Journal of Physical Mathematics

# Liouvilie Type Results for a Class of Quasilinear Parabolic Problem 

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#### Abstract

In this paper, we investigate the parabolic logistic equation with blow-up initial and boundary values on a smooth bounded domain $$
\left\{\begin{array}{l} \Delta u-\frac{\partial u}{\partial t}=a(x, t) f(u) \text { on } \Omega \times(0, T), \\ u=\infty \text { on } \partial \Omega \times(0, T) \cup \bar{\Omega} \times\{0\} \end{array}\right.
$$ where $T>0$. Under suitable assumptions on $a(x, t)$ and $f$, we show that such solution stays bounded in any compact subset of $\Omega$ as increase to $T$. Other asymptotic estimates will given in this work.


Keywords: Parabolic Cauchy problem; Blow-up solutions; Liouville theorems; Comparison principle; Super and subsolution

## Introduction

Let $\Omega$ a bounded domain in $\mathrm{R}^{\mathrm{N}},(\mathrm{N} \geq 3)$ with boundary $\partial \Omega$. In this work we consider the boundary blow-up parabolic problem:

$$
\left\{\begin{array}{l}
\Delta u-\frac{\partial u}{\partial t}=a(x, t) f(u) o n \Omega \times(0, T),  \tag{1.1}\\
u=\infty \text { on } \partial \Omega \times(0, T) \cup \bar{\Omega} \times\{0\}
\end{array}\right.
$$

The nonlinearity f is assumed to fulfil either
(F) $f \in C^{1}(R), f \geq 0$ and $f^{j} \geq 0$
and the Keller-Osserman condition
(KO) $\int_{a}^{\infty}[f(t)]^{-\frac{1}{2}} \leq \infty, \forall a \geq 0$ where $F(t)=\int_{a}^{s}(s) d s$
Throughout this work, we assume that the function $a(x, t)$ is continuous on $\Omega \times[0, \mathrm{~T}]$ and there exist a constant $\gamma>-2$ and positive continuous functions $\alpha_{1}(t)$ and $\alpha_{2}(t)$ on $[0, T]$ such that, for $(x, t) \in[0$, T], we have

$$
\begin{equation*}
\alpha_{1}(t)(d(x, \partial \Omega))^{\gamma} \leq \alpha(x, t) \leq \alpha_{2}(t)(d(x, \partial \Omega))^{\gamma} \tag{1.2}
\end{equation*}
$$

And $a(x, T)=0$ on $\Omega \times\{T\}$
Remark that $\alpha_{1}(T)=0$.
The main purpose of this work is to find proprieties of large solutions (blow-up solutions) of (1.1), that is solutions u satisfying

$$
\begin{equation*}
\lim _{d(x, \partial \Omega) \rightarrow 0} u(x, t)=0 \tag{1.3}
\end{equation*}
$$

uniformly for any $\mathrm{t} \in(0, \mathrm{~T})$. We denote that $\mathrm{u}=\infty$ on $\bar{\Omega} \times\{0\}$ means that

$$
\lim _{t \rightarrow 0} u(x, t)=\infty
$$

uniformly for $\mathrm{x} \in \bar{\Omega}$
The existence of such solution is associated to the existence of large solutions to the stationary equation

$$
\begin{equation*}
\Delta v-a(x, t) f(v)=0, \text { in } R^{n} \times(o, t) \tag{1.4}
\end{equation*}
$$

satisfying

$$
\begin{equation*}
\lim _{d(x, \partial \Omega) \rightarrow 0} v(x)=\infty \tag{1.5}
\end{equation*}
$$

and solutions of the ODE

$$
\begin{equation*}
\varphi^{\prime}-a(x, t) f(\varphi)=0, \quad(0, \infty) \tag{1.6}
\end{equation*}
$$

Subject to the initial blow-up condition:

$$
\begin{equation*}
\lim _{t \rightarrow 0} \varphi(t)=\infty \tag{1.7}
\end{equation*}
$$

The mathematical theory of Blow-up began in the sixties of the latest century, with the works of Fujita and Friedman [1-3]. The two main models considered in those works were the semilinear heat equation with $f(u)=u^{p}$ and $f(u)=e^{u}$. From those days, an increasing interest on blow-up problems has attracted a great number of researchers [4-6]. Equations of the type (1.1) arise especially in population dynamics and ecological models, where the nonnegative quantity u typically stands for the concentration of a species [7-9]. Stationary problems associated to (1.1)1 arises in many nonlinear phenomena, for instance, in the theory of quasi-regular and quasi-conformal mappings [10-12] and in mathematical modeling if non- Newtonian fluids, see [13-15] for a discussion of the physical background. Our study in the present work was partly motivated first by Belhaj et al. [16], in which the authors treated the problem (1.1) in the stationary case and the papers [17,18] concerning the long time asymptotic behavior of a parabolic logistic equation with a degenerate spatial-temporal coefficient, of the form

$$
\left\{\begin{array}{l}
\Delta u-\frac{\partial u}{\partial t}=a(x, t) u-b(x, t) u^{p} \text { on } \Omega \times(0, T),  \tag{1.8}\\
u=\infty \text { on } \partial \Omega \times(0, T) \cup \bar{\Omega} \times\{0\}
\end{array}\right.
$$

In the author studies [6] the existence and the existence of blow-up parabolic problem

$$
\left\{\begin{array}{l}
\Delta u-\frac{\partial u}{\partial t}=f(u) \text { on } \Omega \times(0, T),  \tag{1.9}\\
u=\infty \text { on } \partial \Omega \times(0, T) \cup \bar{\Omega} \times\{0\}
\end{array}\right.
$$

where satisfies the assumption in the following form

$$
\forall(x, y) \in R^{2}, x \geq y \geq m \Rightarrow f(x, y) \geq f(x)+f(y)-L
$$

[^0]In this note, we will also study the qualitative properties of the global equation

$$
\begin{equation*}
\Delta u-\frac{\partial u}{\partial t}=a(x, t) f(u) \text { on } R^{n} \times(o, T) \tag{1.10}
\end{equation*}
$$

Throughout this study, we will make reference to the assumptions:
(F1) $\exists c_{0} \geq 0, y_{o} \geq 0$ and $m \geq 1 f(y) \geq c_{o} y^{m}, \forall y \geq y_{o}$
Under hypothesis (F1), our first result deals with the existence of function depending only on the variable $t$ and $d(x, \partial \Omega)$ which is an upper bound of the solution of the problem (1.1). Precisely, we have

## Theorem 1.1

Assume that f satisfies condition $\left(\mathrm{F}_{2}\right)$ and there exists $\mathrm{c}_{2}>0$ and $\gamma<2$ such that

$$
\begin{equation*}
\frac{c_{2}}{(d(x, \partial \Omega)) \gamma} \leq a(x, t) \tag{1.11}
\end{equation*}
$$

Then there exist a positive constant C such that any solution u of (1.1) satisfies

$$
\begin{equation*}
u(x, t) \leq C\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right)\left[1+\frac{1}{(d(x, \partial \Omega))}\right]^{\frac{1}{m-1}} \tag{1.12}
\end{equation*}
$$

where $\mathrm{q}>1$ and the real b is a positif.
Under the hypothesis of Theorem 1.1, by extending $d(x, \partial \Omega)$ to the infinity, we obtain our second result.

## Theorem 1.2

Assume that f satisfies condition $\left(\mathrm{F}_{2}\right)$ and there exists $\mathrm{c}_{2}>0$ and $<2$ such that

$$
\begin{equation*}
\frac{c_{2}}{(d(x, \partial \Omega)) \gamma} \leq a(x, t) \tag{1.13}
\end{equation*}
$$

Then there exist a positive constant C such that any solution u of (1.10) satisfies

$$
\begin{equation*}
u(x, t) \leq C\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right) \tag{1.14}
\end{equation*}
$$

where $\mathrm{q}>1$ and the real b is a positif.
Note that the proof Theorem 1.1 and Theorem 1.2 and other preliminary result are the aim of the second section. The third paragraph is devoted to the study of explosive solution (1.1). If we set

$$
\begin{aligned}
& V\left(x_{o}, r, a, b\right)=\left\{(x, \mathrm{t}) \in R^{N},\left\|x-x_{o}\right\|\right\} \leq r \text { and } a \leq t \leq b \\
& \text { And } \Xi=\left\{\mid V\left(x_{o} r, a, b\right), x_{o} \in R^{N}, r \geq 0, a, b \in R\right\}
\end{aligned}
$$

It's well known that $\Xi$ is a basis of open sets in $X$. For $V \in \Xi$ we denote by $\partial_{\mathrm{h}} \mathrm{V}$ the heat boundary of V i.e. the following set

$$
\partial_{h} V=\{(x, t) \in \partial V \leq \mathrm{t} b \in R\}
$$

whenever $V=V\left(x_{o}, \mathrm{r}, a, b\right)$
The result interesting existence of explosive solutions is the following

## Theorem 1.3

Suppose that there exists $\mathrm{m}>1$, satisfying

$$
a(x, t) f(y) \geq y^{m},(x, \mathrm{t}) \in R^{N} \times(0, T) \text { and } y \geq 0
$$

Then for every $\mathrm{V} \in \Xi$; there exists a unique positive function u such that

$$
\begin{equation*}
\Delta u-\frac{\partial u}{\partial t}-a(x, t) f(u)=0 \text { in } R^{N} \times(0, T) \tag{1.15}
\end{equation*}
$$

And

$$
\begin{equation*}
\lim _{Y \in V \rightarrow z} u(y)=\infty \text { for every } z \in \partial_{h} V \tag{1.16}
\end{equation*}
$$

## Theorem 1.4

Under condition (1.2),(F1) and (KO), problem (1.1) has a maximal solution $\bar{u}$ and the minimal positive solution $\underline{u}$ in the sense that any positive solution u of (1.1) satisfies $\underline{u} \leq u \leq \bar{u}$ furthermore, for any $t_{0} \in(0, T)$ there exist positive constants $c_{1}$ and $c_{2}$ depending on $t_{0}$, such that, for $(\mathrm{x}, \mathrm{t}) \in \Omega \times[0, \mathrm{~T}]$

$$
\begin{equation*}
\bar{u}(x, t) \leq c_{1}\left(t^{\frac{1}{1-m}}+(d(x, \partial \Omega))^{-\frac{2+\gamma}{m-1}}\right) \tag{1.17}
\end{equation*}
$$

And

$$
\begin{equation*}
\underline{u}(x, t) \geq c_{2}\left(t^{\frac{1}{1-m}}+(d(x, \partial \Omega))^{-\frac{2+\gamma}{m-1}}\right) \tag{1.18}
\end{equation*}
$$

Our result of uniqueness of solution of (1.1) is the following

## Theorem 1.5

Let $f$ be a concave function on $[0, \infty)$. Then under assumptions of Theorem 1.4, the problem (1.1) has a unique positive solution.

## Preliminaries

We start this section by some preliminary results. For the reader's convenience, we recall the defination of supersolution and the subsolution of (1.1). To this end, let $\Omega$ denotes a bounded domain of $\mathrm{R}^{\mathrm{N}}$.

## Definition 2.1

Let $\Omega$ be a bounded domain of $\mathrm{R}^{\mathrm{n}}$. A function $\mathrm{u} \in \mathrm{L}^{2}{ }_{\text {loc }} 2 \mathrm{l} u \in 1$ $\left((0 ; \infty) ; 0 \mathrm{H}_{\text {loc }}{ }^{1}(\Omega)\right) \mathrm{UH}_{\text {loc }}{ }^{1 \mathrm{~lx}}\left(0 ; \infty ; \mathrm{L}^{2}(\Omega)\right)$ such that $\mathrm{f}(\mathrm{u}) \in \mathrm{L}_{\text {loc }}{ }^{2}\left(0, \infty ; \mathrm{L}_{\text {loc }}{ }^{2}(\Omega)\right)$ is a weak supersolution (resp. subsolution) of (1.1) if,

$$
\begin{equation*}
\int_{\Omega}\left[\left(\frac{\partial u}{\partial t}\right) \varphi+\nabla u \nabla \varphi-a(x, t) f(u) \varphi\right](,, t) d x \geq 0 \tag{2.1}
\end{equation*}
$$

for all $\mathrm{t}>0$ and $\varphi \in \mathrm{H}_{0}{ }^{1}(\Omega), \varphi \geq 0$. A function $u$ is a solution of (1.1) if and only if $u$ is a supersolution and a subsolution of (1.1).

In the sequel, we consider a real number $\mathrm{R}>0$ and $\mathrm{x}_{\mathrm{o}} \in \mathrm{R}^{\mathrm{N}}$ : The first preliminary result we need is a standard comparison principle which will be used frequently in this paper.

## Proposition 2.2

Let u be a subsolution of (1.1) and v be a supersolution of (1.1) such that

$$
\begin{equation*}
\lim _{\infty \rightarrow z} \sup [v(x, t)-u(x, t)] \geq 0 \text { for all } \mathrm{z} \in \partial \Omega \text { and } \mathrm{t} \in(0, \mathrm{~T}) \tag{2.2}
\end{equation*}
$$

And
$\lim _{t \rightarrow 0^{+}}\left\|(u-v)^{+}(., t)\right\|^{2}=0$
Suppose that $\mathrm{f}(\mathrm{v}(\mathrm{x}, \mathrm{t})) \geq \mathrm{f}(\mathrm{u}(\mathrm{x}, \mathrm{t}))$ on the set $\mathrm{u}(\mathrm{x}, \mathrm{t}) \leq \mathrm{v}(\mathrm{x}, \mathrm{t})$, then $\mathrm{u}(\mathrm{x}$, $t) \leq v(x, t)$; for all $x \in \Omega$ and $t(0, T)$.

Proof: The function $\varphi=(\mathrm{u}-\mathrm{v})^{+}(. ; \mathrm{t}) \in \mathrm{W}_{0}{ }^{1, \mathrm{p}}(\Omega)$; for all $\mathrm{t} \in(0, \mathrm{t})$.
Using the fact that $u$ is a subsolution and $v$ is a supersolution of (1.1) and testing in (1.1) by ( $\varphi=\mathrm{u}-\mathrm{v})^{+}(. ; \mathrm{t})$; we obtain:

$$
\begin{equation*}
\int_{\Omega} \frac{\partial u}{\partial t} \varphi+\int_{u \geq v+\epsilon} \nabla u \nabla \varphi d x-\int_{u \geq v+\epsilon} a(x, t) f(u) \varphi d x \leq 0 \tag{2.3}
\end{equation*}
$$

Citation: Hsini M, Zahrani EA (2018) Liouvilie Type Results for a Class of Quasilinear Parabolic Problem. J Phys Math 9: 272. doi: 10.4172/20900902.1000272

And
$\int_{\Omega} \frac{\partial u}{\partial t} \varphi+\int_{u \geq v+\epsilon} \nabla v \nabla \varphi d x-\int_{u \geq v+\epsilon} a(x, t) f(v) \varphi d x \leq 0$
Denote by
$I_{1}=\int_{u \geq v+\epsilon}(\nabla v-\nabla u) \nabla(u-v) d x$
and
$I_{2}=\int_{u \geq v+\epsilon} a(x, t)(f(u)-f(v))(u-v-\epsilon) d x$
Combining eqns. (2.3) and (2.4), we get $f_{\Omega} \frac{\partial \varphi}{\partial t} \varphi \leq I_{1}+I_{2}$. On the other hand, we have,
$\mathrm{I}_{1} \leq 0$ and $\mathrm{I}_{2} \leq 0$.
Hence, we get,
$\int_{\Omega} \frac{\partial \varphi}{\partial t} \varphi \leq 0$
That is $\frac{\partial}{\partial t} \int_{\Omega} \varphi^{2}(x, t) d x \leq 0$.
Then the function $t \mapsto \int_{\Omega} \varphi^{2}(x, t) d x$ in non-increasing on the set $(0, T)$ and we get
$\int_{\Omega} \varphi^{2}(x, t) d x \leq \lim _{t \rightarrow 0}\left\|\varphi^{+}(x, t)\right\|^{2}=0$
Since $u$ and $v$ are continuous, therefore $\varphi^{+}=(u-v)^{+}(., t)=0$ and $u(x, t)$ $\leq v(x, t)$, for all $x \in \Omega$ and $t \in(0 ; T)$ : as required.

In the sequel, we introduce the function $h$ on $(0,1)$ which is very useful for the proof of Theorem 1.1.

$$
\begin{equation*}
h(t)=1+[(q-2) b t]^{\frac{1}{2-q}}, q \geq 2 b \geq 0 \tag{2.7}
\end{equation*}
$$

Note the function h satisfies

$$
\begin{align*}
& \text { on } h^{\prime}(t)+b h^{q-1}(t) \geq 0(0, \infty)  \tag{2.8}\\
& \text { And } \lim _{t \rightarrow 0^{+}} h(t)=+\infty \text { and } \lim _{t \rightarrow \infty} h(t)=1
\end{align*}
$$

Remarks 2.3: Proposition 2.2 is well known [3] for the first half of this Proposition. It holds on a more general setting allowing $f$ to depend on $\mathrm{x} ; \mathrm{t}$ and u , and for weak super and sub-solutions of (1.1) which are unbounded but satisfy certain growth conditions near $\mathrm{x}=1$. We refer the interested readers [19] in which Lemma 2.3 is proved as a special case by using a maximum principle in a study [1] (Chapter2, Theorem 9) and the monotone iteration method in Theorem 3.1 of a study [19]. To obtain information about the solution, we will need the following result, which plays an important role in the proof of our main result. To this end, let denote a bounded domain of $\mathrm{R}^{\mathrm{n}}$ and we consider a real $R \geq 1$ and $x_{0} \in R^{n}$.

## Lemma 2.4

There exists $\lambda>0$ and a nondecreasing function $h$ on $(0,1)$ such that the function $v_{\lambda}$ defined by, Let

$$
v_{\lambda}(x, t)=\frac{\lambda h(t)}{\left(R^{2}-\left|x-x_{o}\right|^{2}\right)^{\frac{2}{m-1}}} \text { on } B\left(x_{o}, R\right) \times(0, T)
$$

Satisfies where is a positive constant and h as in (2.7). We have

$$
\begin{align*}
& \lim _{\infty \rightarrow z} v \lambda(x, t)=+\infty \text { for all } z \in \partial B\left(x_{o}, R\right), t \in(0, T)  \tag{2.10}\\
& \text { And }-\frac{\partial u \lambda}{\partial t}+\Delta v_{\lambda}-\beta v_{\lambda} \leq 0 \text { in } B\left(x_{o}, R\right) \times(0, T) \tag{2.11}
\end{align*}
$$

Proof: For each $x \in B\left(x_{0}, R\right)$, we denote $r=\left|x-x_{0}\right|$ and we consider the function $v_{\lambda}$ defined by $v_{\lambda}=\frac{\lambda h(t)}{\left(R^{2}-r^{2}\right)^{\frac{2}{m-1}}}$ for some $\lambda>1$ [20-25]. So it's an easy task to show that $\mathrm{v}_{\lambda}$ fulfil the first part of the Lemma. By mean of a straightforward calculation we verify that (2.11) is equivalent to (2.12)

$$
\begin{equation*}
h^{\prime}(t)\left(R^{2}-r^{2}\right)^{2}+\beta h^{m} \lambda^{m-1} \geq h(t)\left[\frac{4 n}{m-1}\left(R^{2}-r 2\right)+8 \frac{m+1}{(m-1)^{2}} r^{2}\right] \tag{2.12}
\end{equation*}
$$

Using assumption on the function $h$, precisely (2.8), eqn. (2.12) holds true and so (2.11) if

$$
-b h^{q-1}(t)\left(R^{2}-r^{2}\right)^{2}+\beta h^{m} \lambda^{m-1} \geq h(t)\left[\frac{4 n}{m-1}\left(R^{2}-r 2\right)+8 \frac{m+1}{(m-1)^{2}} r^{2}\right]
$$

Since $h>1$ them, (2.12) implies (2.11), if

$$
\begin{align*}
& -b\left(R^{2}-r^{2}\right)^{2}+\beta \lambda^{m-1} \geq\left[\frac{4 n}{m-1}\left(R^{2}-r 2\right)+8 \frac{m+1}{(m-1)^{2}} r^{2}\right]  \tag{2.13}\\
& \lambda \geq\left[\frac{b}{\beta} R^{4}+\left[\frac{4 n}{\beta(m-1)}+8 \frac{m+1}{\beta(m-1)^{2}}\right] R^{2}\right]^{\frac{1}{m-1}}  \tag{2.14}\\
& \text { set }
\end{align*}
$$

$$
\begin{equation*}
\lambda_{o} \geq\left[\frac{b}{\beta} R^{4}+\left[\frac{4 n}{\beta(m-1)}+8 \frac{m+1}{\beta(m-1)^{2}}\right] R^{2}\right]^{\frac{1}{m-1}} \tag{2.15}
\end{equation*}
$$

## Proposition 2.5

Assume that f satisfes condition (F2) and there exist $\mathrm{c}_{2}>0$ and $\gamma<2$ such that

$$
\begin{equation*}
\frac{c_{2}}{(d(x, \partial \Omega))^{\gamma}} \leq a(x, t) \tag{2.16}
\end{equation*}
$$

Then there exist a positive constant $C$ such that any solution $u$ of (1.1) satisfies

$$
\begin{equation*}
u(x, t) \leq C\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right)\left[1+\frac{1}{(d(x, \partial \Omega))^{2}}\right]^{\frac{1}{m-1}} \tag{2.17}
\end{equation*}
$$

Proof: Let $u$ a continuous solution of (1.1) and choose $\eta>0$ such that $\mathrm{u}(\mathrm{x})>\mathrm{y}_{0}$ whenever $\mathrm{d}(\mathrm{x}, \partial \Omega) \leq$ wherever $\mathrm{y}_{0}$ is defined as in (F1). Set

$$
E=\left\{x \in D, d(x, \partial \Omega) \leq \frac{\eta}{2}\right\}
$$

For $x \in E$, we set $B=B(x, R=d(x, \partial \Omega))=2)$. Hence, if $y \in B$, we get from (F1) and (2.19)

$$
\begin{aligned}
& -u_{t}(y, t)+\Delta u(y, t)=a(x, t) f(u)(y, t) \\
& \geq \frac{C_{2} C_{O}}{(\varsigma(y))^{\gamma 2}} u^{m}(y, t) \\
& \geq \frac{C_{2} C_{O}}{(3 R)^{\gamma 2}} u^{m}(y, t)
\end{aligned}
$$

Therefore, the solution $u$ of (1.1) is subsolution of the problem

$$
\begin{equation*}
-\frac{\partial v \lambda}{\partial t}+\Delta v_{\lambda}-\beta v_{\lambda}^{m}=0 \tag{2.18}
\end{equation*}
$$

On the set in $B\left(x_{0}, R\right) \times(0, T)$, where rom (F1) andwhere 2 :

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Let $\lambda_{o}=\left[\frac{b}{\beta} R^{4}+\left(\frac{4 n}{\beta(m-1)}+8 \frac{m+1}{\beta(m-1)^{2}}\right) R^{2}\right]^{\frac{1}{m-1}}$ given in (2.15), then the function $\mathrm{v}_{\lambda_{0}}$ is a sursolution of (2.18) on $\mathrm{B}\left(\mathrm{x}_{0} \cdot \mathrm{R}\right)$ satisfying $\mathrm{v}_{\lambda_{0}}\left(\mathrm{x}_{0} .0\right)=$ $\lim _{t \rightarrow 0} v_{\lambda o}(x, t)=+\infty \geq u(x, 0) \quad \forall x \in B\left(x_{o}, R\right)$ in addition we have

$$
\lim _{x \rightarrow z} \sup \left(v_{\lambda o}(x, \mathrm{t})-u(x, t)\right) \geq 0, \forall t \geq 0
$$

and for all $\mathrm{z} \in \partial \Omega$. Since the function $\mathrm{r} \rightarrow \mathrm{r}^{\mathrm{m}}$ is decreasing on $[0, \infty)$; and $v_{\lambda_{0}} \geq 0$ on the $\operatorname{set}\left\{u \geq v_{\lambda_{0}}\right\}$, it follows from Proposition 2.2 that $u\left(x_{0} ; t\right)$ $\geq \mathrm{v}_{\lambda_{0}}\left(\mathrm{x}_{0} ; \mathrm{t}\right)$ and we get

$$
u\left(x_{o}, t\right) \leq\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right)\left[\frac{b}{\beta}+\left(\frac{4 n}{\beta(m-1)}+\frac{m+1}{\beta(m-1)}\right) \frac{1}{R^{2}}\right]
$$

Since $\mathrm{R}=\mathrm{d}(\mathrm{x}, \partial \Omega) / 2$ then we get,

$$
\begin{equation*}
u(x, t) \leq\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right)\left[\frac{b}{\beta}+\frac{\delta}{d(x, \partial \Omega)^{2}}\right]^{\frac{1}{m-1}} \tag{2.19}
\end{equation*}
$$

Where $\delta=\frac{16 n}{\beta(m-1)}+32 \frac{m+1}{\beta(m-1)^{2}}$ then if we denote $C=\sup \left(\frac{b}{\beta}, \delta\right)$, we can write the following identity

$$
\begin{equation*}
u(x, t) \leq C\left(1+[(q-2) b t]^{\frac{1}{2-q}}\right)\left[\frac{b}{\beta}+\frac{\delta}{d(x, \partial \Omega)^{2}}\right]^{\frac{1}{m-1}} \tag{2.20}
\end{equation*}
$$

## Proof of Theorem 1.3

In what follows, we consider $\mathrm{X}=\mathrm{R}^{\mathrm{N}}$ and $L=\Delta-\frac{\partial}{\partial t}$, the heat operator on X . For every $\mathrm{x}_{\mathrm{o}} \in \mathrm{R}^{\mathrm{N}}, \mathrm{r}>0$ and $\mathrm{a}, \mathrm{b} \in \mathrm{R}$ with $\mathrm{a}<\mathrm{b}$ we denote by
$V\left(x_{o}, r, a, b\right)=\left\{(x, t) \in X,\left\|x-x_{o}\right\| \leq r\right.$ and $\left.a \leq t \leq b\right\}$
And $\Xi=\left\{V\left(x_{o}, r, a, b\right), x_{o} \in R^{N}, r \geq 0, a, b \in R\right\}$
It's well known that $\Xi$ is a basis of open sets in $X$. For $V \in \Xi$ we denote by $\partial_{\mathrm{h}} \mathrm{V}$ the heat boundary of V i.e. the following set

$$
\partial_{h} V=\{(x, t) \in \partial V, a \leq t \leq b \in R\}
$$

whenever $\mathrm{V}=\mathrm{V}\left(\mathrm{x}_{0}, \mathrm{r}, \mathrm{a}, \mathrm{b}\right)$. By a study [2], we have the following Lemma.

## Lemma 3.1

For every $\mathrm{V} \in \Xi$ there exists a positive function $u \in \mathrm{~V}$ with the following properties:

1) u and all derivatives $\frac{\partial u}{\partial x_{i}}, \frac{\partial u}{\partial x^{2}}, \frac{\partial u}{\partial t}$ for $\mathrm{i} \in\{1,2, \ldots \mathrm{~N}\}$ are uniformly holder continuous in V.
2) $\Delta-\frac{\partial}{\partial t} \leq 0$ on $V$
3) limy $\rightarrow z \in V u(y)=0$ for every $z \in \partial h_{V}$.

Remarks 3.2: The function $u$ of the previous Lemma satisfies the following properties [26,27]: There exists a constant $M$ which is an upper bound to $u(0<u<M)$;

$$
\begin{align*}
& \|\nabla u\|^{2}=\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_{i}}\right)^{2} \leq M^{2}  \tag{3.1}\\
& 0 \leq \frac{\partial u}{\partial t}-\Delta u \leq M \tag{3.2}
\end{align*}
$$

## Proposition 3.3

Let $\mathrm{m}>1$. For every $\mathrm{V} \in \Xi$, there exists a function $\mathrm{w}=\mathrm{w}(\mathrm{x}, \mathrm{t}) \in \mathrm{C}^{+}(\mathrm{V})$
such that
$L w=\Delta w-\frac{\partial w}{\partial t} \leq a(x, t) f(v)$
And $\lim _{(x, t) \rightarrow(z, s)} w(x, t)=\infty$ for all $(z, s) \in \partial_{h} V$
Proof: Let u be a function as in Lemma 3.1 and $w=\frac{1}{(c u)^{\beta}}$. We obtain the statement if we can choose c and $\beta$ such that
$L w=\Delta w-\frac{\partial w}{\partial t} \leq a(x, t) f(v)$
We have $\Delta w=c^{2} \beta(\beta+1)(c u)^{-\beta-2}|\nabla u|^{2}-c \beta(c u)^{-\beta-1} \Delta u$
And hence $\Delta w-\frac{\partial w}{\partial t}=c \beta(c u)^{-\beta-1}\left(\frac{\partial u}{\partial t}-\Delta u\right)+c^{2} \beta(\beta+1)(c u)^{-\beta-2}|\nabla u|^{2}$
Using remark (40), we get $\Delta w-\frac{\partial w}{\partial t} \leq(c u)^{-\beta-2}\left[c \beta M c u+c^{2} \beta(\beta+1) M^{2}\right]$
Then we get
$\Delta w-\frac{\partial w}{\partial t} \leq(c u)^{-\beta-2}\left[c \beta M c u+c^{2} \beta(\beta+1) M^{2}\right]$
for $\beta=\frac{2}{m-1}$ and $c=\frac{1}{M(\beta+2)}$, we obtain
$\Delta w-\frac{\partial w}{\partial t} \leq w^{w}$
And using the fact that $w^{m} \leq a(x, t) f(w)$, we get
$\Delta w-\frac{\partial w}{\partial t} \leq a(x, t) f(w)$
The property 3 in Remark (3.2) yields the statement about the limit of $w$ at the heat boundary.

## Proof of Theorem 1.3 Completed

By a study [1], there exists for every $n \in N$ a unique function $u_{n}$ on V such that

$$
\left\{\begin{array}{l}
\Delta u_{n}-\frac{\partial u_{n}}{\partial t}=a(x, t) f\left(u_{n}\right)  \tag{3.5}\\
u_{n}=n
\end{array}\right.
$$

moreover, $\left(\mathrm{u}_{\mathrm{n}}\right)$ is increasing.
Let v be a function on V with the properties of the Proposition 3.3. By the maximum principle, we have $u_{n} \leq v$ for every $n$ and we get $u=\operatorname{supu}_{\mathrm{n}}$ satisfies on V the required equation. Let $\mathrm{v} \in \mathrm{C}^{1}(\mathrm{~V})$ satisfying $\Delta v-\frac{\partial u}{\partial t}=a(x, t) f(v)$ with infinite limit at the heat boundary, by minimum principle, we have $\mathrm{u}_{\mathrm{n}} \leq \mathrm{v}$ for every $\mathrm{n} \in \mathrm{N}$. By passing to the limit as $\mathrm{n} \rightarrow \infty$, we get $\mathrm{u} \leq \mathrm{v}$.

Let $\left(x_{0}, t_{0}\right) \in V$ and since $V$ is convex, it's a star domain at $\left(x_{0}, t_{0}\right)$ and $\left(x_{0}+r\left(x-x_{0}\right) ; t_{0}+r 2\left(t-t_{0}\right)\right)$ is in $V$ for every $(x, t) 2 V$ and $r \in[0,1]$. Let us consider,

$$
w(x)=\beta v\left(x_{o}+r\left(x-x_{o}\right) t_{o}+r^{2}\left(t-t_{o}\right)\right)
$$

We can verify that
$\Delta w-\frac{\partial u}{\partial t}=w^{m} \leq a(x, t) f(w)$
For $\beta=r^{2}$, we obtain
$\Delta w-\frac{\partial u}{\partial t}=w^{m} \leq a(x, t) f(w)$
Using minimum principle we get,
$\left(r^{2}\right)^{\frac{1}{m-1}} v\left(x_{o}+r\left(x-x_{o}\right), t_{o}+r^{2}\left(t-t_{o}\right)\right) \leq u(x, t)$ for every $r \in[0,1]$
By passing to the limit as $\mathrm{r} \uparrow 1$, we get
$v(x, t) \leq u(x, t)$.
Thus $\mathrm{u}=\mathrm{v}$, which is the required uniqueness.

## Proof of Theorem 1.4

The proof of Theorem 1.4 is based on various comparison arguments [28-31]. We begin this section by some intermediary results which play an important role to the proof of the existence of minimal and maximal positive solution of the problem (1.1).

## Lemma 4.1

The unique solution $\mathrm{z} *$ of the problem
$-\Delta z=-d(x, \partial \Omega)^{\gamma} z^{m}, z=\infty o n \partial \Omega$
Satisfies $\lim _{\infty \rightarrow \infty \Omega} \frac{z^{*}(x)}{(d(x))^{-\frac{2+y}{m-1}}}=\left[\frac{(2+\gamma)(1+\gamma+m)}{(m-1)^{2}}\right]^{\frac{1}{m-1}}$
Proof: It suffices to apply Theorem 6.15 in a study [8].
The second intermediary result is the following, note that it's proof is immediately.

## Lemma 4.2

The auxiliary problem
$v+v=0 \quad v(0)=\infty$
has a unique solution v * given by the following formula

$$
\begin{equation*}
v^{*}(t)=\frac{1}{[(1-m) t]^{\frac{1}{m-1}}}, t \geq 0 \tag{4.4}
\end{equation*}
$$

Remarks 4.3: 1) There exist two positive constants $c_{1}$ and $c_{2}$ such that the unique solution $\mathrm{z}^{\star}$ of (4.1) satisfies

$$
\begin{equation*}
c_{1}(d(x))^{-\frac{2+\gamma}{m-1}} \leq x^{+}(x) \leq c_{2}(d(x))^{-\frac{2+\gamma}{m-1}} \tag{4.5}
\end{equation*}
$$

2) Due to the fact that the fact that the function $\alpha_{1}$ (defined in eqn. $(1.2))$ is continuous and positive on $[0, \mathrm{~T}]$, we may suppose that
$\alpha_{1} \geq m_{\epsilon}$ in $[0, T-\epsilon]$ for $\in \ll 1$ small
3) If $\gamma \in(-2,0)$, then we can and $c_{3}>0$ such that $c_{3}(d(x)) \geq 1$, in this case we obtain

$$
\begin{equation*}
\left(v^{+}\right)^{\prime} \geq-c_{a}(d(x))^{\gamma}\left(v^{+}\right)^{m} \tag{4.7}
\end{equation*}
$$

For $\mathrm{n} \in \mathrm{N}^{*}$ and $\varepsilon>0$, we consider the problem
$\left\{\begin{array}{l}\Delta u_{n}-\frac{\partial u_{n}}{\partial t}=a(x, t) f\left(u_{n}\right) \quad \Omega \times[0, T-\epsilon] \\ u_{n}=n \text { on } \partial \Omega \times[0, T-\epsilon] \cup \bar{\Omega} \times\{0\}\end{array}\right.$
It's clear that (4.8) has a unique positive solution $u_{n}$. Using the classical comparison principle, we get $u_{n}<u_{n}+1$. That is the sequence $u_{n}$ is strictly increasing on the set $\bar{\Omega} \times[0, T-\epsilon]$. Now, we try to find a supersolution of (4.8) which is independent of the integer $n$.

## Lemma 4.4

The problem (4.8) possess a super-solution $\mathrm{U}^{*}$.
Proof: We begin by the case $\gamma \in(-2,0)$. Using eqns. (4.5) and (4.7),
we can find a constant $M>1$ sufficiently large such that $\mathrm{U}^{*}:=\mathrm{M}\left(\mathrm{v}^{*}+\mathrm{z}^{*}\right)$ satisfies,

$$
U_{t}^{+}-\Delta U^{+} \geq-m_{\varepsilon}(d(x))^{\gamma}\left(U^{*}\right)^{m}, x \in \Omega, t \geq 0
$$

Using assumption (F1) on the function $\mathrm{f},(4.6)$ and (1.2), we obtain
$U^{*}{ }_{t}-\Delta U^{*} \geq-a(x, t) f\left(U^{*}\right)$, for $x \in \Omega, t \in[0, T-\varepsilon]$
Therefore $\mathrm{U}^{*}$ is a super-solution of (4.8) and $\mathrm{U}^{*}$ is independent on n . Now we prove the existence of super- solution for the case $\gamma \geq 0$ Set $\mathrm{U}_{0}=\mathrm{M}_{\mathrm{o}}\left(\mathrm{v}^{*}+\mathrm{Z}^{*}\right)$ and $m_{\gamma}=\inf \left(m, \frac{2 m+\gamma}{2+\gamma}\right)$, then there exists a constant $\mathrm{c}_{5}>0$ such that
$U_{o}{ }^{\underline{m} \gamma} \leq c_{5} d(x)^{\gamma \frac{m-1}{2+\gamma}}$
Indeed $U_{o} \geq z^{*} \geq c_{1} d(x)^{-\frac{2+\gamma}{m-1}}$. Due to left term of inequality (4.5) there exist $\mathrm{c}_{4}>0$, such that
$d(x)^{\gamma} \leq c 4 z^{+}(x)^{-\gamma \frac{m-1}{2+\gamma}}$
And thus $-\Delta z^{*} \geq-c_{4}\left(z^{*}\right)^{\underline{m} \gamma}, m_{\gamma}=\inf \left(m, \frac{2 m+\gamma}{2+\gamma}\right)$
Consequently, there exists $M_{0} \gg 1$ large enough such that $\mathrm{U}_{0}=\mathrm{M}_{0}\left(\mathrm{v}^{*}+\mathrm{Z}^{*}\right)$ satisfying $\left(U_{o}\right) t-\Delta U_{o} \frac{-c_{o} m_{\in}}{c_{5}} U_{o}{ }^{\underline{m}}$
where $c_{0}$ is the constant defined in (F1) and $m_{\epsilon}$ as in (4.6). Then,
$\left(U_{o}\right) t-\Delta U_{o} \frac{-c_{o} m_{\epsilon}}{c_{5}} d(x)^{\gamma} U_{o}{ }^{\underline{m}}$
By assumptions (F1), (1.2) and (4.6) we have
$\left(U_{o}\right)_{t}-\Delta U_{o} \geq-a(x, t) f\left(U_{o}\right)$
Therefore, $\mathrm{U}^{*}=\mathrm{U}_{\mathrm{o}}$ satisfies (4.9), as required.
Now we are able to prove Theorem 1.4. The next Propositions deal the existence of minimal and maximal solutions of the problem (1.1).

## Proposition 4.5

Assume the condition (1.2),(F1) and (KO) are satisfied then the problem (1.1) has a minimal positive solution $u$, in the sens that any positive solution $u$ of (1.1) satisfies $\underline{u} \leq u$.

Proof: First, it must be mentioned that we conserve notations used in Lemma 4.1, 4.2 and 4.10.

Let $n$ be an integer number. Then $U^{*}$ a super-solution and $u_{n}$ a solution of (4.8), we have $\mathrm{U}^{*}(\mathrm{x} ; \mathrm{t})>\mathrm{u}_{\mathrm{n}}(\mathrm{X} ; \mathrm{t})$ on $\partial \Omega \times(0 ; \mathrm{T}-\varepsilon)$ and on $\bar{\Omega}$ for all small $\epsilon>0$. Using the comparison principle, we obtain

$$
u_{n}(X, \mathrm{t}) \leq U^{+}(x, t) \text { on } \bar{\Omega} \times(0, T-\epsilon), \epsilon \gg 1
$$

It should be noticed that, for any compact $K$ of and for $\epsilon \ll 1$, the function $U^{*}$ is bounded on the set $\mathrm{K} \times(\epsilon, \mathrm{T}-\varepsilon)$. By standard regularity arguments, $\mathrm{u}_{\mathrm{n}}(\mathrm{x} ; \mathrm{t}) \rightarrow \mathrm{u}(\mathrm{x} ; \mathrm{t})$ as $\mathrm{n} \rightarrow \infty$ uniformly on any compact subset of $\Omega \times(0 ; \mathrm{T})$, where u satisfies (1.1). Thus $\underline{u}$, is a solution to (1.1) $[32,33]$. Note that $u$ is the minimal positive solution of (1.1). Indeed, let $u$ a positive solution of (1.1) and apply the comparision principle we ge:

By passing to the limit as $\mathrm{n} \rightarrow \infty$, we obtain
$\underline{u} \leq u \quad$ on $\Omega \times(0, T)$
Since $\epsilon$ is arbitrary small, then
$\underline{u} \leq u \quad$ on $\Omega \times(0, T)$
Thus u is a the minimal positive solution of the problem (1.1).

Citation: Hsini M, Zahrani EA (2018) Liouvilie Type Results for a Class of Quasilinear Parabolic Problem. J Phys Math 9: 272. doi: 10.4172/20900902.1000272

## Proposition 4.6

Assume that conditions (1.2), (F1) and (KO) are satisfied. Then problem (1.1) has a maximal positive solution $u$, in the sens that any positive solution u of (1.1) satisfies $u \leq \bar{u}$

Proof: Now we will prove the existence of the maximal solution of the problem (1.1). For this aim, we set

$$
\Omega_{\epsilon}:=\{x \in \Omega ; d(x, \partial \Omega) \geq \in\}
$$

where $\epsilon$ is a positive small real and we consider the following problem

$$
\left\{\begin{array}{l}
\Delta u-\frac{\partial u}{\partial t}=a(x, t) f(u) o n \Omega_{\epsilon} \times(\in, T)  \tag{4.10}\\
u=\infty \text { on } \partial \Omega \times(\epsilon, T) \cup \Omega_{\epsilon} \times\{\epsilon\}
\end{array}\right.
$$

let us denote $u^{\epsilon}$ the minimal positive solution of (4.10). If we apply the parabolic comparison principle, we get

$$
\underline{u}^{\epsilon 1} \geq \underline{u}^{\epsilon 2} \geq \underline{u} \text { if } \epsilon_{1} \geq \epsilon_{2} \geq 0
$$

Moreover, we can extract a decreasing sequence $\epsilon_{\mathrm{n}} \rightarrow 0$ satisfying $\underline{u}^{\epsilon_{n}} \rightarrow \bar{u}$ as $\in_{\mathrm{n}} \rightarrow 0$ and $\bar{u}$ is a solution of (1.1). Indeed, for any positive solution $u$ of (1.1) and using the parabolic comparison principle we obtain

$$
\underline{u}^{\in n} \geq u \text { in } \Omega_{n} \times\left(\epsilon_{n}, T\right)
$$

By passing to the limit as $\mathrm{n} \rightarrow \infty$, we get $\bar{u} \geq u$, which proves that u is the maximal solution of (1.1).

## Proof of Theorem 1.4 Completed

## First step: Proof of the inequality (1.18)

Using the previous arguments, we have

$$
\begin{equation*}
\underline{u} \geq \underline{u} \leq M\left(v_{\delta}+z_{\delta}^{+}\right) \text {in } \Omega_{\delta} \times(\delta, T-\epsilon) \tag{4.11}
\end{equation*}
$$

Where $\mathrm{M} \geq 1$ is a constant independent of $\delta, \mathrm{v}_{\delta}(\mathrm{t})=\mathrm{v}^{*}(\mathrm{t}-\delta)$, and the function $\mathrm{z}^{*}$ is the unique positive solution of the problem

$$
\begin{equation*}
-\Delta z=-d(x, \partial \Omega)^{\gamma} z^{m}, z=\infty \quad \text { on } \partial \Omega_{\delta} \tag{4.12}
\end{equation*}
$$

In the case $\gamma \in(-2,0)$. In the other case i.e. $\gamma \geq 0$, the function $z{ }^{*}{ }_{\delta}$ will be considered as the unique positive solution of

$$
\begin{equation*}
-\Delta z=-d(x, \partial \Omega)^{\gamma} z^{m}, z=\infty \quad \text { on } \partial \Omega_{\delta} \tag{4.13}
\end{equation*}
$$

Note that the existence and uniqueness of $\mathrm{z} *{ }_{\delta}$ was treated in a study [8] precisely Theorem 6.15.

Passing to the limit as $\delta \rightarrow 0$ in (4.11), we get $\mathrm{z}^{*} \rightarrow \mathrm{z}_{-}$and

$$
\begin{equation*}
\underline{u} \leq \bar{u} \leq M\left(v^{+}+z^{+}\right) \text {in } \Omega \times(0, T-\epsilon) \tag{4.14}
\end{equation*}
$$

Combining eqns. (4.17), (4.5) and (4.14), we get immediately the inequality (1.17) in Theorem 1.4.

## Second step: Proof of the inequality (17)

To reach this aim, we pursue the same method used in the proof of
(1.17) and we need the following two Lemma $[34,35]$.

## Lemma 4.7

The auxiliary problem

$$
\begin{equation*}
-\Delta z=-d(x, \partial \Omega)^{\gamma} z^{m}, z=n \quad \text { on } \partial \Omega \tag{4.15}
\end{equation*}
$$

has a unique solution $\mathrm{z}_{\mathrm{n}}$, which converges uniformly on any compact set of $\Omega$ to the solution $\mathrm{z}^{*}$ of (4.1).

## Lemma 4.8

The auxiliary problem

$$
\begin{equation*}
v^{\prime}+v^{m}=0, t \geq \mu \text { and } v(-\mu)=\infty \tag{4.16}
\end{equation*}
$$

has a unique solution $v^{*}$ given by the following formula

$$
\begin{equation*}
v_{\mu}^{+}(t)=\frac{1}{[(1-m)(t+\mu)]^{\frac{1}{m-1}}}, t \geq 0 \tag{4.17}
\end{equation*}
$$

Remarks 4.9: 1) Remark that $\mathrm{v}^{*} \rightarrow \mathrm{v}^{*}$ as $\mu \rightarrow 0$ where $\mathrm{v}^{*}$ is the solution of (4.3). For the proof, we refer to Theorem 6.15 in a study [8].
2) Due to the fact that the fact that the function $\alpha_{2}$ (defined in (1.2)) is continuous and positive on [0, T], we may suppose that there exists a constant $\mathrm{K}_{\epsilon}>0$ such that

$$
\begin{equation*}
\alpha_{2} \leq K_{\epsilon} \text { in }[0, T-\in] \text { for } \in \leq \leq 1 \tag{4.18}
\end{equation*}
$$

3) Due to eqn. (4.5), there exists a constant $\mathrm{c}_{7}>0$ such that

$$
\begin{equation*}
c_{7}^{-1}(d(x))^{-\frac{2+\gamma}{m-1}} \leq z^{+}(x) \leq c_{7}(d(x))^{-\frac{2+\gamma}{m-1}} \tag{4.19}
\end{equation*}
$$

4) If $\gamma>0$, then we can find $c_{8}>0$ such that $c_{8}\left(d(x)^{\gamma} \leq 1\right.$ in In this case we obtain

$$
\begin{equation*}
\left(v^{+} \mu\right) \leq-c_{8}(d(x))^{\gamma}\left(v^{+} \mu\right)^{m} \tag{4.20}
\end{equation*}
$$

## Lemma 4.10

The problem (4.8) possess a sub-solution $\mathrm{U}_{*}$
Proof: We begin by the case $\gamma \geq 0$. For $\mathrm{k}_{\epsilon}>0$, small enough, we set

$$
U_{+}=c_{o} k_{\in}\left(\left(v^{+} \mu+z^{+}\right)\right.
$$

where $\mathrm{c}_{0}$ as in (F1).
Using equations (4.19) and (4.20), we get
$\left(U_{*}\right)_{t}-\Delta U_{*} \leq-c_{o} k_{\epsilon}(d(x))^{\gamma} U_{*}^{m}$, for $x \in \Omega$ and $t \geq-\mu$
Shrinking the constant $\mathrm{k}_{\epsilon}$ and using assumption (F1), we can write
$\left(U_{*}\right)_{t}-\Delta U_{*} \leq a(x, t) f\left(U_{*}\right)$
Therefore the function fufill the proposition in this case.
If $\gamma \in(-2,0)$. Due to eqn. (4.5), there exists $c_{9}>0$ such that
$(d(x))^{\gamma} \geq c_{9}\left(z^{+}\right)^{\bar{m}_{r}-m}$ where $\bar{m}_{\gamma}=\sup \left(m, \frac{2 m+1}{m+2}\right)$
For $\mathrm{k}_{1}>0$ small, we get
$U_{+}^{1}=c_{o} k_{1}\left(\left(v_{\mu}^{+}+z^{+}\right)\right.$
$\left(\Delta U_{*}^{1}\right)_{\mathrm{t}}-\Delta U_{*}^{1} \leq-k_{1}(d(x))^{\gamma}\left(U_{*}^{1}\right)^{m}$, for $x \in \Omega$ and $t \geq-v$
Choose $\mathrm{k}_{1}$, such that $\mathrm{k}_{1}(\mathrm{~d}(\mathrm{x}))^{\mathrm{r}} \leq \mathrm{a}(\mathrm{x} ; \mathrm{t})$, then we get by $(\mathrm{F} 1)$
$\left(\Delta U_{*}^{1}\right)_{\mathrm{t}}-\Delta U_{*}^{1} \leq k_{1}(d(x))^{\gamma}\left(U_{*}^{1}\right)^{m}$
Now we are ready to prove desisable inequality (1.17). Let ${ }^{\bar{u}}$ is a minimal solution of (1.1) and $\mathrm{z}_{\mathrm{n}}$ is solution of (4.16). By shrinking the real k1 (defined as (4.21), we can write,

$$
\underline{u}(x, t) \geq 2 k_{1} z_{n}(x) \text { for } x \in \Omega \text { and } t \in(0, t-\in)
$$

Due to Lemma 4.7 we have also

$$
\underline{u}(x, t) \geq 2 k_{1} z^{*}(x) \text { for } x \in \Omega \text { and } t \in(0, t-\epsilon)
$$

Therefore if we set $\lceil\varepsilon=\partial \Omega \times(0, \mathrm{t}-\varepsilon] \cup \Omega \times\{0\}$, we obtain

$$
\lim _{(\infty, t) \rightarrow \Gamma \epsilon} \inf \left(u(x, t)-U_{+}(x, t)\right) \geq 0
$$

If we apply maximum principle, we get
$\underline{u}(x, t) \geq k_{1}\left(v_{\mu}{ }^{+}(t)+z^{+}(x)\right)$, for $x \in \Omega$ and $t \in(0, t-\in)$
Combining eqns. (4.19), (4.20) and (4.22), we obtain inequality (1.17)

## Proof of Theorem 1.5

Proof: It suffices to show that $\underline{u}=\bar{u}$ in $\Omega \times(0, T,-\varepsilon)$ for $0<\epsilon<T$. Using inequalities (1.17) and (1.18), there exists a positive constant $\mu>1$ such that
$\underline{u} \leq \bar{u} \leq \mu u$ in $\Omega \times(0, T-\epsilon)$
Suppose that $\underline{u}=\bar{u}$ in the set $\Omega \times(0, \mathrm{~T}-\varepsilon)$ and set
$V=\underline{u}-\frac{1}{2 \mu}(\bar{u}-\underline{u})$
It's easy to verify that
$\frac{2 \mu}{2 \mu+1} V+\frac{1}{2 \mu+1} \bar{u}=\bar{u}$
And $\underline{u} \geq V$ in $\Omega \times(0, T-\epsilon)$
Using the fact that - f is convex we get,
$f(\underline{u}) \geq \frac{2 \mu}{2 \mu+1} f(V)+\frac{1}{2 \mu+1} f(\bar{u})$
Consequently,
$V_{t}-\Delta V=a(x, t)\left[\frac{2 \mu+1}{2 \mu} f(V)+\frac{1}{2 \mu} f(\bar{u})\right] \geq a(x, t) f(V)$
Hence V is supersolution of
$\left\{\begin{array}{l}\frac{\partial V}{\partial t}-\Delta V \geq a(x, t) f(V \text { on } \Omega \times(0, T-\epsilon)), \\ V=\infty \quad \text { on } \quad \partial \Omega \times(0, T-\in \bigcup \bar{\Omega} \times\{0\})\end{array}\right.$
Using the comparison principle, we get $u \leq V$, which is contradiction with (5.1) then $\underline{u}=\bar{u}$ and consequently the result of theorem holds.

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    Received May 05, 2018; Accepted June 6, 2018; Published June 15, 2018
    Citation: Hsini M, Zahrani EA (2018) Liouvilie Type Results for a Class of Quasilinear Parabolic Problem. J Phys Math 9: 272. doi: 10.4172/2090-0902. 1000272
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