Local envelopes on CR manifolds [1]

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Abstract

We study the problem whether CR functions on a sufficiently pseudoconcave CR manifold M extend locally across a hypersurface of M. The sharpness of the main result will be discussed by way of a counter-example.

2000 MSC: 32V20, 32F10, 32D10, 35N10.

1 Introduction

It is a very classical fact that for a strictly (pseudo)convex real hypersurface H in $\mathbb{C}^n$, $n \geq 2$, holomorphic functions extend from the concave side across H. In the present note we will study the corresponding question for CR functions on embedded CR manifolds from a strictly local point of view.

All manifolds will be assumed to be $\mathcal{C}^\infty$-smooth. Recall that a submanifold M of $\mathbb{C}^n$ is called *CR manifold* if the dimension of the complex tangent space $T_p^cM = T_pM \cap JT_pM$ does not depend on $p \in M$ ($J = J_p$ denoting multiplication by the complex unit of $T_p\mathbb{C}^n$). In this case, the complex tangent spaces form a bundle $T^cM \subset TM$, whose complex rank m is called *CR dimension of* M, shortly $m = \text{CRdim}M$. A CR manifold $M \subset \mathbb{C}^n$ is called *generic* if its CR dimension is as small as real/complex linear algebra allows, i.e. if $m = n - \text{codim}M$. A $\mathcal{C}^1$-function f on M is called CR function if $df|_{T^cM}$ is J-linear. Locally one may express this by a system of m independent linear first-order differential equations, allowing to interpret the CR property in distributional sense. The space of continuous CR distributions on M will be denoted by $CR(M)$.

The nonintegrability of T^cM is measured by the nonvanishing of the Levi form. For $X \in T_p^cM$ define the *vector-valued Levi form* by $\mathcal{L}(X) = [J\tilde{X}, \tilde{X}] \bmod T_p^cM$, where $\tilde{X}$ is an arbitrary smooth section of T_p^cM extending X. It is easily verified that the expression is tensorial and yields a well defined mapping $\mathcal{L} : T_p^cM \to T_pM/T_p^cM$. Let $\Sigma_p = (T_p^cM)^\perp = \{\eta \in T_p^*M : \eta|_{T_p^cM} \equiv 0\}$ be the fiber of the *characteristic bundle* Σ *of* M. For nonzero $\eta \in \Sigma_p$, we define the *directional Levi form* by $\mathcal{L}(\eta, X) = \langle \eta, \mathcal{L}(X) \rangle$. Now a generic CR manifold M is called *strictly/weakly q-concave* if for every nonzero $\eta \in \Sigma$ the hermitian form $\mathcal{L}(\eta, \cdot)$ has at least q negative/nonpositive eigenvalues. Finally M is called *strictly pseudoconvex* if $\mathcal{L}(\eta, \cdot)$ is strictly definite for some nonzero $\eta \in \Sigma$ (see [9] for more on CR geometry).

Throughout we will work in the following setting: M will denote a smooth generic CR manifold passing through the origin and H a smooth real hypersurface of $\mathbb{C}^n$ intersecting M transversally in the origin. Hence $H_M = H \cap M$ is a smooth hypersurface of M near 0. For simplicity we will assume that the intersection is even *J-generic*, meaning that T_0^cM and T_0^cH are transverse, or equivalently, that H_M is itself a generic CR manifold near 0. For a distinguished local side H^+ of H, we will consider subdomains U^+ of $M \cap H^+$ whose boundary ∂U^+ contains

[1]Presented at the 3rd Baltic-Nordic Workshop “Algebra, Geometry, and Mathematical Physics“, Göteborg, Sweden, October 11–13, 2007.

a neighborhood of 0 in H_M. We ask whether CR functions on U^+ extend to a uniform M-neighborhood of 0.

Such extension cannot hold for strictly pseudoconvex M. In this case M can be, after a convenient holomorphic coordinate change, locally imbedded into some strictly convex hypersurface. This gives us plentiful functions with isolated peak points, destroying any hope for extension (independently of the shape of H). If we assume H to be strictly pseudoconvex, a similar reason excludes extension from domains U^+ lying on the convex side. Our aim is to strive for weak assumptions on M guaranteing extension under the hypothesis H is strictly pseudoconvex and U^+ lies on the concave side.

It can be seen that M is weakly 1-concave precisely if it is nowhere strictly pseudoconvex [4]. It is known that in this case one has extension phenomena for certain Dirichlet-type problems [3, 4, 10]. Interestingly, weak 1-concavity of M is *not* enough for our Cauchy-type problem (see Section 3). Our main result is the following.

Theorem 1.1. *Let $M \subset \mathbb{C}^n$ be a smooth generic weakly 2-concave CR manifold of CR dimension m intersecting a smooth strictly pseudoconvex hypersurface $H \subset \mathbb{C}^n$ J-generically in the origin. Let $U^+ \subset M$ be a relative domain, lying on the pseudoconcave side of H and containing in its closure a neighborhood of the origin in H_M. Then there is an open neighborhood V of the origin in M such that every continuous CR functions on U^+ uniquely extends to a continuous CR function on $U^+ \cup V$.*

In the strictly 2-concave case, this was proved in [8] by means of adapted integral formulas. Our approach will be very different, focusing on the geometry of related envelopes of holomorphy. In the *weakly* 2-concave case, Theorem 1.1 is even new for hypersurfaces. Here the reader may consult [10] for refinements for J-degenerate intersections.

In Section 3 we will see that Theorem 1.1 fails if M is only weakly 1-concave. Note that the CR orbits of M near 0 may be very complicated (see [9]). It is worth observing that in our situation we need no assumption on CR orbits. Compare this to global results in [2, 10], where the situation is very different. Finally we remark that it should be a subtle task to sharpen the condition on H_M significantly. Our arguments extend to the case where H is weakly pseudoconvex but satisfies a certain finite-type condition at 0 (see Remark 2.1). However, even for extendability of *holomorphic* functions from a given side of a real hypersurface of $\mathbb{C}^n$, finding a geometric characterization is a long-standing open problem.

2 Proof of the main result

After a quadratic holomorphic coordinate change, we may assume that H is strictly convex near the origin. After a unitary rotation, M writes as a smooth graph $y'' = h(z', x'')$, where $z' = (z_1, \ldots, z_m)$, $z'' = (z_{m+1}, \ldots, z_n) = (x_{m+1} + iy_{m+1}, \ldots, x_n + iy_n)$, $h(0) = 0$, $dh(0) = 0$. The strategy is first to prove an extension result for *holomorphic* functions, and to conclude then by approximation techniques.

Part 1: Holomorphic extension. First we assume that we are to extend functions holomorphic in a thin ambient domain $V^+ \subset \mathbb{C}^n$ containing U^+. Let (X, π) be the envelope of holomorphy of V^+. Recall that X is an n-dimensional complex manifold, $\pi : X \to \mathbb{C}^n$ a locally biholomorphic map, and V^+ can be viewed as a subdomain of X via a canonical embedding $\alpha : V^+ \hookrightarrow X$ satisfying the lifting property $\pi \circ \alpha = \mathrm{id}_{V^+}$. The fact that X is the maximal domain over $\mathbb{C}^n$ to which all holomorphic functions on V^+ extend simultaneously translates as follows: (i) $f \mapsto f \circ \alpha$ is a topological isomorphism from $\mathcal{O}(X)$ onto $\mathcal{O}(V^+)$ *(extension)* and (ii) X is a Stein manifold *(maximality)*. Since X is Stein there is a strictly plurisubharmonic function $\rho \in \mathcal{C}^\infty_{\mathbb{R}}(X)$ such that $\{\rho < r\}$ is relatively compact in X for all $r \in \mathbb{R}$ (*see* [6], [7] [9] for envelopes). Holomorphic extension from V^+ to a neighborhood of 0 is the content of the following claim: *The mapping* α *extends as a lifting to an* M*-neighborhood* V *of the origin whose size depends on* U^+*, but not on the particular shape of* V^+*.*

Let $h(z)$ be a complex linear defining function of $T_0^c H$ such that $T_0 H = \{\mathrm{Re}(h) = 0\}$ and $\mathrm{Re}(h)$ increases along the direction pointing into the convex side. For $\epsilon > 0$ small, we consider the family $B_c = \{h(z) = c, |z| < \epsilon\}$. If $\delta > 0$ is very small, then convexity of H and J-genericity of $M \cap H$ yield for $|c| < \delta$: **(i)** $M_c = B_c \cap M$ is a weakly 1-concave generic CR submanifold of B_c of CR dimension $m - 1 > 0$ (topologically an $(\dim M - 2)$-ball), **(ii)** $M_c \backslash U^+$ is either empty, an isolated point or a compact ball. The latter means in particular that the boundaries of M_c stay in U^+. We may furthermore assume $M_c \subset U^+$ for $-\delta < c < 0$. The idea is now to use a version of the continuity principle for subfamilies of the M_c.

To prove the claim it suffices to show that, for $|\hat{c}| < \delta$, the union $\bigcup_{c \in [-\delta/2, \hat{c}]} M_c$ lifts to X ($[-\delta/2, \hat{c}]$ denoting the straight segment in $\mathbb{C}$ between $-\delta/2$ and $\hat{c}$). If this is not the case, then there is a maximal half-open segment $[-\delta/2, \tilde{c})$, where $\tilde{c} < \hat{c}$, such that $\bigcup_{c \in [-\delta, \tilde{c})} M_c$ lifts to X. Maximality of $\tilde{c}$ implies that $\sup \rho \circ \alpha|_{M_c} \to \infty$ if $[-\delta, \tilde{c}) \ni z \to \tilde{c}$. Since the distance of the boundaries of the M_c to ∂X is positive, this implies that $\rho \circ \alpha|_{M_c}$ is nonconstant and has a maximum in the interior whenever c is close to $\tilde{c}$. This contradicts the subsequent maximum principle, and the claim follows.

Lemma 2.1. *Let* $\mathcal{D}$ *be a relatively compact domain in a smooth generic weakly 1-concave CR manifold* $\mathcal{M} \subset \mathbb{C}^n$*. If* ϕ *is a smooth strictly plurisubharmonic function defined near* $\overline{\mathcal{D}}$*, then we have* $\max_{\overline{\mathcal{D}}} \phi \leq \max_{\partial \mathcal{D}} \phi$*.*

This follows from [5]. For the sake of completeness we provide a short argument: If $\max_{\overline{\mathcal{D}}} \phi > \max_{\partial \mathcal{D}} \phi$, the same holds for a generic Morse perturbation ψ, which we may choose such that ψ has no critical points on $\mathcal{M}$ and $\psi|_{\mathcal{M}}$ is also Morse. Then $\psi|_{\mathcal{D}}$ has somewhere a quadratic maximum z_0. Thus $\mathcal{M}$ touches the strictly pseudoconvex hypersurface $\{\psi = \psi(z_0)\}$ in z_0 from the convex side and is therefore itself strictly pseudoconvex near z_0. The lemma follows.

In the sequel, we will need a simple a-priori estimate: Pick $\eta > 0$ such that the intersection B^+ of $B_\eta(0)$ with the concave side of M is contained in U^+ and that $(\partial B^+ \backslash H_M) \subset U^+$. Applying the claim to the points of $\partial B^+ \cap H_M$, we see that the restriction of any $f \in \mathcal{O}(U^+)$ to B^+ is bounded. Applying the claim with B^+ instead of U^+, we obtain extension from U^+ to $U^+ \cup V$ together with an estimate $\sup_V |\tilde{f}| \leq \sup_{B^+} |f|$ ($\tilde{f}$ denoting the extension). The estimate immediately follows from the inclusion $\tilde{f}(V) \subset f(B^+)$. In fact, if we had $c \in \tilde{f}(V) \backslash f(B^+)$, then $(f(z) - c)^{-1}$ would still be holomorphic near B^+ without being extendable along V.

Part 2: Approximation. CR extension will now be derived by an application of the Baouendi-Treves approximation theorem ([1], see also [11]). Since H_M is generic near the origin, there is a smooth totally real n-dimensional submanifold $R \subset H_M$. We may include R into a smooth foliation $R_{\hat{s}} = \{s_1 = \hat{s}_1, \ldots, s_m = \hat{s}_m\}$ of an M-neighborhood of the origin such that **(i)** $s_1, \ldots, s_m$ are smooth real functions with independent differentials, **(ii)** the parameter $\hat{s}$ ranges over some ball U_s around the origin in $\mathbb{R}^m$ and **(iii)** s_1 is a local defining function of H_M positive on the (+)-side of H_M. Supplementing functions, we obtain real coordinates $s_1, \ldots, s_m, t_1, \ldots, t_n$ on an M-neighborhood of 0. By [1], there are arbitrarily small open balls U_s and $B_1 \subset\subset B_2 \subset \mathbb{R}^m$, all centered in 0, such that continuous CR functions on $\{s_1 > 0\} \times B_2$ can be approximated by the restrictions of polynomials in $z_1, \ldots, z_n$, locally uniformly on compact subsets of $\{s_1 > 0\} \times B_1$.

For U^+ given as in Theorem 1.1, we may arrange that $\{s_1 > 0\} \times B_2 \subset U^+$. Pick furthermore a slightly smaller ball $U_s' \subset\subset U_s$. The constructions in Part 1 depend continuously on the data. If $\lambda > 0$ is sufficiently small, we can find an M-neighborhood V of 0 such that every function f holomorphic near $\{s \in U_s' : s_1 > \lambda\} \times B_2$ possesses a holomorphic extension $\tilde{f}$ to an ambient neighborhood of $V \cup \{s \in U_s' : s_1 > \lambda\} \times B_2$ satisfying $\sup_V |\tilde{f}| \leq \sup_{\{s_1 > \lambda\} \times B_1} |f|$.

Let now P_j be polynomials approximating $g \in CR(U^+)$. Then $P_j|_{\{s\in U_s':s_1>\lambda\}\times B_1}$ converges uniformly, hence also its restriction to V by the a-priori estimate. Thus the limit defines a continuous CR function g_V on V. Since P_j approaches g locally uniformly on $\{s \in U_s' : s_1 > 0\} \times B_1$, g and g_V glue into the desired extension. Uniqueness follows from general structure theorems [11] or from a closer inspection of the approximation process. The proof of Theorem 1.1 is complete.

Remark 2.1. a) It is not very essential to work with *continuous* CR functions. If g is a CR distribution on U^+, we may use a method from [1], [11], to represent it on $\{s \in U_s' : s_1 > 0\} \times B_2$ as $g = \Delta_M^k f$, where f is a continuous CR function on $\{s \in U_s' : s_1 > 0\} \times B_2$, Δ_M is a CR variant of the Laplace operator and k is a sufficiently large integer. Now one first extends f by Theorem 1.1 and obtains the desired extension as $\Delta_M^k \tilde{f}$. We omit the details.

b) The argument still works if H is only weakly pseudoconvex but possesses a supporting holomorphic hyperplane touching it (from the concave side) with finite-order contact at the origin. But, as mentioned in the introduction, it should be hard to obtain a sharp result.

c) One can reduce the number of strictly pseudoconvex directions required for H if one assumes weak q-concavity for M with $q > 2$ (compare [8]).

3 Weakly 1-concave counter-example

Our example will be a modification of the weakly but not strongly 1-concave hypersurface

$$M_0 = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : y_3 = |z_2|^2\}$$

Note that M_0 is foliated by complex lines and the Levi form[2] $\mathcal{L}$ has one zero eigenvalue at every $z \in M_0$. Pick a smooth function $g : \mathbb{R} \to \mathbb{R}_0^+$ which vanishes identically for $t \leq 0$ and is strictly convex for $t > 0$. We claim that the hypersurface

$$M = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : y_3 = |z_2|^2 - g(x_1 - |z_2|^2)\}$$

is weakly 1-concave in a neighborhood of the origin. To see this, we observe first that the term $|z_2|^2$ implies that the Levi form[3] of $y_3 - |z_2|^2 - g(x_1 - |z_2|^2)$ is positive in the z_2-direction, which is contained in $T_0^c M$. Hence we have a positive direction at any $z \in M$ close to the origin. Secondly, we note that the slices

$$M_c = \{(z_1, z_3) \in \mathbb{C}^2 : y_3 = |c|^2 - g(x_1 - |c|^2)\} \cong M \cap \{z_2 = c\}$$

are concave graphs over the real (z_1, x_3)-hyperplane in $\mathbb{C}^2_{z_1,z_3}$. Consequently the Levi form of $y_3 - |z_2|^2 - g(x_1 - |z_2|^2)$ must have a nontrivial nonpositive eigenvector tangent to M at any $z \in M$. This implies the claim.

Next we verify that the hypersurface $H_M = M \cap \{x_1 = |z_2|^2\}$ can be embedded into a strictly pseudoconvex hypersurface H transverse to M. Note that the simplest candidate $H_0 = \{x_1 = |z_2|^2\}$ is only weakly pseudoconvex. Instead we try to construct H as a graph $x_1 = h(y_1, z_2, z_3)$ satisfying $dh(0) = 0$. The desired h is hence prescribed along $\pi_{y_1,z_2,z_3}(H_M)$. These partial data already imply that H will have positive Levi curvature in the z_2-direction. But now it is standard that we can produce a strictly pseudoconvex H by bending H_0 near the origin strongly enough along the y_3-direction into the pseudoconvex side (without changing H_M).

[2]For hypersurfaces the characteristic bundle is one-dimensional. Hence the total and the directional Levi forms coincide essentially.

[3]The Levi form of a function is $\mathcal{L}_\phi(X) = \frac{i}{2}\partial\overline{\partial}\phi(X, \overline{X})$. The Levi form of a regular level set $M = \{\phi = c\}$ is the restriction of $\mathcal{L}_\phi$ to $T^c M \cong T^{1,0} M$.

As a matter of fact, the complex hyperplanes $E_t = \{z_3 = it\}$, $t < 0$, do not intersect $M^+ = M \cap \{x_1 < |z_2|^2\}$. On the other hand, the intersection $E_t \cap M$ contains points in an arbitrarily given neighborhood of the origin, if $t < 0$ is sufficiently close to 0. Hence the functions $f_t = (z_3 - it)^{-1}$, $t < 0$, show that there is no local CR extension from M^+ to a uniform neighborhood of the origin.

We conclude with a remark on the envelope of V^+.

Remark 3.1. Fix a domain $U^+ \subset M$ as in Theorem 1.1, and consider ambient open neighborhoods V^+ of U^+. We observe that there is always holomorphic extension from V^+ through some part of H. To this end, we construct small Bishop discs attached to the generic CR manifold H_M (*see* [9] for the disc method). Because of the strict pseudoconvexity of H the interior of the discs will lie in the pseudoconvex side of H. If we deform H_M together with the attached discs into V^+, we obtain a one-sheeted part of X (the envelope of holomorphy of V^+) which passes through H into the pseudoconvex side and contains the origin in its closure.

Note that the size of this part of X depends sensitively on the thickness of V^+. However the disc argument shows that for every V^+ the projection of the envelope to $\mathbb{C}^n$ contains points on the pseudoconvex side of distance to H bounded from below by some uniform positive constant. Of course the above arguments show that there is no M-neighborhood of the origin lifting simultaneously to all possible X. Intuitively speaking, the trouble is that the X lose the contact to M at the origin.

Acknowledgement

This research builds on a part of the author's habilitation thesis, supported by the Humboldt University Berlin. I would like to thank C. Denson Hill, Christine Laurent-Thiébaut and Jürgen Leiterer for inspiring discussions on the topic.

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Received January 10, 2007
Revised March 27, 2008