x,t\right) f\left(V\right)$$

Hence V is supersolution of

$$\begin{cases} \frac{\partial V}{\partial t} - \Delta V \ge a(x,t) f\left(V \text{ on } \Omega \times (0,T-\epsilon)\right), \\ V = \infty \text{ on } \partial \Omega \times (0,T-\epsilon \cup \overline{\Omega} \times \{0\}) \end{cases}$$

Using the comparison principle, we get $u \le V$, which is contradiction with (5.1) then $\underline{u} = \overline{u}$ and consequently the result of theorem holds.

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