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Loops in Noncompact Groups of Hermitian Symmetric Type and Factorization

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Abstract

In studies of Pittmann, we showed that a loop in a simply connected compact Lie group has a unique Birkhoff (or triangular) factorization if and only if the loop has a unique root subgroup factorization (relative to a choice of a reduced sequence of simple reflections in the affine Weyl group). In this paper our main purpose is to investigate Birkhoff and root subgroup factorization for loops in a noncompact semisimple Lie group of Hermitian symmetric type. In literature of caine, we showed that for an element of, i.e. a constant loop, there is a unique Birkhoff factorization if and only if there is a root subgroup factorization. However for loops in, while a root subgroup factorization implies a unique Birkhoff factorization, the converse is false. As in the compact case, root subgroup factorization is intimately related to factorization of Toeplitz determinants.

Keywords: Noncompact groups; Birkhoff factorization; Weyl group

Introduction

Finite dimensional Riemannian symmetric spaces come in dual pairs, one of compact type and one of noncompact type. Given such a pair, there is a diagram of finite dimensional groups

$$\begin{tikzcd} & \dot{G} & \\ \dot{G}_0 \arrow[ur] & & \dot{U} \arrow[ul] \\ & \dot{K} \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad (0.1)$$

where $\dot{U}$ is the universal covering of the identity component of the isometry group of the compact type symmetric space $\dot{X} \simeq \dot{U}/\dot{K}$, $\dot{G}$ is the complexification of $\dot{U}$, and $\dot{G}_0$ is a covering of the isometry group for the dual noncompact symmetric space $\dot{X}_0 = \dot{G}_0/\dot{K}$.

The main purpose of this paper is to investigate Birkhoff (or triangular) factorization and "root subgroup factorization" for the loop group of $\dot{G}_0$, assuming $\dot{G}_0$ is of Hermitian symmetric type so that X_0 and X are Hermitian symmetric spaces. Birkhoff factorization is investigated in studies of Caine and Wisdom [1-15], from various points of view. In particular Birkhoff factorization for $L\dot{U} := C^\infty(S^1, \dot{U})$ is developed in Chapter 8 of Wisdom [15], using the Grassmannian model for the homogeneous space $L\dot{U}/\dot{U}$. Root subgroup factorization for generic loops in $\dot{U}$ appeared more recently in literature of Pickrell [11] (for $\dot{U} = SU(2)$, the rank one case) and Pittmann [13]. The Birkhoff decomposition for $L\dot{G}_0 := C^\infty(S^1, \dot{G}_0)$, i.e., the intersection of the Birkhoff decomposition for $L\dot{G}$ with $L\dot{G}_0$, is far more complicated than for $L\dot{U}$. With respect to root subgroup factorization, beyond loops in a torus (corresponding to imaginary roots), in the compact context the basic building blocks are exclusively spheres (corresponding to real roots), and in the Hermitian symmetric noncompact context the building blocks are a combination of spheres and disks. This introduces additional analytic complications, and perhaps the main point of this paper is to communicate the problems that arise from noncompactness.

For $g \in L\dot{U}$, the basic fact is that g has a unique triangular factorization if and only if g has a unique "root subgroup factorization" (relative to the choice of a reduced sequence of simple reflections in the affine Weyl group). This is also true for elements of $\dot{G}_0$ (constant loops); [4]. However, somewhat to our surprise, this is far from true for loops in $\dot{G}_0$.

Relatively little sophistication is required to state the basic results in the rank one noncompact case. This is essentially because (in addition to loops in a torus) the basic building blocks are exclusively disks, and there is essentially a unique way to choose a reduced sequence of simple reflections in the affine Weyl group, so that the dependence on this choice can be suppressed.

The Rank 1 Case

We consider the data determined by the Riemann sphere and the Poincaré disk. For this pair, the diagram (0.1) becomes

$$\begin{tikzcd} & \mathrm{SL}(2,\mathbb{C}) & \\ \mathrm{SU}(1,1) \arrow[ur] & & \mathrm{SU}(2) \arrow[ul] \\ & \mathrm{S}(\mathrm{U}(1) \times \mathrm{U}(1)) \arrow[ul] \arrow[ur] & \end{tikzcd} \qquad (0.2)$$

Let $L_{fin}\mathrm{SL}(2,\mathbb{C})$ denote the group consisting of maps $S^1 \to SL(2,\mathbb{C})$ having finite Fourier series, with pointwise multiplication. The subset of those functions having values in SU(1,1) is then a subgroup, denoted $L_{fin}\mathrm{SU}(1,1)$.

Example 0.1 *For each* $\zeta \in \Delta := \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ *and* $n \in \mathbb{Z}$, *the function* $S^1 \to \mathrm{SU}(1,1)$ *defined by*

$$z \mapsto \mathbf{a}(\zeta)\begin{pmatrix} 1 & \zeta z^{-n} \\ \bar{\zeta} z^n & 1 \end{pmatrix}, \text{where } \mathbf{a}(\zeta) = (1 - |\zeta|^2)^{-1/2}, \qquad (0.3)$$

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is in $L_{fin}\mathrm{SU}(1,1)$.

$L_{fin}\mathrm{SU}(2)$ and $L_{fin}\mathrm{SU}(1,1)$ are dense in the smooth loop groups $L\mathrm{SU}(2) := C^\infty(S^1,\mathrm{SU}(2))$ and $L\mathrm{SU}(1,1) := C^\infty(S^1,\mathrm{SU}(1,1))$, respectively. This is proven in the compact case in Proposition 3.5.3 of [15], and the argument applies also for SU(1,1), taking into account the obvious modifications.

For a Laurent series $f(z)=\sum f z$, let $f^*(z)=\sum \overline{f}_n z^{-n}$. If Ω is a domain on the Riemann sphere, we write $H^0(\Omega)$ for the vector space of holomorphic scalar valued functions on Ω. If $f\in H^0(\Delta)$, then $f^*\in H^0(\Delta^*)$, where Δ^* denotes the open unit disk at ∞.

Theorem 0.1 *Suppose that* $g_1\in L_{fin}\mathrm{SU}(1,1)$ *and fix* $n > 0$. *Consider the following three statements:*

(I.1) g_1 *is of the form*

$$g_1(z)=\begin{pmatrix} a(z) & b(z)\\ b^*(z) & a^*(z)\end{pmatrix},\quad z\in S^1,$$

where a and b are polynomials in z of order n-1 and n, respectively, with $a(0) > 0$.

(I.2) g_1 *has a "root subgroup factorization" of the form*

$$g_1(z)=\mathbf{a}(\eta_n)\begin{pmatrix} 1 & \overline{\eta}_n z^n\\ \eta_n z^{-n} & 1\end{pmatrix}\ldots\mathbf{a}(\eta_0)\begin{pmatrix} 1 & \overline{\eta}_0\\ \eta_0 & 1\end{pmatrix},$$

for some sequence $(\eta_i)_{i=0}^n$ *in Δ and* a : Δ → ℝ *is the function in Example 0.1.*

(I.3) g_1 *has triangular factorization of the form*

$$\begin{pmatrix} 1 & 0\\ \sum_{j=0}^{n}\overline{y}_j z^{-j} & 1\end{pmatrix}\begin{pmatrix} a_1 & 0\\ 0 & a_1^{-1}\end{pmatrix}\begin{pmatrix} \alpha_1(z) & \beta_1(z)\\ \gamma_1(z) & \delta_1(z)\end{pmatrix},$$

where $a_1 > 0$, *the third factor is a matrix valued polynomial in z which is unipotent upper triangular at z* = 0.

Statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3). If g_1 *is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e.,* g_1 *has a root subgroup factorization as in (I.2).*

There is a similar set of implications for $g_2\in L_{fin}SU(1,1)$ *and the following statements:*

(II.1) g_2 *is of the form*

$$g_2(z)=\begin{pmatrix} d^*(z) & c^*(z)\\ c(z) & d(z)\end{pmatrix},\quad z\in S^1,$$

where c and d are polynomials in z of order n and n-1, respectively, with $c(0) = 0$ *and* $d(0) > 0$.

(II.2) g_2 *has a "root subgroup factorization" of the form*

$$g_2(z)=\mathbf{a}(\zeta_n)\begin{pmatrix} 1 & \zeta_n z^{-n}\\ \overline{\zeta}_n z^n & 1\end{pmatrix}\ldots\mathbf{a}(\zeta_1)\begin{pmatrix} 1 & \zeta_1 z^{-1}\\ \overline{\zeta}_1 z & 1\end{pmatrix},$$

for some sequence $(\zeta_k)_{k=1}^n$ *in Δ and* $\mathbf{a}$: Δ → ℝ *is the function in Example 0.1.*

(II.3) g_2 *has a triangular factorization of the form*

$$\begin{pmatrix} 1 & \sum_{j=1}^{n}x_j z^{-j}\\ 0 & 1\end{pmatrix}\begin{pmatrix} a_2 & 0\\ 0 & a_2^{-1}\end{pmatrix}\begin{pmatrix} \alpha_2(z) & \beta_2(z)\\ \gamma_2(z) & \delta_2(z)\end{pmatrix},$$

where $a_2 > 0$, *and the third factor is a matrix valued polynomial in z which is unipotent upper triangular at z = 0.*

When g_1 *and* g_2 *have root subgroup factorizations, the scalar entries determining the diagonal factor have the product form*

$$a_1=\prod_{i=0}^{n}\mathbf{a}(\eta_i)\text{ and }a_2^{-1}=\prod_{k=1}^{n}\mathbf{a}(\zeta_k),\text{ respectively.}\qquad(0.4)$$

In general we do not know how to describe the connected component in the first and third conditions. The following example shows how disconnectness arises in the simplest nontrivial case.

Example 0.2. Consider the case $n = 2$ and g_2 *as* in II.3 with $x = x_1 z + x_2 z^2$, $1-x_2\overline{x}_2\neq 0$,

$$\alpha_2=1-a_2^{-2}\overline{x}_1 x_2 z,\quad \beta_2=-\frac{a_2^{-2}\overline{x}_1^2 x_2}{1-x_2\overline{x}_2}$$

$$\gamma_2=\frac{x_1}{1-x_2\overline{x}_2}z+x_2z^2,\quad \delta_2=1+\frac{\overline{x}_1x_2}{1-x_2\overline{x}_2}z$$

and

$$a_2^2=\frac{(1-x_2\overline{x}_2)^2-x_1\overline{x}_1}{1-x_2\overline{x}_2}.$$

It is straightforward to check that this g_2 does indeed have values in $SU(1,1)$. In order for $a_2^2>0$, there are two possibilities: the first is that both the numerator and denominator are positive, in which case there is a root subgroup factorization (with $\zeta_1=\overline{x}_1/(1-|x_2|^2)$ and $\zeta_2=\overline{x}_2$), and the second is that both the top and bottom are negative, in which case root subgroup factorization fails (because when there is a root subgroup factorization, we must have $|\zeta_1|,|\zeta_2| < 1$).

In order to formulate a general factorization result, we need a C^∞ version of Theorem 0.1.

Theorem 0.2. *Suppose that* $g_1\in LSU(1,1)$. *The following conditions are equivalent:*

(I.1) g_1 *is of the form*

$$g_1(z)=\begin{pmatrix} a(z) & b(z)\\ b^*(z) & a^*(z)\end{pmatrix},\quad z\in S^1,$$

where a and b are holomorphic in Δ and have C^∞ *boundary values, with* $a(0) > 0$.

(I.3) g_1 *has triangular factorization of the form*

$$\begin{pmatrix} 1 & 0\\ y^* & 1\end{pmatrix}\begin{pmatrix} a_1 & 0\\ 0 & a_1^{-1}\end{pmatrix}\begin{pmatrix} \alpha_1(z) & \beta_1(z)\\ \gamma_1(z) & \delta_1(z)\end{pmatrix},$$

where y is holomorphic in Δ with C^∞ *boundary values,* $a_1 > 0$, *and the third factor is a matrix valued polynomial in z which is unipotent upper triangular at z* = 0.

Similarly if $g_2\in LSU(1,1)$, *the following statements are equivalent:*

(II.1) g_2 *is of the form*

$$g_2(z)=\begin{pmatrix} d^*(z) & c^*(z)\\ c(z) & d(z)\end{pmatrix},\quad z\in S^1,$$

where c and d are holomorphic in Δ and have C^∞ boundary values, with $c(0) = 0$ and $d(0) > 0$.

(II.3) g_2 has a triangular factorization of the form

$$\begin{pmatrix} 1 & x^*\\ 0 & 1\end{pmatrix}\begin{pmatrix} a_2 & 0\\ 0 & a_2^{-1}\end{pmatrix}\begin{pmatrix} \alpha_2(z) & \beta_2(z)\\ \gamma_2(z) & \delta_2(z)\end{pmatrix},$$

where $a_2>0$, x is holomorphic in Δ and has C^∞ boundary values, $x(0) = 0$, and the third factor is a matrix valued function which is holomorphic in Δ, has C^∞ boundary values, and is unipotent upper triangular at $z = 0$.

Let $\sigma : SL(2,\mathbb{C}) \to SL(2,\mathbb{C})$ denote the anti-holomorphic involution of $SL(2,\mathbb{C})$ which fixes $SU(1,1)$; explicitly

$$\sigma(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} d^* & c^* \\ b^* & a^* \end{pmatrix}.$$

The following theorem is the analogue of Theorem 0.2 of [11] (the notation is taken from Section 1 of [11], and reviewed below the statement of the theorem).

Theorem 0.3. *Suppose* $g\in LSU(1,1)_{(0)}$, *the identity component. Then g has a unique "partial root subgroup factorization" of the form*

$$g(z) = \sigma(g_1^{-1}(z))\begin{pmatrix} e^{\chi(z)} & 0 \\ 0 & e^{-\chi(z)} \end{pmatrix} g_2(z)$$

where $\chi \in C^\infty(S^1, i\mathbb{R})/2\pi i\mathbb{Z}$ *and* g_1 *and* g_2 *are as in Theorem 0.2, if and only if g has a triangular factorization g = lmau (0.5) below) such that the boundary values of* l_{21}/l_{11} *and* u_{21}/u_{22} *are <1 in magnitude on* S^1.

The following example shows that the unaesthetic condition on the boundary values is essential.

Example 0.3. Consider g_2 as in Theorem 0.1. The loop $g = g_2^*$ (the Hermitian conjugate of g_2 around the circle) has triangular factorization

$$g = \begin{pmatrix} \alpha_2^* & \gamma_2^* \\ \beta_2^* & \delta_2^* \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \sum_{j=1}^{n} x_j z^j & 1 \end{pmatrix}.$$

If $n = 2$, then $x_1 = \overline{\zeta}_1(1-|\zeta_2|^2)$ and $x_2 = \overline{\zeta}_2$, and this loop will often not satisfy the condition $|x_1 z + x_2 z^2| < 1$ on S^1. In this case g will not have a partial root subgroup factorization in the sense of Theorem 0.3.

The group $LSL(2,\mathbb{C})$ has a Birkhoff decomposition

$$LSL(2,\mathbb{C}) = \coprod_{w\in W} \Sigma_w^{LSL(2,\mathbb{C})}$$

where W (an affine Weyl group, and in this case the infinite dihedral group) is a quotient of a discrete group of unitary loops

$$W = \{\mathbf{w} = \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{\varepsilon} : n\in\mathbb{Z}, \varepsilon\in\mathbb{Z}_4\}/\{\pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$$
$$= \langle r_0, r_1 \mid r_0^2 = r_1^2 = 1\rangle,$$

where

$$r_0 = \left[\begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix}\right], r_1 = \left[\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right]$$

(the reflections corresponding to the two simple roots for the Kac-Moody extension of $\mathfrak{sl}(2,\mathbb{C})$. The set $\Sigma_w^{LSL(2,\mathbb{C})}$ consists of loops which have a (Birkhoff) factorization of the form

$$g = l \cdot w \cdot m \,.\, a \,.\, u, \quad (0.5)$$

where $w = [\mathrm{w}]$,

$$l = \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \in H^0(\Delta^*, G), \quad l(\infty) = \begin{pmatrix} 1 & 0 \\ l_{21}(\infty) & 1 \end{pmatrix},$$

l has smooth boundary values on S^1, $m = \begin{pmatrix} m_0 & 0 \\ 0 & m_0^{-1} \end{pmatrix}$, $m_0 \in S^1$, $a = \begin{pmatrix} a_0 & 0 \\ 0 & a_0^{-1} \end{pmatrix}$, $a_0 > 0$,

$$u = \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \in H^0(\Delta, G), \quad u(0) = \begin{pmatrix} 1 & u_{12}(0) \\ 0 & 1 \end{pmatrix},$$

and u has smooth boundary values on S^1. If $w = 1$, the generic case, then we say (as in Section 1 of [11]) that g has a triangular factorization, and in this case the factors are unique.

Next, let $LSU(1,1)_{(n)}$ denote the connected component containing $\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \in Hom(S^1;\dot{T})$, and let

$$\Sigma_w^{LSU(1,1)} := \Sigma_w^{LSL(2,\mathbb{C})} \cap LSU(1,1)$$

and

$$\Sigma_w^{LSU(1,1)(n)} := \Sigma_w^{LSL(2,\mathbb{C})} \cap LSU(1,1)_{(n)}.$$

Since $SU(1,1)$ is homotopy equivalent to the torus $\dot{T} \simeq U(1)$, the connected components of $LSU(1,1)$ are homotopy equivalent to the connected components of $L\dot{T} \simeq LU(1)$ which are indexed by winding number. Write $L\dot{T}_{(n)}$ for the connected component indexed by an integer n. Then it is known that the intersection $\Sigma_w^{LSL(2,\mathbb{C})} \cap L\dot{T} = L\dot{T}_{(n)}$ when $w = \left[\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}\right]$ and empty otherwise (refer Section 8.4 of [15]); in particular this intersection is contractible to w, modulo multiplication by $\dot{T}$. Based partly on the finite dimensional results in [4], one might expect the following to be true:

(1) Modulo $\dot{T}$, it should be possible to contract $\Sigma_w^{LSU(1,1)(n)}$ down to w; in particular $\Sigma_w^{LSU(1,1)}$ should be empty unless w is represented by a loop in $SU(1,1)$.

(2) $\Sigma_1^{LSU(1,1)(0)} = LSU(1,1)_{(0)}$.

(3) Each $\Sigma_w^{LSU(1,1)(n)}$ should admit a relatively explicit parameterization.

Statements (1) and (2) are definitely false; statement (3) is very elusive, if not doubtful.

Proposition 0.1.

(a) $\Sigma_w^{LSU(1,1)(n)}$ *can be nonempty even if w is not represented by a loop in* $SU(1,1)$. *For example,* $\Sigma_{r_1}^{LSU(1,1)(1)}$ *is nonempty.*

(b) $\Sigma_1^{LSU(1,1)(0)}$ *is properly contained in* $LSU(1,1)_{(0)}$.

To summarize one surprise, the set of loops having a root subgroup factorization is properly contained in the set of loops in the identity component which have a triangular factorization which, in turn, is a proper subset of the identity component of $LSU(1,1)$. It seems plausible that all of the intersections $\Sigma_w^{LSU(1,1)(n)}$ are nonempty, and topologically nontrivial. Unfortunately we lack a geometric explanation for why these intersections are so complicated.

Toeplitz determinants

The group $LSU(1,1)$ acts by bounded multiplication operators on the Hilbert space $H := L^2(S^1;\mathbb{C}^2)$. As in literature of Widom [15], this defines a homomorphism of $LSU(1,1)$ into the restricted general linear group of H defined relative to the Hardy polarization $H = H_+ \oplus H_-$, where H_+ is the subspace of boundary values of functions in $H^0(\Delta,\mathbb{C})$ and H_- is the subspace of boundary values of functions in $H^0(\Delta^*, \mathbb{C})$. For a loop g, let $A(g)$ (respectively, $A_1(g)$) denote the corresponding

Toeplitz operator, i.e., the compression of multiplication by g to H_+ (resp., the shifted Toeplitz operator, i.e. the compression to $H_+ \ominus \mathbb{C}\begin{pmatrix}0\\1\end{pmatrix}$). It is well known that $A(g)\,A(g^{-1})$ and $A_1(g)\,A_1(g^{-1})$ are determinant class operators (i.e., of the form 1+ trace class).

Theorem 0.4. *Suppose that* $g \in LSU(1,1)_{(0)}$ *has a root subgroup factorization as in part (b) of Theorem 0.3. Then*

$$\det(A(g)A(g^{-1})) = \left(\prod_{i=0}^{\infty}\frac{1}{(1-|\eta_i|^2)^i}\right)\times\left(\prod_{j=1}^{\infty}e^{-2j|\chi_j|^2}\right)\times\left(\prod_{k=1}^{\infty}\frac{1}{(1-|\zeta_k|^2)^k}\right),$$

$$\det(A_1(g)A_1(g^{-1})) = \left(\prod_{i=0}^{\infty}\frac{1}{(1-|\eta_i|^2)^{i+1}}\right)\times\left(\prod_{j=1}^{\infty}e^{-2j|\chi_j|^2}\right)\times\left(\prod_{k=1}^{\infty}\frac{1}{(1-|\zeta_k|^2)^{k-1}}\right)$$

and if $g = lmau$ is the triangular factorization as in (0.5) (with $\mathbf{w} = 1$*), then*

$$a_0^2 = \frac{\det(A_1(g)A_1(g^{-1}))}{\det(A(g)A(g^{-1}))} = \frac{\prod_{k=1}^{\infty}(1-|\zeta_k|^2)}{\prod_{i=0}^{\infty}(1-|\eta_i|^2)}.$$

When $(\eta_i)_{i=0}^{\infty}$ and $(\zeta_k)_{k=1}^{\infty}$ are the zero sequences (the abelian case), the first formula specializes to a result of Szego and Widom (Theorem 7.1 of [16]). Estelle Basor pointed out to us that this result, for g as in (0.3), can be deduced from Theorem 5.1 of [16].

Additional motivation

There is a developing analogue of root subgroup factorization for the group of homeomorphisms of a circle, a group which (in some ways) is similar to a noncompact type Lie group [12]; there are other analogues as well [1]. It is important to identify potential pitfalls. In this paper our primary contribution is perhaps to identify what can go wrong with Birkhoff and root subgroup factorization for loops into a noncompact target; these lessons are potentially valuable in other contexts.

From another point of view, it is expected that root subgroup factorization is relevant to finding Darboux coordinates for homogeneous Poisson structures on $L\dot{U}$ and $L\dot{G}_0$ [10]. As of this writing, this is an open question.

Plan of the paper

This paper is essentially a sequel to studies of Pittmann and Pressley [4,13]. We will refer to the latter paper as the 'finite dimensional case', and we note the differences as we go along.

Section 1 is on background for finite dimensional groups (which is identical to [4]) and loop groups. In section 2 we consider the intersection of the Birkhoff decomposition for $L\dot{G}$ with $L\dot{G}_0$. Unfortunately for loops in $\dot{G}_0$, there does not exist an analogue of "block (or coarse) triangular decomposition", a key feature of the finite dimensional case. Consequently there does not exist a reduction to the compact type case, as in finite dimensions. One might still naively expect that there could be a relatively transparent way to parameterize the intersections of the Birkhoff components with $L\dot{G}_0$ (as in the finite dimensional case, and in the case of loops into compact groups, e.g., using root subgroup factorization). But these intersections turn out to be topologically nontrivial. Most of the section is devoted to rank one examples which illustrate this.

In Section 3 we consider root subgroup factorization for generic loops in $\dot{G}_0$. Our objective in this section is to prove analogues of Theorems 4.1, 4.2, and 5.1 of studies of Pittmann [13], for generic loops in (the Kac-Moody central extension of) $L\dot{G}_0$ (when $\dot{G}_0$ is of Hermitian symmetric type). As in the rank one case above, all of the statements have to be severely modified. The structures of the arguments in this noncompact context are roughly the same as in literature of Pittmann [13], but there are many differences in the details (reflected in the more complicated statements of theorems).

Notation and Background

In this paper, we will make use of the fact that (certain extensions of) loop algebras of complex semisimple Lie algebras and finite dimensional complex semisimple Lie algebras fit into the common framework of Kac-Moody Lie algebras. To distinguish data associated the finite dimensional Lie algebras from the analogous information for the infinite dimensional loop algebra of such a Lie algebra, we will adhere to a convention of Kac and label the data associated with finite dimensional data by an overhead dot.

Finite dimensional groups and algebras

We consider the data (0.1) determined as follows from a compact Hermitian symmetric space $\dot{X}$. We consider the isometry group of $\dot{X}$ and let $\dot{U}$ denote the universal covering group. Then $\dot{U}$ is a simply connected compact group and we let $\dot{K}$ be the stability subgroup of a point in $\dot{X}$, so that $\dot{X} \simeq \dot{U}/\dot{K}$. This determines an involution $\dot{\Theta}$ of $\dot{U}$ which we extend holomorphically to the complexification $\dot{G}$ of $\dot{U}$. The composition $g \mapsto \dot{\Theta}(g^{-*})$ of $\dot{\Theta}$ with the Cartan involution $(\cdot)^{-*}$ fixing $\dot{U}$ inside of $\dot{G}$ is then an antiholomorphic involution of $\dot{G}$ fixing a real form $\dot{G}_0$ which is $\dot{\Theta}$-stable. The fixed point set of $\dot{\Theta}$ in $\dot{G}_0$ is $\dot{K}$ and the coset space $\dot{G}_0/\dot{K}$ is a model for the non-compact Hermitian symmetric space $\dot{X}_0$ dual to $\dot{X}$.

Remark 1.1. The notation $(\cdot)^{-*}$ for the Cartan involution fixing $\dot{U}$ inside of $\dot{G}$ is suggestive of the matrix operation of inverse conjugate transpose which fixes $SU(n)$ inside of $SL(n,\mathbb{C})$. Likewise, we will use $(\cdot)^*$ to denote the operation $g \mapsto (g^{-1})^{-*}$.

Thus, we obtain the diagram of finite dimensional groups (0.1). Correspondingly, we obtain an analogous diagram of finite dimensional Lie algebras $\dot{\mathfrak{g}}_0$, $\dot{\mathfrak{k}}$, $\dot{\mathfrak{g}}$, $\dot{\mathfrak{u}}$, and we use $\dot{\Theta}$ and $(\cdot)^*$to also denote the corresponding infinitesimal involutions. Let $\dot{\mathfrak{g}}_0 = \dot{\mathfrak{k}} + \dot{\mathfrak{p}}$ denote the eigenspace decomposition of $\dot{\mathfrak{g}}_0$ under $\dot{\Theta}$. Then $\dot{\mathfrak{u}} = \dot{\mathfrak{k}} + i\dot{\mathfrak{p}}$ is the eigenspace decomposition of $\dot{\mathfrak{u}}$ under $\dot{\Theta}$.

Choose a Cartan subalgebra $\dot{\mathfrak{t}}$ in $\dot{\mathfrak{k}}$. Then $\dot{\mathfrak{t}}$ is a Cartan subalgebra of $\dot{\mathfrak{g}}_0$ since $\dot{\mathfrak{g}}_0$ is of Hermtian symmetric type. The centralizer $\dot{\mathfrak{h}}$ of $\dot{\mathfrak{t}}$ in $\dot{\mathfrak{g}}$ is then a $\dot{\Theta}$-stable Cartan subalgebra of the complex semisimple Lie algebra $\dot{\mathfrak{g}}$. Furthermore, $\dot{\mathfrak{h}} = \dot{\mathfrak{t}} + \dot{\mathfrak{a}}$, where $\dot{\mathfrak{a}} = i\dot{\mathfrak{t}}$, is the eigenspace decomposition of $\dot{\mathfrak{h}}$ with respect to the involution $(\cdot)^{-*}$.

We will write $\dot{W} = N_{\dot{G}}(\dot{H})/\dot{H}$ for the Weyl group of the pair $(\dot{\mathfrak{g}},\dot{\mathfrak{h}})$. Choose a Weyl chamber $\dot{C} \subset \dot{\mathfrak{a}}$. This determines a choice of positive roots for the action of $\dot{\mathfrak{h}}$ on $\dot{\mathfrak{g}}$. Let $\dot{\mathfrak{n}}^{\pm}$ denote the sum of the positive (resp. negative) root spaces. Then

$$\dot{\mathfrak{g}} = \dot{\mathfrak{n}}^- + \dot{\mathfrak{h}} + \dot{\mathfrak{n}}^+$$

is a $\dot{\Theta}$-stable triangular decomposition of $\dot{\mathfrak{g}}$. An important consequence of $\dot{\Theta}$-stability is that $\dot{\mathfrak{n}}^+$ and $\dot{\mathfrak{n}}^-$ are interchanged by the action of $(\cdot)^{-*}$, i.e., $(\dot{\mathfrak{n}}^{\pm})^{-*} = \dot{\mathfrak{n}}^{\mp}$.

Let $\dot{\theta}$ denote the highest root and normalize the Killing form so that (for the dual form) $\langle\dot{\theta},\dot{\theta}\rangle = 2$. For each root $\dot{\alpha}$ let $h_{\dot{\alpha}} \in \dot{\mathfrak{a}}$ denote the associated coroot. The Hermitian symmetric type assumption, together with the Θ-stability of $\dot{\mathfrak{h}}$, implies that each root space $\dot{\mathfrak{g}}_{\dot{\alpha}}$ is contained in either $\dot{\mathfrak{k}}^{\mathbb{C}}$ or in $\dot{\mathfrak{p}}^{\mathbb{C}}$ and thus the roots can be sorted into two types. A root $\dot{\alpha}$ is of *compact type* if the root space $\dot{\mathfrak{g}}_{\dot{\alpha}}$ is a subset of $\dot{\mathfrak{k}}^{\mathbb{C}} \subset \dot{\mathfrak{g}}$ and of *noncompact type* otherwise, i.e., when $\dot{\mathfrak{g}}_{\dot{\alpha}} \subset \dot{\mathfrak{p}}^{\mathbb{C}}$. The following proposition is an elementary fact.

Proposition 1.1. *For each simple positive root* $\dot{\gamma}$ *there exists a Lie algebra homomorphism* $\iota_{\dot{\gamma}} : \mathfrak{sl}(2,\mathbb{C}) \to \dot{\mathfrak{g}}$ *which carries the standard triangular decomposition of* sl(2,ℂ) *into the triangular decomposition* $\dot{\mathfrak{g}} = \dot{\mathfrak{n}}^- + \dot{\mathfrak{h}} + \dot{\mathfrak{n}}^+$ *and:*

(a) in any case $\iota_{\dot{\gamma}}$ restricts to a homomorphism $\iota_{\dot{\gamma}} : \mathfrak{su}(2) \to \dot{\mathfrak{u}}$;

(b) when $\dot{\gamma}$ is of compact type then $\iota_{\dot{\gamma}}$ restricts to $\iota_{\dot{\gamma}} : \mathfrak{su}(2) \to \dot{\mathfrak{k}}$;

(c) when $\dot{\gamma}$ is of noncompact type then $\iota_{\dot{\gamma}}$ restricts to $\iota_{\dot{\gamma}} : \mathfrak{su}(1,1) \to \dot{\mathfrak{g}}_0$.

We denote the corresponding group homomorphism by the same symbol. Note that if $\dot{\gamma}$ is of noncompact type, then $\iota_{\dot{\gamma}}$ induces an embedding of the rank one diagram (0.2) into the finite dimensional group diagram (0.1). For each simple positive root $\dot{\gamma}$, we use the group homomorphism to set

$$\mathbf{r}_{\dot{\gamma}} = \iota_{\dot{\gamma}} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in N_{\dot{U}}(\dot{T}) \tag{1.1}$$

and obtain a specific representative for the associated simple reflection $r_{\dot{\gamma}} \in W = N_{\dot{U}}(\dot{T})/\dot{T}$ corresponding to $\dot{\gamma}$. (We will adhere to the convention of using boldface letters to denote representatives of Weyl group elements).

Remark 1.2**.** Throughout this paper we regard the homomorphism ι_{γ} corresponding to a simple positive root *γ* as fixed. If *η* is another positive root, then there is a Weyl group element *w* such that *η* = *w*·; by choosing a representative $\in N\ (T)$ for *w*, we obtain a homomorphism $\iota_{\eta}(\cdot) = \mathbf{w}\iota_{\gamma}(\cdot)\mathbf{w}^{-1}$ with the same properties as in the proposition. This homomorphism will depend on the choice of *w* and its representative **w**, but the dependence will be relatively insignificant in this paper.

Let $\dot{\alpha}_1,\ldots,\dot{\alpha}_{\mathrm{rank}(\dot{\mathfrak{g}})}$ denote the simple positive roots and write $\dot{h}_1,\ldots,\dot{h}_{\mathrm{rank}(\dot{\mathfrak{g}})}$ for the corresponding coroots. Then $\dot{h}_1,\ldots,\dot{h}_{\mathrm{rank}(\dot{\mathfrak{g}})}$ form a basis for $\dot{\mathfrak{a}}$ and the dual basis elements $\dot{\Lambda}_1,\ldots,\dot{\Lambda}_{\mathrm{rank}(\dot{\mathfrak{g}})}$ are the fundamental weights. For the coroot lattice, we write

$$\check{\dot{T}} = \bigoplus_{1\le j\le \mathrm{rank}(\dot{\mathfrak{g}})} \mathbb{Z}\dot{h}_j \subset \dot{\mathfrak{a}}.$$

The affine Weyl group for $\dot{\mathfrak{g}}$ is the semidirect product $\dot{W} \ltimes \check{\dot{T}}$. For the action of $\dot{W}$ on $\dot{\mathfrak{a}}$, a fundamental domain is the Weyl chamber $\dot{C}$. For the natural affine action of $\dot{W} \ltimes \check{\dot{T}}$ on $\dot{\mathfrak{a}}$, a fundamental domain is the convex set

$$C_0 = \{x \in \dot{C} : \dot{\theta}(x) < 1\}$$

known as the fundamental alcove. Since C_0 will play the role for an infinite dimensional group *G* extending $L\dot{G}$ that $\dot{C}$ plays for the finite dimensional group $\dot{G}$, we purposely omit an overhead dot from the label C_0.

Loop algebras and extensions

Let $L\dot{\mathfrak{g}} = C^{\infty}(S^1, \dot{\mathfrak{g}})$ and let $L_{fin}\dot{\mathfrak{g}}$ denote the subspace of functions with finite Fourier series. Then $L_{fin}\dot{\mathfrak{g}}$ is a subalgebra of $L\dot{\mathfrak{g}}$ with respect to the the point-wise bracket. There is a universal central extension

$$0 \to \mathbb{C}c \to \tilde{L}\dot{\mathfrak{g}} \to L\dot{\mathfrak{g}} \to 0, \tag{1.2}$$

where $\tilde{L}\dot{\mathfrak{g}} = L\dot{\mathfrak{g}} \oplus \mathbb{C}c$ as a vector space, and

$$[X + \lambda c, Y + \lambda' c]_{\tilde{L}\dot{\mathfrak{g}}} := [X,Y]_{L\dot{\mathfrak{g}}} + \frac{i}{2\pi}\int_{S^1} \langle X \wedge dY \rangle c. \tag{1.3}$$

The smooth completion of the untwisted affine Kac-Moody Lie algebra corresponding to $\dot{\mathfrak{g}}$ is

$$\hat{L}\dot{\mathfrak{g}} = \mathbb{C}d \ltimes \tilde{L}\dot{\mathfrak{g}} \qquad \text{(thesemidirectsum)},$$

where the derivation *d* acts by $d(X + \lambda c) = \frac{1}{i}\frac{d}{d\theta}X$, for $X \in L\dot{\mathfrak{g}}$, and [*d*,*c*] = 0.

Proposition 1.2. *For both* $L\dot{\mathfrak{u}}$ *and* $L\dot{\mathfrak{g}}_0$*, the cocycle* $(X,Y) \to \int_{S^1}\langle X \wedge dY\rangle$ *is real-valued. In particular the affine extension (1.2) induces a unitary central extension*

$$0 \to i\mathbb{R}c \to \tilde{L}\dot{\mathfrak{g}}_0 \to L\dot{\mathfrak{g}}_0 \to 0$$

and a real form $\hat{L}\dot{\mathfrak{g}}_0 = i\mathbb{R}d \ltimes \tilde{L}\dot{\mathfrak{g}}_0$ for $\hat{L}\dot{\mathfrak{g}}$ *(and similarly for* $\hat{L}\dot{\mathfrak{u}} = i\mathbb{R}d \ltimes \tilde{L}\dot{\mathfrak{u}}$ *as in the compact case* [13]*).*

We identify $\dot{\mathfrak{g}}$ with the constant loops in $L\dot{\mathfrak{g}}$. Because the extension is trivial over $\dot{\mathfrak{g}}$, there are embeddings of Lie algebras $\dot{\mathfrak{g}} \to \tilde{L}\dot{\mathfrak{g}} \to \mathfrak{g} = \hat{L}\dot{\mathfrak{g}}$. The involution $\dot{\Theta}$ on $\dot{\mathfrak{g}}$ induces an involution on $L\dot{\mathfrak{g}}$ by post-composition. We extend this to an involution Θ on $\mathfrak{g} = \tilde{L}\dot{\mathfrak{g}}$ by declaring that Θ(*c*) = *c*, and similarly extend it to $\hat{L}\dot{\mathfrak{g}}$ by declaring that Θ(*d*) = *d*.

Let $\mathfrak{g} = \tilde{L}\dot{\mathfrak{g}}$, $\mathfrak{g}_0 = \tilde{L}\dot{\mathfrak{g}}_0$, $\mathfrak{u} = \tilde{L}\dot{\mathfrak{u}}$, and $\mathfrak{k} = \tilde{L}\dot{\mathfrak{k}}$. We set $\mathfrak{t} = i\mathbb{R}c \oplus \dot{\mathfrak{t}}$ and $\mathfrak{a} = \mathfrak{h}_{\mathbb{R}} = \mathbb{R}c \oplus \dot{\mathfrak{a}}$. Then, the decompositions

$$\mathfrak{g} = \tilde{L}\dot{\mathfrak{g}} = \mathfrak{n}^- + \mathfrak{h} + \mathfrak{n}^+ \quad \text{and} \quad \hat{L}\dot{\mathfrak{g}} = \mathfrak{n}^- + (\mathbb{C}d + \mathfrak{h}) + \mathfrak{n}^+, \tag{1.4}$$

where $\mathfrak{h} = \dot{\mathfrak{h}} + \mathbb{C}c$ and $\mathfrak{n}^{\pm}$ denotes the smooth completion of $\dot{\mathfrak{n}}^{\pm} + \dot{\mathfrak{g}}(z^{\pm 1}\mathbb{C}[z^{\pm 1}])$, respectively, are triangular decompositions. The simple positive roots for the pair $(\hat{L}_{fin}\dot{\mathfrak{g}}, \mathbb{C}d + \mathfrak{h})$ are $\alpha_0, \alpha_1, \ldots, \alpha_{\mathrm{rank}(\dot{\mathfrak{g}})}$, where

$$\alpha_0 = d^* - \dot{\theta}, \quad \alpha_j = \dot{\alpha}_j \text{for} j > 0,$$

$d^*(d) = 1$, $d^*(c) = 0$, $d^*(\dot{\mathfrak{h}}) = 0$, and the $\dot{\alpha}_j$ are extended to ℂ*d*+h by requiring $\dot{\alpha}_j(c) = \dot{\alpha}_j(d) = 0$. The simple coroots are $h_0, h_1, \ldots, h_{\mathrm{rank}(\dot{\mathfrak{g}})}$, where

$$h_0 = c - \dot{h}_{\dot{\theta}} \text{and} h_j = \dot{h}_j \text{for} j > 0.$$

For *i* > 0, the root homomorphism ι_{α_i} is simply $\iota_{\dot{\alpha}_i}$ followed by the inclusion $\dot{\mathfrak{g}} \subset L\dot{\mathfrak{g}}$. For *i* = 0

$$\iota_{\alpha_0}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = e_{\dot{\theta}}z^{-1}, \quad \iota_{\alpha_0}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = f_{\dot{\theta}}z, \tag{1.5}$$

where $\{f_{\dot{\theta}}, \dot{h}_{\dot{\theta}}, e_{\dot{\theta}}\}$ satisfy the sl(2,ℂ) -commutation relations, and $e_{\dot{\theta}}$ is a highest root vector for $\dot{\mathfrak{g}}$. The fundamental dominant integral functionals on h are Λ_j, *j* = 0,..,*r*.

Loop groups and extensions

Let $\Pi : \tilde{L}\dot{G} \to L\dot{G}$ ($\Pi : \tilde{L}\dot{G}_0 \to L\dot{G}_0$) denote the universal central $\mathbb{C}^*$ (resp., 𝕋) extension of the smooth loop group $L\dot{G}$ (resp. $L\dot{G}_0$).

Proposition 1.3. Π *induces a central circle extension*

$$1 \to \mathbb{T} \to \tilde{L}\dot{G}_0 \to L\dot{G}_0 \to 1$$

(and similarly for unitary loops as in literature of Pittmann [13]*).*

Proof. This follows from Proposition 1.2.

Let $G = \tilde{L}\dot{G}$ and let $N^{\pm}$ denote the subgroups corresponding to n±. Since the restriction of Π to $N^{\pm}$ is an isomorphism, we will always identify $N^{\pm}$ with its image, e.g., $l \in N^{+}$ is identified with a smooth loop in $\dot{G}$ having a holomorphic extension to Δ satisfying $l(0) \in \dot{N}^{+}$. Also, set *T* = exp(t) and *A* = exp(a).

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As in the finite dimensional case, for $\tilde{g} \in N^- \cdot TA \cdot N^+ \subset G = \tilde{L}\dot{G}$, there is a unique triangular decomposition

$$\tilde{g} = l \cdot ma \cdot u, \quad \text{where} \quad ma = \prod_{j=0}^{\text{rank}\dot{\mathfrak{g}}} \sigma_j(\tilde{g})^{h_j}, \tag{1.6}$$

and $\sigma_j = \sigma_{\Lambda_j}$ is the fundamental matrix coefficient for the highest weight vector corresponding to Λ_j. If $\Pi(\tilde{g}) = g$, then because $\sigma_0^{h_0} = \sigma_0^{c - \dot{h}_{\dot{\theta}}}$ projects to $\sigma_0^{-\dot{h}_{\dot{\theta}}}$, g = $l \cdot \Pi(d) \cdot u$ $g = l \cdot \Pi(d) \cdot u$, where

$$(\Pi(d))(g) = \sigma_0(\tilde{g})^{-\dot{h}_{\dot{\theta}}} \prod_{j=1}^{\text{rank}\dot{\mathfrak{g}}} \sigma_j(\tilde{g})^{\dot{h}_j} = \prod_{j=1}^{\text{rank}\dot{\mathfrak{g}}} \left(\frac{\sigma_j(\tilde{g})}{\sigma_0(\tilde{g})^{\breve{a}_j}} \right)^{\dot{h}_j}, \tag{1.7}$$

and the $\breve{a}_j$ are positive integers such that $\dot{h}_{\dot{\theta}} = \sum \breve{a}_j \dot{h}_j$ (these numbers are also compiled in Section 1.1 of [7]).

Proposition 1.4.

(a) *$N^\pm$ are stable with respect to Θ, whereas $N^\pm$ are interchanged by $(\cdot)^*$. If $\tilde{g} \in G = \tilde{L}\dot{G}$ has triangular factorization $\tilde{g} = l \cdot m(\tilde{g}) a(\tilde{g}) \cdot u$ as in* (1.6), *then*

$$\Theta(\tilde{g}) = \Theta(l) \cdot m(\tilde{g}) a(\tilde{g}) \cdot \Theta(u)$$

and

$$(\tilde{g})^* = u^* \cdot m(\tilde{g})^* a(\tilde{g}) \cdot l^*$$

are triangular factorizations.

(b) *If $\tilde{g} \in G = \tilde{L}\dot{G}$, then $\sigma_j(\Theta(\tilde{g})) = \sigma_j(\tilde{g})$ and $\sigma_j(\tilde{g}^*) = \sigma_j(\tilde{g})^*$.*

(c) *If $\tilde{g} \in G_0$ $\tilde{L}G_0$, then | (̃)| depends only on $g = \Pi(\tilde{g}) \in LG$, and*

$$|\sigma_j|(g) := |\sigma_j(\tilde{g})| = (\sigma_j(\tilde{g}) \sigma_j(\tilde{g}^{-1}))^{1/2}. \tag{1.8}$$

(d) *For $\tilde{g} \in \tilde{L}\dot{G}_0$ and $g = \Pi(\tilde{g}) \in LG_0$, $\tilde{g}$ has a triangular factorization if and only if g has a triangular factorization. The restriction of the projection $\Pi : \tilde{L}\dot{G}_0 \to L\dot{G}_0$ to elements with $m(\tilde{g}) = 1$ is injective.*

Proof. (a) and (b) follow from the compatibility of the triangular factorization with respect to Θ and u. The first part of (c) follows from the fact that the induced extension $\tilde{L}\dot{G}_0$ is unitary. The formula 1.8 in (c) follows from the fact that if $\lambda \in \mathbb{T}$, then

$$\sigma_j(\tilde{g}\lambda) = \lambda^l \sigma_j(\tilde{g})$$

where l is the level.

A note on the rank one case

In this subsection we will freely use the notation in Section 1 of [11] and [15] (as in section 1 of [11], we denote the Toeplitz and shifted Toeplitz operators by A and A_1, respectively).

In the rank one case σ_0 and σ_1 can be concretely realized as "regularized Toeplitz determinants." In the notation of section 6.6 of [15], a concrete model for the central extension is

$$\tilde{L}G = \{[g,q] : (g,q) \in L\dot{G} \times GL(H_+), A(g)q^{-1} = 1 + \text{traceclass}\}$$

(here $\dot{G} = SL(2,\mathbb{C})$, $H = L^2(S^1, \mathbb{C}^2)$, and H_+ is the subspace of boundary values of holomorphic functions on the disk). In this realization

$$\sigma_0([g,q]) = \det(A(g)q^{-1}).$$

Proposition 1.5. *For $g \in L\dot{G}_0$, using the notation in Proposition 1.4,*

$$|\sigma_0|^2(g) = \det(A(g)A(g^{-1})) \text{and} |\sigma_1|^2(g) = \det(A_1(g)A_1(g^{-1})).$$

Proof. This follows from (c) of Proposition 1.4.

Reduced sequences in the affine weyl group

The Weyl group W for $(\hat{L}\dot{\mathfrak{g}}, \mathbb{C}d + \mathfrak{h})$ acts by isometries of $(\mathbb{R}d + \text{h}\mathbb{R}, \langle\cdot,\cdot\rangle)$. The action of W on $\mathbb{R}c$ is trivial. The affine plane $d + \dot{\mathfrak{h}}_{\mathbb{R}}$ is W-stable, and this action identifies W with the affine Weyl group $\dot{W} \ltimes \breve{\dot{T}}$ of $\dot{\mathfrak{g}}$ and its affine action on $\dot{\mathfrak{h}}_{\mathbb{R}}$ (Chapter 5 of [15]). In this realization, the simple reflection r_{α_0} is a reflection in $\dot{\mathfrak{h}}_{\mathbb{R}}$ followed by a translation in $\dot{\mathfrak{h}}_{\mathbb{R}}$, specifically

$$r_{\alpha_0} = \dot{h}_{\dot{\theta}} \circ r_{\dot{\theta}}, \quad \text{and} \quad r_{\alpha_i} = r_{\dot{\alpha}_i}, \quad i > 0. \tag{1.9}$$

In general, we can present a given $w \in W$ as

$$w = h_w \circ \dot{w}, \quad h_w \in \breve{\dot{T}}, \quad \dot{w} \in \dot{W}.$$

We let $Inv(w)$ denote the inversion set of w, i.e., the set of positive roots which are mapped to negative roots by w.

Remark 1.3. In the finite dimensional context [4], the root subgroup factorization of generic elements of $\dot{G}_0$ depended on a reduced expression for $\dot{w}_0$, the longest element of the Weyl group $\dot{W}$, i.e., a finite reduced word in simple reflections. In this infinite dimensional context, where there is no longest element of W, we must allow the possibility that root subgroup factorization of generic elements will depend on a possibly infinite sequence $(r_j)_{j=1}^\infty$ of simple reflections.

Definition 1. We will say that an infinite sequence $(r_j)_{j=1}^\infty$ of simple reflections in W is *reduced* if each partial product $w_j := r_j r_{j-1} \cdots r_1$ is a reduced expression for each j.

Remark 1.4. In the rank one case, there are only two possible reduced sequences since W is the infinite dihedral group. As a result, there are only two forms for the root subgroup factorization of generic elements of $LSU(1,1)$. This is the reason for the structure of the theorems stated in the Introduction, involving two sets of analogous implications. In the higher rank setting, however, there are infinitely many forms the factorization could take.

Lemma 1.1. *Let $(r_j)_{j=1}^\infty$ be a reduced sequence of simple reflections in W and let $(\gamma_j)_{j=1}^\infty$ denote the sequence of corresponding simple positive roots of* g. *Then for each n:*

(a) *the inversion set of the partial product w_n is*

$$Inv(w_n) = \{\tau_j := w_{j-1}^{-1} \cdot \gamma_j = r_1..r_{j-1} \cdot \gamma_j : j = 1,..,n\};$$

(b) $w_k \tau_n > 0$, *for each* $k < n$.

Definition 2. A reduced sequence of simple reflections $(r_j)_{j=1}^\infty$ is affine periodic if, in terms of the identification of W with the affine Weyl group,

1. There exists l such that the partial product w_l is in $\breve{\dot{T}}$, i.e., acts as a translation on $\dot{\mathfrak{h}}_{\mathbb{R}}$, and

2. $w_{s+l} = w_s \circ w_l$, for all s. For the minimal such l, we will refer to w_l^{-1} as the period, and l as the length of the period.

Remark 1.5. The second condition is equivalent to periodicity of the associated sequence of simple roots $(\gamma_j)_{j=1}^\infty$, i.e., $\gamma_{s+l} = \gamma_s$ for each s. Through the affine action, the sequence of reflections applied to the fundamental alcove C_0 determines a non-terminating walk through the alcoves in $\dot{\mathfrak{a}}$. In these terms, affine periodicity of the sequence $(r_j)_{j=1}^\infty$ means that the walk from step l+1 to 2l is the original walk up to step l translated by , and so on.

We now recall Theorem 3.5 of [13] (this is what we will need in Section 3 for root subgroup factorization of generic loops in $\dot{G}_0$).

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Theorem 1.1.

(a) *There exists an affine periodic reduced sequence* $(r_j)_{j=1}^{\infty}$ *of simple reflections such that, in the notation of Lemma 1.1,*

$$\{\tau_j : 1 \le j < \infty\} = \{qd^* - \dot{\alpha} : \dot{\alpha} > 0, q = 1, 2, \ldots\}, \tag{1.10}$$

i.e., such that the span of the corresponding root spaces is $(z\quad[z])$. *The period can be chosen to be any point in* $C \cap \breve{T}$.

(b) *Given a reduced sequence as in (a), and a reduced expression for* $\dot{w}_0 = r_{-N} \cdots r_0$ *(where* $\dot{w}_0$ *is the longest element of* $\dot{W}$ *), the sequence*

$$r_{-N}, \ldots, r_0, r_1, \ldots$$

is another reduced sequence. The corresponding set of positive roots mapped to negative roots is

$$\{qd^* + \dot{\alpha} : \dot{\alpha} > 0, q = 0, 1, \ldots\},$$

i.e., the span of the corresponding root spaces is $\dot{\mathfrak{n}}^+(\mathbb{C}[z])$.

Many examples illustrating this theorem appear in the dissertation of Pittmann-Polletta ([?]).

4.6 Contrast with finite dimensions

In literature of Caine [4] we considered $\dot{G}_0$ (constant loops). The key fact (depending on the Hermitian type assumption) was that

$$\dot{\mathfrak{n}}^{\pm} = \dot{\mathfrak{n}}^{\pm}_{\mathfrak{k}} + \dot{\mathfrak{n}}^{\pm}_{\mathfrak{p}}$$

where the latter summand, $\dot{\mathfrak{n}}^{\pm}_{\mathfrak{p}} = \dot{\mathfrak{n}}^{\pm} \cap \dot{\mathfrak{p}}^{\mathbb{C}}$, is an abelian ideal in the parabolic subalgebra $\dot{\mathfrak{k}}^{\mathbb{C}} + \dot{\mathfrak{n}}^{\pm}_{\mathfrak{p}}$ of $\dot{\mathfrak{g}}$. This led to a block (coarse) triangular factorization, which largely reduces the (finite dimensional) Hermitian noncompact case to the compact case.

In the present context there is an analogous decomposition

$$\mathfrak{n}^{\pm} = \mathfrak{n}^{\pm}_{\mathfrak{k}} + \mathfrak{n}^{\pm}_{\mathfrak{p}}$$

where $g_0 = k + p$ is the eigenspace decomposition of g_0 under $\dot{\Theta}$. In this case

$$\mathfrak{n}^{+}_{\mathfrak{p}} = (\mathfrak{n}^{+} \cap L^{+}\dot{\mathfrak{n}}^{-}_{\mathfrak{p}}) + (\mathfrak{n}^{+} \cap L^{+}\dot{\mathfrak{n}}^{-}_{\mathfrak{p}})$$

where each of the two summands is a subalgebra, but the sum is not a Lie algebra (let alone an abelian ideal in a parabolic subalgebra). The fundamental difficulty is that in the finite dimensional case $\dot{N}^+$ is a nilpotent group, and hence whenever the Lie algebra is a sum of subalgebras, there is a corresponding global decomposition at the group level. However, in the loop case N^+ is a profinite nilpotent group, and the corresponding result is not true, e.g., a holomorphic map from from the disk to the Lie algebra has a pointwise triangular decomposition, but pointwise triangular factorization fails very badly at the group level. For example, the $SL(2, \mathbb{C})$ -valued holomorphic function $\begin{pmatrix} 1-2z & 1 \\ -2z & 1 \end{pmatrix}$ does not have a pointwise triangular factorization because the (1,1) entry vanishes at $z = 1/2$.

Compact vs noncompact type roots in g

As in the finite dimensional setting, a root of h on g is said to be of *compact type* if the corresponding root space belongs to k$\mathbb{C}$, and said to be of *noncompact type* if the corresponding root space belongs to p$^{\mathbb{C}}$. Here $\mathfrak{k}^{\mathbb{C}} = \tilde{L}\dot{\mathfrak{k}}^{\mathbb{C}}$ and $\mathfrak{p}^{\mathbb{C}} = \mathfrak{n}^{-}_{\mathfrak{p}} + \mathfrak{n}^{+}_{\mathfrak{p}}$ (so this terminology is perhaps less than ideal).

Remark 1.6. In rank one, the compact type roots are the imaginary roots and the noncompact type roots are the real roots. This is yet another special feature of the rank one case.

The basic framework and notation

In the remainder of the paper we will mainly be concerned with the loop analogue of (0.1):

$$\begin{tikzcd} & G & \\ G_0 \arrow[ur] & & U \arrow[ul] \\ & K \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{1.11}$$

where $U := \tilde{L}\dot{U}$, the (simply connected) central circle extension of $L\dot{U}$, $G := \tilde{L}\dot{G}$, the (simply connected) central $\mathbb{C}^*$ extension of $L\dot{G}$, $G_0 := \tilde{L}\dot{G}_0$, the central circle extension of $L\dot{G}_0$, and $K := \tilde{L}\dot{K}$, the central circle extension of $L\dot{K}$. There is a corresponding diagram of Lie algebras, where the Lie algebra of G is $\mathfrak{g} = \tilde{L}\dot{\mathfrak{g}}$, and so on.

It will often happen that we can more simply work at the level of loops, rather than at the level of central extensions. We will often state results, for example, in terms of G, but in proving results it is often possible and easier to work with $L\dot{G}$.

Birkhoff Decomposition for Loops

By definition the Birkhoff decomposition of $G = \tilde{L}\dot{G}$ is

$$G = \coprod_{W} \Sigma_w^G \text{where} \Sigma_w^G := N^- w B^+. \tag{2.1}$$

If we fix a representative $\mathbf{w} \in N_U(T)$ for $w \in W$, then each $g \in \Sigma_w^G$ has a unique Birkhoff factorization

$$g = l\mathbf{w}mau, \quad l \in N^- \cap wN^- w^{-1}, \quad ma \in TA, u \in N^+. \tag{2.2}$$

As in the finite dimensional case, for fixed $m_0 \in T$, $\{g \in \Sigma_w^G : m(g) = m_0\}$ is a stratum (diffeomorphic to the product of the Birkhoff stratum for the flag space G / B^+ corresponding to w with N^+); refer Theorem 8.7.2 of [15]. We will refer to Σ_w^G as the "(isotypic) component of the Birkhoff decomposition of G corresponding to $w \in W$."

One virtue of root subgroup factorization is that it generates many explicit examples of Birkhoff factorizations.

Birkhoff decomposition for $L\dot{U}$

Given $w \in W$, define

$$\Sigma_w^U := \Sigma_w^G \cap U.$$

Theorem 2.1 *Fix a representative* $\mathbf{w} \in N_U(T)$ *for w. For* $g \in \Sigma_w^G$ *the unique factorization (2.2) induces a bijective correspondence*

$$\Sigma_w^U \leftrightarrow \left(N^- \cap wN^- w^{-1}\right) \times T\text{givenby}g \mapsto (l, m).$$

We refer to Σ_w^U as the isotypic component of the Birkhoff decomposition for U; each component consists of a union of strata permuted by the action of T. The theorem provides an explicit parameterization for these strata. We have recalled this result simply for the sake of comparison. Our primary objective is to investigate the Birkhoff decomposition for $L\dot{G}_0$.

Birkhoff decomposition for $(L\dot{G}_0)_0$, the identity component

Given $w \in W$, define

$$\Sigma_w^{G_0} := \Sigma_w^G \cap G_0$$

$$\Sigma_w^{L\dot{G}_0} := \Sigma_w^{L\dot{G}} \cap L\dot{G}_0$$

and so on.

As we stated in the introduction (where we focused on the rank one case), our original expectation was that each of these components would be (modulo a torus) contractible to w. Our main objective in this subsection is to provide examples in the rank one case, for the identity component, which illustrate why this is not true.

Proposition 2.1. *$\Sigma_1^{LSU(1,1)_{(0)}}$ is properly contained in $LSU(1,1)_{(0)}$.*

Proof. For any $g \in LSU(1,1)$ there is a pointwise polar decomposition

$$g = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} a & b \\ b^* & a \end{pmatrix}$$

where $a = \sqrt{1+|b|^2}$, and $\lambda : S^1 \to S^1$.

If $g \in LSU(1,1)_{(0)}$, then λ has degree zero, and thus λ has a triangular factorization

$$\lambda = e^{\psi_-} e^{\psi_0} e^{\psi_+}$$

where $\psi_- = -\psi_+^*$ and $\psi_0 \in i\mathbb{R}$. Because a is a positive periodic function, it will have a triangular factorization

$$a = e^{\chi_-} e^{\chi_0} e^{\chi_+}$$

where $\chi_- = \chi_+^*$ and $\chi_0 \in \mathbb{R}$.

We can always multiply g on the left (right) by something in B^- (B^+, respectively) without affecting the question of whether g has a triangular factorization. For example in determining whether g has a triangular factorization, we can ignore the factor $\exp(\psi_- + \psi_0)$ in λ, because this can be factored out on the left. We will use this observation repeatedly (note that we can recover ψ_- from ψ_+, and the zero mode is inconsequential).

There is a factorization of

$$\begin{pmatrix} a & b \\ \bar{b} & a \end{pmatrix}$$

as the product

$$\begin{pmatrix} e^{\chi_-} & 0 \\ 0 & e^{-\chi_-} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b^* e^{(\chi_- - \chi_0 - \chi_+)} & 1 \end{pmatrix} \begin{pmatrix} e^{\chi_0} & 0 \\ 0 & e^{-\chi_0} \end{pmatrix} \begin{pmatrix} 1 & b e^{(-\chi_- - \chi_0 + \chi_+)} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\chi_+} & 0 \\ 0 & e^{-\chi_+} \end{pmatrix}.$$

To obtain g we have to multiply this on the left by λ. It follows after some calculation that g will have a triangular factorization if and only if

$$\begin{pmatrix} 1 & 0 \\ b^* e^{(\chi_- - \chi_0 - \chi_+)} e^{-2\psi_+} & 1 \end{pmatrix} \begin{pmatrix} e^{\chi_0} & 0 \\ 0 & e^{-\chi_0} \end{pmatrix} \begin{pmatrix} 1 & b e^{(-\chi_- - \chi_0 + \chi_+)} e^{2\psi_+} \\ 0 & 1 \end{pmatrix}$$

has a triangular factorization.

At this point, to simplify notation, we let $b_1 := b e^{(-\chi_- - \chi_0 + \chi_+)}$. Note that $b_1 b_1^* = b b^* e^{(-2\chi_0)}$. Thus g has a triangular factorization if and only if the loop

$$\begin{pmatrix} 1 & 0 \\ b_1^* e^{-2\psi_+} & 1 \end{pmatrix} \begin{pmatrix} e^{\chi_0} & 0 \\ 0 & e^{-\chi_0} \end{pmatrix} \begin{pmatrix} 1 & b_1 e^{2\psi_+} \\ 0 & 1 \end{pmatrix} = e^{\chi_0} \begin{pmatrix} 1 & b_1 e^{2\psi_+} \\ b_1^* e^{-2\psi_+} & b_1 b_1^* + e^{-2\chi_0} \end{pmatrix}$$

has a triangular factorization. Note that the (2,2) entry of the right hand side equals $aa^* e^{-2\chi_0}$.

We directly calculate the kernel of the Toeplitz operator associated to this loop. We obtain the equations (for $f_1, f_2 \in H^0(D)$)

$$f_1 + (b_1 e^{2\psi_+} f_2)_+ = 0, \text{and} \left(b_1^* e^{-2\psi_+} f_1 + (b_1 b_1^* + e^{-2\chi_0}) f_2 \right)_+ = 0.$$

We can solve the first equation for f_1. The second equation becomes

$$\left((e^{-2\chi_0} + b_1 b_1^*) f_2 - b_1^* e^{-2\psi_+} (b_1 e^{2\psi_+} f_2)_+ \right)_+ = 0.$$

If we set $b_2 = b_1 e^{\chi_0} = b e^{-\chi_- + \chi_+}$, then this can be rewritten as

$$\left(f_2 + b_2^* e^{-2\psi_+} (b_2 e^{2\psi_+} f_2)_- \right)_+ = 0.$$

If we set $F = e^{2\psi_+} f_2$, then we see that there exists a nontrivial kernel if and only if there exists nonzero $F \in H_+$ such that

$$\left(e^{-2\psi_+} (F + b_2^* (b_2 F)_-) \right)_+ = 0. \tag{2.3}$$

It is easy to find ψ_+ and b_2 such that there does exist a nonzero F satisfying this condition.

Example 2.1. $b_2 = \frac{1}{z} - 1$, $e^{-2\psi_+} = e^{2z}$, *and* $F = \frac{1-e^{-2z}}{z} - 1$. In other words if

$$g = \begin{pmatrix} e^{\frac{1}{z} - z} & 0 \\ 0 & e^{-\frac{1}{z} + z} \end{pmatrix} \begin{pmatrix} (3 - \frac{1}{z} - z)^{1/2} & e^{-\chi_- + \chi_+} (\frac{1}{z} - 1) \\ e^{+\chi_- - \chi_+} (z - 1) & (3 - \frac{1}{z} - z)^{1/2} \end{pmatrix}$$

where

$$(3 - \frac{1}{z} - z)^{1/2} = e^{\chi_-} e^{\chi_0} e^{\chi_+}$$

then g is a loop in the identity component of $LSU(1,1)$ and does not have a Riemann-Hilbert factorizaton, hence also does not have a triangular factorization.

Birkhoff decomposition for nonidentity components of $L\dot{G}_0$

Consider the rank one case and the problem of finding the Birkhoff factorization for g which is of the form $g = \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} g_0$, where g_0 is in the identity component and has a known triangular factorization (as for example in Theorem 0.1), and $n > 0$. Write

$$g = \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} \begin{pmatrix} m_0 a_0 & 0 \\ 0 & (m_0 a_0)^{-1} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}.$$

Factor l as

$$\begin{pmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} \\ \sum_{k=-\infty}^{-2n} \alpha_k z^k & L_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sum_{k=-2n+1}^{-1} x_k z^k & 1 \end{pmatrix}.$$

Then g will have the form

$$g = L' \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sum_{k=-2n+1}^{0} x_k z^k & 1 \end{pmatrix} \begin{pmatrix} m_0 a_0 & 0 \\ 0 & (m_0 a_0)^{-1} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$$

where $L' \in N^-$. Consequently to find the Birkhoff factorization for g, it suffices to find the factorization for the triangular matrix valued function

$$\begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sum_{k=-2n+1}^{0} x_k z^k & 1 \end{pmatrix} = \begin{pmatrix} z^{-n} & 0 \\ \sum_{k=-n+1}^{n} c_k z^k & z^n \end{pmatrix}. \tag{2.4}$$

Remark 2.1. What we are doing here is factoring N^- as $N^- \cap wN^-w^{-1}$ times $N \cap wN^+w^{-1}$. So this is very general. The problem of understanding Birkhoff factorization for triangular matrix valued functions is considered in literature of Clancey [5].

Example 2.2 When $n = 1$, we could take $g_0 = g_1$ in Theorem 0.1. Then

$$g=\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\mathbf{a}(\eta_1)\begin{pmatrix} 1 & \overline{\eta}_1 \\ \eta_1 & 1 \end{pmatrix}\mathbf{a}(\eta_0)\begin{pmatrix} 1 & \overline{\eta}_0 z \\ \eta_0 z^{-1} & 1 \end{pmatrix},$$

$$=\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \overline{y}_0+\overline{y}_1 z^{-1} & 1 \end{pmatrix}\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}\begin{pmatrix} \alpha_1(z) & \beta_1(z) \\ \gamma_1(z) & \delta_1(z) \end{pmatrix}$$

where $y_1=-\overline{\eta}_1$ and $y_0=-\overline{\eta}_0(1-\eta_1\overline{\eta}_1)$ (note $|y_0|,|y_1|<1$).

Lemma 2.1 *Fix n>0. For a triangular matrix valued function as in (2.4),*

(a) *the Toeplitz operator A is invertible if and only if the Toeplitz matrix*

$$A'=\begin{pmatrix} c_0 & c_{-1} & .. & c_{-n+1} \\ c_1 & c_0 & .. & c_{-n+2} \\ .. & .. & .. & .. \\ c_{n-1} & .. & .. & c_0 \end{pmatrix} \tag{2.5}$$

is invertible, and

(b) *the shifted Toeplitz operator A_1 is invertible if and only if the Toeplitz matrix*

$$A''=\begin{pmatrix} c_1 & c_0 & .. & c_{-n+2} \\ c_2 & c_1 & .. & c_{-n+3} \\ .. & .. & .. & .. \\ c_n & .. & .. & c_1 \end{pmatrix} \tag{2.6}$$

is invertible.

Proof. The Fredholm indices for both operators are zero, so we need to check the kernels.

Part (a): Suppose that

$$\begin{pmatrix} f \\ h \end{pmatrix}=\begin{pmatrix} \sum_{k=0}^{\infty} f_k z^k \\ \sum_{k=0}^{\infty} h_k z^k \end{pmatrix}$$

is in the kernel of A. Then $(z^{-n}f)_+=0$, implying $f=\sum_{k=0}^{n-1} f_k z^k$, and

$$\left(\left(\sum_{k=-n+1}^{n} c_k z^k\right)\left(\sum_{k=0}^{n-1} f_k z^k\right)+\sum_{k=0}^{\infty} h_k z^{k+n} \right)_+=0.$$

This equation implies $h_k = 0$ for $k \geq n$. These equations have the matrix form

$$\begin{pmatrix} A' & 0 \\ C' & 1_{n\times n} \end{pmatrix}\begin{pmatrix} \vec{f} \\ \vec{h} \end{pmatrix}=\vec{0}$$

where $\vec{f}$ (resp. $\vec{h}$) is the vector of coefficients of f (resp. h) and A' is the $n \times n$ Toeplitz matrix in (2.5). This implies part (a).

The proof of part (b) is similar.

Example 2.3. Suppose $n = 1$. When $c_0 \neq 0$ there is a Riemann-Hilbert factorization (because A is invertible)

$$\begin{pmatrix} z^{-1} & 0 \\ c_0+c_1 z & z \end{pmatrix}=\begin{pmatrix} 1 & z^{-1}/c_0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -c_1/c_0 & -1/c_0 \\ c_0 & 0 \end{pmatrix}\begin{pmatrix} 1+\frac{c_1}{c_0}z & z/c_0 \\ \frac{-c_1^2}{c_0}z & 1-\frac{c_1}{c_0}z \end{pmatrix}.$$

When $c_0, c_1 \neq 0$, there is a triangular factorization (because A and A_1 are invertible),

$$\begin{pmatrix} z^{-1} & 0 \\ c_0+c_1 z & z \end{pmatrix}=\begin{pmatrix} 1-\frac{c_0}{c_1}z^{-1} & \frac{1}{c_0}z^{-1} \\ -\frac{c_0^2}{c_1} & 1 \end{pmatrix}\begin{pmatrix} -\frac{c_1}{c_0} & 0 \\ 0 & -\frac{c_0}{c_1} \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{c_0} \\ -\frac{c_1^2}{c_0}z & 1-\frac{c_1}{c_0}z \end{pmatrix}.$$

In this case $g \in \Sigma_1^{LSU(1,1)_{(-1)}}$.

When $c_1 \to 0$ this "degenerates" to a Birkhoff factorization

$$\begin{pmatrix} z^{-1} & 0 \\ c_0 & z \end{pmatrix}=\begin{pmatrix} 1 & \frac{1}{c_0}z^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{c_0} & 0 \\ 0 & c_0 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{c_0}z \\ 0 & 1 \end{pmatrix}.$$

In this case $g \in \Sigma_{r_1}^{LSU(1,1)_{(-1)}}$.

When $c_0 \to 0$ this "degenerates" to a Birkhoff factorization

$$\begin{pmatrix} z^{-1} & 0 \\ c_1 z & z \end{pmatrix}=\begin{pmatrix} 1 & \frac{1}{c_1}z^{-2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -z^{-1} \\ z & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{c_1} & 0 \\ 0 & c_1 \end{pmatrix}\begin{pmatrix} 1 & \frac{1}{c_1} \\ 0 & 1 \end{pmatrix}.$$

In this case $g \in \Sigma_{r_0}^{LSU(1,1)_{(-1)}}$.

When both $c_0, c_1 \to 0$ this goes to $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}$. In this case $g \in \Sigma_{r_0 r_1}^{LSU(1,1)_{(-1)}}$, where in the Weyl group $\begin{pmatrix} z^{-1} & 0 \\ 0 & z \end{pmatrix}=r_0 r_1$.

These calculations show that we are obtaining loops in the corresponding strata, despite the fact that neither r_0 nor r_1 are represented by loops in $\dot{K}=S^1$. Moreover the conditions on c_0, c_1 above show that the intersection of the Σ_{r_1} component with the $n = -1$ connected component is topologically nontrivial. However we do not know how to quantify this.

Root Subgroup Factorization for Generic Loops in $\dot{G}_0$

Our objective in this section is to prove analogues of Theorems 4.1, 4.2, and 5.1 of [13], for generic loops in $\dot{G}_0$ (which is always assumed to be of Hermitian symmetric type). The structure of the proofs in this noncompact context is basically the same as in literature of Pittmann [13]. But there are important differences. In order to obtain formulas for determinants of Toeplitz operators, as in Theorem 0.4, we have to work with the central extension $\tilde{L}\dot{G}$.

Throughout this section we choose a reduced sequence $\{r_j\}_{j=1}^{\infty}$ as in Theorem 1.1, part (a). We set $\mathbf{w}_j = \mathbf{r}_j \ldots \mathbf{r}_1$ and

$$i_{\tau_n}=\mathbf{w}_{n-1} i_{\gamma_n} \mathbf{w}_{n-1}^{-1}, \qquad n=1,2,\ldots$$

$$i_{\tau'_{-N}}=i_{\gamma_{-N}}, \quad i_{\tau'_{-(N-1)}}=\mathbf{r}_{-N} i_{\gamma_{-(N-1)}} \mathbf{r}_{-N}^{-1},\ldots, \quad i_{\tau'_0}=\dot{\mathbf{w}}_0 i_{\gamma_0} \dot{\mathbf{w}}_0^{-1}$$

and for $n > 0$

$$i_{\tau'_n}=\dot{\mathbf{w}}_0 \mathbf{w}_{n-1} i_{\gamma_n} \mathbf{w}_{n-1}^{-1} \dot{\mathbf{w}}_0^{-1}.$$

As in studies of Caine [4], for $\zeta \in \mathbb{C}$, let $\mathbf{a}_+(\zeta)=(1+|\zeta|^2)^{-1/2}$ and

$$k(\zeta)=\mathbf{a}_+(\zeta)\begin{pmatrix} 1 & -\overline{\zeta} \\ \zeta & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_+(\zeta) & 0 \\ 0 & \mathbf{a}_+(\zeta)^{-1} \end{pmatrix}\begin{pmatrix} 1 & -\overline{\zeta} \\ 0 & 1 \end{pmatrix}\in SU(2). \tag{3.1}$$

For $|\zeta|<1$, let $\mathbf{a}_-(\zeta)=(1-|\zeta|^2)^{-1/2}$ and

$$q(\zeta)=\mathbf{a}_-(\zeta)\begin{pmatrix} 1 & \overline{\zeta} \\ \zeta & 1 \end{pmatrix}=\begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix}\begin{pmatrix} \mathbf{a}_-(\zeta) & 0 \\ 0 & \mathbf{a}_-(\zeta)^{-1} \end{pmatrix}\begin{pmatrix} 1 & \overline{\zeta} \\ 0 & 1 \end{pmatrix}\in SU(1,1). \tag{3.2}$$

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Generalizations of Theorem 3.1

Theorem 3.1. *Suppose that* $\tilde{g}_1 \in \tilde{L}_{fin}\dot{G}_0$ *and* $\Pi(\tilde{g}_1) = g_1$. *Consider the following three statements:*

(I.1) $m(\tilde{g}_1) = 1$, *and for each complex irreducible representation V() for* $\dot{G}$, *with lowest weight vector* $\varphi \in V(\pi)$, $\pi(g_1)^{-1}(\phi)$ *is a polynomial in z (with values in V), and is a positive multiple of* φ at $z = 0$.

(I.2) $\tilde{g}_1$ *has a factorization of the form*

$$\tilde{g}_1 = i_{\tau'_n}(g(\eta_n))\ldots i_{\tau'_{-N}}(g(\eta_{-N})) \in \tilde{L}_{fin}\dot{G}_0$$

where $g(\eta_j) = k(\eta_j)$ *for some* $\eta_j \in \mathbb{C}$ *(resp.* $g(\eta_j) = q(\eta_j)$ *for some* $\eta_j \in \Delta$*) when* τ_j *is a compact type (resp. non-compact type) root.*

(I.3) $\tilde{g}_1$ *has triangular factorization of the form* $\tilde{g}_1 = l_1 a_1 u_1$ *where* $l_1 \in \dot{N}^-(\mathbb{C}[z^{-1}])$.

Then statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3).

Moreover, in the notation of (I.2),

$$a_1 = \prod_{j=-N}^{n} \mathbf{a}(\eta_j)^{h_{\tau'_j}}.$$

Similarly, suppose that $\tilde{g}_2 \in \tilde{L}_{fin}\dot{G}_0$ *and* $\Pi(\tilde{g}_2) = g_2$. *Consider the following three statements:*

(II.1) $m(\tilde{g}_2) = 1$, *and for each complex irreducible representation* $V(\pi)$ *for* $\dot{G}$, *with highest weight vector* $v \in V(\pi)$, $\pi(g_2)^{-1}(v)$ *is a polynomial in z (with values in V), and is a positive multiple of v at* $z = 0$.

(II.2) $\tilde{g}_2$ *has a factorization of the form*

$$\tilde{g}_2 = i_{\tau_n}(g(\zeta_n))\ldots i_{\tau_1}(g(\zeta_1))$$

for some $\zeta_j \in \Delta$.

(II.3) $\tilde{g}_2$ *has triangular factorization of the form* $\tilde{g}_2 = l_2 a_2 u_2$, *where* $l_2 \in \dot{N}^+(z^{-1}\mathbb{C}[z^{-1}])$.

Then statements (II.1) and (II.3) are equivalent. (II.2) implies (II.1) and (II.3).

Also, in the notation of (II.2),

$$a_2 = \prod_{j=1}^{n} \mathbf{a}(\zeta_j)^{h_{\tau_j}}. \tag{3.3}$$

Remark 3.1. Note that we are not making any attempt to characterize the set of l_1 that arise in (I.3) (and similarly for the set of l_2 in (II.3)).

Conjecture 3.1. *If* g_1 *is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e.,* g_1 *has a root subgroup factorization as in (I.2). If* g_2 *is in the identity connected component of the sets in (II.1) and (II.3), then the converse holds, i.e.,* g_2 *has a root subgroup factorization as in (II.2).*

In the course of the following proof of Theorem 3.1, we will prove a version of this conjecture, in the rank one case, which completes the proof of Theorem 0.1 (Remark 3.2 below).

Proof. The two sets of implications are proven in the same way. We consider the second set.

We first want to argue that (II.2) implies (II.3). We recall that the subalgebra $\mathfrak{n}^- \cap \mathbf{w}_{n-1}^{-1}\mathfrak{n}^+\mathbf{w}_{n-1}$ is spanned by the root spaces corresponding to negative roots $-\tau_j$, $j=1,..,n$. The calculation is the same as in the proof of Theorem 2.5 in [4]. In the process we will also prove the product formula for a_2.

The equation (3.1) implies that

$$\iota_{\tau_j}(\;(\zeta_j)) = \iota_{\tau_j}\left(\begin{pmatrix} 1 & 0 \\ & \end{pmatrix}\right)\;(\zeta_j)^{\;j}\,\iota_{\tau_j}\left(\begin{pmatrix} & \pm\bar{\ }_j \\ 0 & 1 \end{pmatrix}\right)$$

$$= \exp(\zeta_j f_{\tau_j})\mathbf{a}(\zeta_j)^{h_{\tau_j}}\mathbf{w}_{j-1}^{-1}\exp(\pm\bar{\zeta}_j e_{\gamma_j})\mathbf{w}_{j-1}$$

is a triangular factorization. Here, $\mathbf{a}(\zeta_j) = \mathbf{a}_{\pm}(\zeta_j)$ and the plus/minus case is used when τ_j is a compact/noncompact type root, respectively.

Let $g^{(n)} = \iota_{\tau_n}(g(\zeta_n))\ldots\iota_{\tau_1}(g(\zeta_1))$. First suppose that $n = 2$. Then

$$g^{(2)} = \exp(\zeta_2 f_{\tau_2})\mathbf{a}(\zeta_2)^{h_{\tau_2}}\mathbf{r}_1\exp(\pm\bar{\zeta}_2 e_{\gamma_2})\mathbf{r}_1^{-1}\exp(\zeta_1 f_{\gamma_1})\mathbf{a}(\zeta_1)^{h_{\gamma_1}}\exp(\pm\bar{\zeta}_1 e_{\gamma_1}). \tag{3.4}$$

The key point is that

$$\mathbf{r}_1\exp(\pm\bar{\zeta}_2 e_{\gamma_2})\mathbf{r}_1^{-1}\exp(\zeta_1 f_{\gamma_1}) = \mathbf{r}_1\exp(\pm\bar{\zeta}_2 e_{\gamma_2})\exp(\zeta_1 e_{\gamma_1})\mathbf{r}_1^{-1}$$

$$= \mathbf{r}_1\exp(\zeta_1 e_{\gamma_1})\tilde{u}\mathbf{r}_1^{-1}, \quad (\text{forsome} \quad \tilde{u} \in N^+ \cap r_1 N^+ r_1^{-1})$$

$$= \exp(\zeta_1 f_{\gamma_1})\mathbf{u}, \quad (\text{forsome} \quad \mathbf{u} \in N^+).$$

Insert this calculation into (3.4). We then see that $g^{(2)}$ has a triangular factorization $g^{(2)} = l^{(2)}a^{(2)}u^{(2)}$, where

$$a^{(2)} = \mathbf{a}(\zeta_1)^{h_{\tau_1}}\mathbf{a}(\zeta_2)^{h_{\tau_2}}$$

and

$$l^{(2)} = \exp(\zeta_2 f_{\tau_2})\exp(\zeta_1\mathbf{a}(\zeta_2)^{-\tau_1(h_{\tau_2})}f_{\tau_1}) \tag{3.5}$$

$$= \exp(\zeta_2 f_{\tau_2} + \zeta_1\mathbf{a}(\zeta_2)^{-\tau_1(h_{\tau_2})}f_{\tau_1})$$

(the last equality holds because a two dimensional nilpotent algebra is necessarily commutative).

To apply induction, we assume that $g^{(n-1)}$ has a triangular factorization $g^{(n-1)} = l^{(n-1)}a^{(n-1)}u^{(n-1)}$ with

$$l^{(n-1)} = \exp(\zeta_{n-1}f_{\tau_{n-1}})\tilde{l} \in N^- \cap w_{n-1}^{-1}N^+w_{n-1} = \exp(\sum_{j=1}^{n-1}\mathbb{C}f_{\tau_j}), \tag{3.6}$$

for some $\tilde{l} \in N^- \cap w_{n-2}^{-1}N^+w_{n-2} = \exp(\sum_{j=1}^{n-2}\mathbb{C}f_{\tau_j})$, and

$$a^{(n-1)} = \prod_{j=1}^{n-1}\mathbf{a}(\zeta_j)^{h_{\tau_j}}.$$

We have established this for $n1 = 1, 2$. For $n \geq 2$

$$g^{(n)} = \exp(\zeta_n f_{\tau_n})\mathbf{a}(\zeta_n)^{h_{\tau_n}}\mathbf{w}_{n-1}^{-1}\exp(\pm\bar{\zeta}_n e_{\gamma_n})\mathbf{w}_{n-1}\exp(\zeta_{n-1}f_{\tau_{n-1}})\tilde{l}a(g^{(n-1)})u(g^{(n-1)})$$

$$= \exp(\zeta_n f_{\tau_n})\mathbf{a}(\zeta_n)^{h_{\tau_n}}\mathbf{w}_{n-1}^{-1}\exp(\pm\bar{\zeta}_n e_{\gamma_n})\tilde{u}\mathbf{w}_{n-1}a(g^{(n-1)})u(g^{(n-1)}),$$

where $\tilde{u} = \mathbf{w}_{n-1}\exp(\zeta_{n-1}f_{\tau_{n-1}})\tilde{l}\mathbf{w}_{n-1}^{-1} \in \mathbf{w}_{n-1}N^-\mathbf{w}_{n-1}^{-1} \cap N^+$. Now write

$$\exp(\pm\bar{\zeta}_n e_{\gamma_n})\tilde{u} = \tilde{u}_1\tilde{u}_2,$$

relative to the decomposition

$$N^+ = \left(N^+ \cap w_{n-1}N^-w_{n-1}^{-1}\right)\left(N^+ \cap w_{n-1}N^+w_{n-1}^{-1}\right).$$

Let

$$\mathbf{l} = \mathbf{a}(\zeta_n)^{h_{\tau_n}}\mathbf{w}_{n-1}^{-1}\tilde{u}_1\mathbf{w}_{n-1}\mathbf{a}(\zeta_n)^{-h_{\tau_n}} \in N^- \cap \mathbf{w}_{n-1}^{-1}N^+\mathbf{w}_{n-1}.$$

Then $g^{(n)}$ has triangular decomposition

$$g^{(n)} = \left(\exp(\zeta_n f_{\tau_n})\mathbf{l}\right)\left(\mathbf{a}(\zeta_n)^{h_{\tau_n}}a^{(n-1)}\right)\left((a^{(n-1)})^{-1}\tilde{u}_2 a^{(n-1)}u^{(n-1)}\right).$$

This implies the induction step.

This calculation shows that (II.2) implies (II.3). It also implies the product formula for (3.3) a_2.

Remark 3.2. In reference to Conjecture 6.1, we observe that the preceding calculation shows that we have a map (using the notation we have established above)

$$\{(\zeta_j): j=1,..,n\} \to \exp(\oplus_{j=1}^{n}\mathbb{C}f_{\tau_j}):(\zeta_j)\to l(g^{(n)}) \tag{3.7}$$

where each ζ_j ranges over either the complex plane or a disk, depending on whether the jth root is of compact or noncompact type. The calculation above also shows that the map is 1-1 and open. We claim that the image of this map is closed in

$$\{l_2 \in \exp(\oplus_{j=1}^{n}\mathbb{C}f_{\tau_j}) : \exists\ \tilde{g}_2 \text{havingtriangularfactorization} \tilde{g}_2 = l_2 a_2 u_2\}.$$

This follows from the product formula for a_2, which shows that as the parameters tend to the boundary, the triangular factorization fails. This implies that the image of the map is the connected component which contains $l_2 = 1$. This proves the implication (II.2) implies (II.3) in the special case of Theorem 3.1, because n is fixed in the statement of that theorem, but this does not complete the proof of Conjecture 6.1. The difficulty is that we do not know how to formulate statements (I.1) and (II.1) in the general case in a way that regards n as fixed.

It is obvious that (II.3) implies (II.1). In fact (II.3) implies a stronger condition. If (II.3) holds, then given a highest weight vector v as in (II.1), corresponding to highest weight $\dot{\Lambda}$, then

$$\pi(g_2^{-1})v = \pi(u_2^{-1}a_2^{-1}l^{-1})v = a_2^{-\dot{\Lambda}}\pi(u_2^{-1})v, \tag{3.8}$$

implying that $\pi(g_2^{-1})v$ is holomorphic in Δ and nonvanishing at all points. However we do not need to include this nonvanishing condition in (II.1), in this finite case.

It remains to prove that (II.1) implies (II.3). Because $\tilde{g}_2$ is determined by g_2, as in Lemma 1.4, it suffices to show that g_2 has a triangular factorization (with trivial $\dot{T}$ component). Hence we will slightly abuse notation and work at the level of loops in the remainder of this proof.

To motivate the argument, suppose that g_2 has triangular factorization as in (II.3). Because $u_2(0)\in \dot{N}^+$, there exists a pointwise $\dot{G}$-triangular factorization

$$u_2(z)^{-1} = \dot{l}(u_2(z)^{-1})\dot{d}(u_2(z)^{-1})\dot{u}(u_2(z)^{-1}) \tag{3.9}$$

which is certainly valid in a neighborhood of $z = 0$; more precisely, (3.9) exists at a point $z \in \mathbb{C}$ if and only if

$$\dot{\sigma}_i(u_2(z)^{-1}) \neq 0, \quad i=1,..,r.$$

When (3.9) exists (and using the fact that g_2 is defined on $\mathbb{C}^*$ in this algebraic context),

$$g_2(z) = \left(l_2(z)a_2\dot{u}(u_2(z)^{-1})^{-1}a_2^{-1}\right)\left(a_2\dot{d}(u_2(z)^{-1})^{-1}\right)\dot{l}(u_2(z)^{-1})^{-1}.$$

This implies

$$g_2(z)^{-1} = \dot{l}(u_2(z)^{-1})\left(\dot{d}(u_2(z)^{-1})a_2^{-1}\right)\left(a_2\dot{u}(u_2(z)^{-1})a_2^{-1}l_2(z)^{-1}\right). \tag{3.10}$$

This is a pointwise $\dot{G}$-triangular factorization of g_2^{-1}, which is certainly valid in a punctured neighborhood of $z = 0$. The important facts are that (1) the first factor in (3.10)

$$\dot{l}(g_2^{-1}) = \dot{l}(u_2(z)^{-1}) \tag{3.11}$$

does not have a pole at $z = 0$; (2) for the third (upper triangular) factor in (3.10), the factorization

$$\dot{u}(g_2^{-1})^{-1} = l_2(z)\left(a_2\dot{u}(u_2(z)^{-1})a_2^{-1}\right) \tag{3.12}$$

is a $L\dot{G}$-triangular factorization of $\dot{u}(g_2^{-1})^{-1}\in L\dot{N}^+$, where we view $\dot{u}(g_2^{-1})^{-1}$ as a loop by restricting to a small circle surrounding $z = 0$; and (3) because there is an a priori formula for a_2 in terms of g_2 (refer 1.7), we can recover l_2 and (the pointwise triangular factorization for) u_2^{-1} from (3.10)-(3.12): $l_2 = l(\dot{u}(g_2^{-1})^{-1})$ (by (3.12)), and

$$\dot{l}(u_2(z)^{-1}) = \dot{l}(g_2(z)^{-1}), \quad \dot{d}(u_2(z)^{-1}) = \dot{d}(g_2(z)^{-1})a_2, \tag{3.13}$$

and $\quad \dot{u}(u_2(z)^{-1}) = a_2^{-1}u(\dot{u}(g_2(z)^{-1}))a_2.$

We remark that this uses the fact that g_2 is defined in $\mathbb{C}^*$ in an essential way.

Now suppose that (II.1) holds. In particular (II.1) implies that $\dot{\sigma}_i(g_2^{-1})$ has a removable singularity at $z = 0$ and is positive at $z = 0$, for i=1,..,r. Thus g_2^{-1} has a pointwise $\dot{G}$-triangular factorization as in (3.10), for all z in some punctured neighborhood of $z = 0$.

We claim that (3.11) does not have at pole at $z = 0$. To see this, recall that for an $n \times n$ matrix $g = (g_{ij})$ having an LDU factorization, the entries of the factors can be written explicitly as ratios of determinants:

$$\dot{d}(g) = \mathrm{diag}(\sigma_1, \sigma_2/\sigma_1, \sigma_3/\sigma_2, .., \sigma_n/\sigma_{n-1})$$

where σ_k is the determinant of the k^{th} principal submatrix, $\sigma_k = \det((g_{ij})_{1\le i,j\le k})$; for $i > j$,

$$l_{ij} = \det\begin{pmatrix} g_{11} & g_{12} & .. & g_{1j} \\ g_{21} & & & \\ . & & & \\ . & & & \\ g_{j-1,1} & & & g_{j-1,j} \\ g_{i,1} & & & g_{ij}\end{pmatrix}/\sigma_j = \frac{\langle g\varepsilon_1\wedge..\wedge\varepsilon_j, \varepsilon_1\wedge..\wedge\varepsilon_{j-1}\wedge\varepsilon_i\rangle}{\langle g\varepsilon_1\wedge..\wedge\varepsilon_j, \varepsilon_1\wedge..\wedge\varepsilon_j\rangle} \tag{3.14}$$

and for $i > j$,

$$u_{ij} = \det\begin{pmatrix} g_{11} & g_{12} & .. & g_{1,i-1} & g_{1,j} \\ . & & & & g_{2,j} \\ . & & & & \\ g_{i,1} & & & & g_{i,j}\end{pmatrix}/\sigma_i.$$

Apply this to $g = g_2^{-1}$ in a highest weight representation. Then (3.14), together with (II.1), implies the claim.

The factorization (3.12) is unobstructed. Thus it exists. We can now read the calculation backwards, as in (3.13), and obtain a triangular factorization for g_2 as in (II.3) (initially for the restriction to a small circle about 0; but because g_2 is of finite type, this is valid also for the standard circle). This completes the proof.

In the C^∞ analogue of Theorem 3.1, it is necessary to add further hypotheses in parts I.1 and II.1; (3.8). To reiterate, we are now assuming that the sequence $\{r_j\}_{j=1}^{\infty}$ is affine periodic.

Theorem 3.2. *Suppose that* $\tilde{g}_1 \in \tilde{L}\dot{G}_0$ *and* $\Pi(\tilde{g}_1) = g_1$. *Consider the following three statements:*

(I.1) $m(\tilde{g}_1) = 1$, *and for each complex irreducible representation* $V(\pi)$ *for* $\dot{G}$, *with lowest weight vector* $\varphi \in V(\pi)$, $\pi(g_1)^{-1}(\varphi)$ *has holomorphic extension to* Δ, *is nonzero at all* $z \in \Delta$, *and is a positive multiple of* v *at* $z = 0$.

(I.2) $\tilde{g}_1$ *has a factorization of the form*

$$\tilde{g}_1 = \lim_{n\to\infty} i_{\tau_n'}(g(\eta_n))...i_{\tau_{-N}'}(g(\eta_{-N})),$$

where $g(\eta_j) = k(\eta_j)$ *for some* $\eta_j \in \mathbb{C}$ *(resp.* $g(\eta_j) = q(\eta_j)$ *for some* $\eta_j \in \Delta$*) when* τ_j *is a compact type (resp. non-compact type) root and the sequence* $(\eta_j)_{j=-N}^{\infty}$ *is rapidly decreasing.*

(I.3) $\tilde{g}_1$ *has triangular factorization of the form* $\tilde{g}_1 = l_1 a_1 u_1$ *where* $l_1 \in H^0(\Delta^*, \dot{N}^-)$ *has smooth boundary values.*

Then statements (I.1) and (I.3) are equivalent. (I.2) implies (I.1) and (I.3).

Moreover, in the notation of (I.2),

$$a_1 = \prod_{j=-N}^{\infty} \mathbf{a}(\eta_j)^{h_{\tau_j}}. \quad (3.15)$$

Similarly, suppose that $\tilde{g}_2 \in \tilde{L}\dot{G}_0$ *and* $\Pi(\tilde{g}_2) = g_2$. *Consider the following three statements:*

(II.1) $m(\tilde{g}_2) = 1$; *and for each complex irreducible representation* $V(\pi)$ *for* $\dot{G}$, *with highest weight vector* $v \in V(\pi)$, $\pi(g_2)^{-1}(v) \in H^0(\Delta;V)$ *has holomorphic extension to* Δ, *is nonzero at all* $z \in \Delta$, *and is a positive multiple of* v *at* $z = 0$.

(II.2) $\tilde{g}_2$ *has a factorization of the form*

$$\tilde{g}_2 = \lim_{n\to\infty} i_{\tau_n}(g(\zeta_n)) \ldots i_{\tau_1}(g(\zeta_1))$$

where $g(\zeta_j) = k(\zeta_j)$ for some $\zeta_j \in \mathbb{C}$ (resp. $g(\zeta_j) = q(\zeta_j)$ for some $\zeta_j \in \Delta$) when τ_j is a compact type (resp. non-compact type) root and the sequence $(\zeta_j)_{j=1}^{\infty}$ is rapidly decreasing.

3. $\tilde{g}_2$ has triangular factorization of the form $\tilde{g}_2 = l_2 a_2 u_2$, where $l_2 \in H^0(\Delta^*, \infty; \dot{N}^+, 1)$ has smooth boundary values.

Then statements (II.1) and (II.3) are equivalent. (II.2) implies (II.1) and (II.3).

Also, in the notation of (II.2),

$$a_2 = \prod_{j=1}^{\infty} \mathbf{a}(\zeta_j)^{h_{\tau_j}}. \tag{3.16}$$

Conjecture 3.2. *If* g_1 *is in the identity connected component of the sets in (I.1) and (I.3), then the converse holds, i.e.* g_1 *has a root subgroup factorization as in (I.2). If* g_2 *is in the identity connected component of the sets in (II.1) and (II.3), then the converse holds, i.e.* g_2 *has a root subgroup factorization as in (II.2).*

In Remark 3.3, at the end of the following proof, we will indicate how we envision proving this conjecture. The issue in this C^∞ context involves analysis, and we are not as confident in the truth of this Conjecture 3.2.

Proof. The two sets of equivalences and implications are proven in the same way. We consider the second set.

Suppose that (II.1) holds. To show that (II.3) holds, it suffices to prove that g_2 has a triangular factorization with l_2 of the prescribed form (Lemma 1.4). By working in a fixed faithful highest weight representation for $\dot{\mathfrak{g}}$, without loss of generality, we can suppose $\dot{G}_0$ is a matrix subgroup of SL(n,$\mathbb{C}$) (where $\dot{\mathfrak{n}}_+$ consists of upper triangular matrices). We will assume that this representation is the complexified adjoint representation, or some subrepresentation of the exterior algebra of the adjoint representation, so that we can suppose that $\dot{G}_0$ fixes a (indefinite) Hermitian form (in the case of the adjoint representation, this is derived from the Killing form).

For the purposes of this proof, we will use the terminology in Section 1 of literature of Pickrell [11]. We view $g_2 \in L\dot{G}_0$ as a multiplication operator on the Hilbert space $\mathcal{H} = L^2(S^1;\mathbb{C}^n)$, and we write

$$M_{g_2} = \begin{pmatrix} A(g_2) & B(g_2) \\ C(g_2) & D(g_2) \end{pmatrix}$$

relative to the Hardy polarization $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$, where $A(g_2)$ is the compression of M_{g_2} to $\mathcal{H}^+$, the subspace of functions in $\mathcal{H}$ with holomorphic extension to Δ. To show that g_2 has a Birkhoff factorization, we must show that $A(g_2)$ is invertible (Theorem 1.1 of [11]).

Let $C_1,\ldots,C_n$ denote the columns of g_2^{-1}, and let $C_1^*,\ldots,C_n^*$ denote the rows of g_2. We can regard these as dual bases with respect to the pairing given by matrix multiplication, i.e., $C_i^* C_j = \delta_{ij}$.

The hypothesis of (II.1) implies that both C_1 and C_n^* have holomorphic extensions to Δ (in the latter case, by considering the dual representation). Now suppose that $f \in \mathcal{H}^+$ is in the kernel of $A(g_2)$. Then

$$(C_j^* f)_+ = 0, \quad j = 1,\ldots,n, \tag{3.17}$$

where $()_+$ denotes projection to $\mathcal{H}^+$. Since C_n^* has holomorphic extension to Δ, $(C_n^* f)_+ = C_n^* f$ and therefore $C_n^* f$ is identically zero on S^1 by (3.17). This implies that for $z \in S^1$, $f(z)$ is a linear combination of the $n-1$ columns $C_j(z)$, $j<n$. We write

$$f = \lambda_1 C_1 + \ldots + \lambda_{n-1} C_{n-1}$$

where the coefficients are functions on the circle (defined a.e.). Now consider the pointwise wedge product of $\mathbb{C}^n$ vectors

$$f \wedge C_1 \wedge \ldots \wedge C_{n-2} = \pm \lambda_{n-1} C_1 \wedge \ldots \wedge C_{n-1}.$$

The vectors $C_1 \wedge \ldots \wedge C_j$ extend holomorphically to Δ, and never vanish, for any j, by (II.1) (by considering the representation $\wedge^j(\mathbb{C}^n)$). Since f also extends holomorphically, this implies that $_{n-1}$ has holomorphic extension to . Now

$$C_{n-1}^* f = \lambda_{n-1} C_{n-1}^* C_{n-1} = \lambda_{n-1}$$

by (3.17) and duality.

Since the right hand side is holomorphic in Δ, by (3.17) (for $j = n-1$) λ_{n-1} vanishes identically. This implies that in fact f is a (pointwise) linear combination of the first $n-2$ columns of g_2^{-1}. Continuing the argument in the obvious way (by next wedging f with $C_1 \wedge \ldots \wedge C_{n-3}$ to conclude that λ_{n-2} must vanish), we conclude that f is zero. This implies that $\ker(A(g_2)) = 0$. Since $\dot{G}$ is simply connected, $(A(g_2)$ has index zero. Hence $(A(g_2)$ is invertible. This implies (II.3).

It is obvious that (II.3) implies (II.1); see (3.8). Thus (II.1) and (II.3) are equivalent.

Before showing that (II.2) implies (II.1) and (II.3), we need to explain why the C^∞ limit in (II.2) exists. We first consider the projection of the product in $L\dot{K}$. Because $g(\zeta_j) = 1 + O(|\zeta_j|)$ as $\zeta_j \to 0$, the condition for the product in (II.2) to converge absolutely is that $\sum \zeta_n$ converges absolutely. So g_2 certainly represents a continuous loop.

We will now calculate the derivative formally. In this calculation, we let $g_2^{(n)}$ denote the product up to n, and $\tau_n = q(n)d^* - \dot{\alpha}(n)$ $(q(n) > 0$, and $\dot{\alpha}(n) > 0$). Then

$$g_2^{-1}(\frac{\partial g_2}{\partial \theta}) = \Pi(\sum_{n=1}^{\infty} Ad(g^{(n-1)})^{-1}\left(\iota_{\tau_n}(g(\zeta_n))^{-1} \frac{\partial}{\partial \theta} \iota_{\tau_n}(g(\zeta_n))\right)) \tag{3.18}$$

$$= \sum_{n=1}^{\infty} Ad(g^{(n-1)})^{-1}\left(\sqrt{-1}\frac{q(n)}{1 \pm |\zeta_n|^2}(\mp|\zeta_n|^2 h_{\dot{\alpha}(n)} - \zeta_n e_{\dot{\alpha}(n)} z^{-q(n)} \mp \bar{\zeta}_n f_{\dot{\alpha}(n)} z^{q(n)})\right).$$

Because we are using an affine periodic sequence of simple reflections (with period $w_l^{-1} \in C \subset \dot{h}_{\mathbb{R}}$), $\tau_{l+1} = w_l^{-1} \cdot \tau_1$, $\tau_{l+2} = w_l^{-1} \tau_2$, and so on. In general, writing $\tau_j = k(j)d^* - \dot{\alpha}(j)$ as above, and using Proposition (4.9.5) of [15] to calculate the coadjoint action,

$$\tau_{nl+j} = w_l^{-n} \cdot \tau_j = (q(j) + n\dot{\alpha}(j)(w_l))d^* - \dot{\alpha}(j). \tag{3.19}$$

Because $\dot{\alpha}(w_l) > 0$, for all $\dot{\alpha} > 0$, it follows that $q(n)$ is asymptotically n. Together with Bessel's inequality, (3.18) implies that

$$\int |g_2^{-1}(\frac{\partial g_2}{\partial\theta})|^2\, d\theta \le \sum_{n=1}^{\infty} \| \mathrm{Ad}(g^{(n-1)}) \|_2^2 \frac{q(n)^2}{(1\pm|\zeta_n|^2)^2}(|\zeta_n|^4+|\zeta_n|^2)|h_{\dot\alpha(n)}|^2$$

The right hand side is comparable to $\sum_{n=1}^{\infty} n^2|\zeta_n|^2$ because $\|\mathrm{Ad}(g^{(n-1)})\|_2^2$ is uniformly bounded in n. Thus g_2 is W^1 (the L^2 Sobolev space) whenever $(\zeta_j) \in w^1$. Higher derivatives can be similarly calculated. This shows that if w^n, then $g_2 \in W^n$. Hence if $\zeta \in c^\infty$, the Frechet space of rapidly decreasing sequences, then $g_2 \in C^\infty$. Together with Proposition 1.4, this implies that the product in (II.2) converges in $\tilde{L}\dot{G}_0$.

Now suppose that (II.2) holds. The map from ζ to $\tilde{g}_2$ is continuous, with respect to the standard Frechet topologies for rapidly decreasing sequences and smooth functions. The product (3.16) is also a continuous function of ζ, and hence is nonzero. This implies that $\tilde{g}_2$ has a triangular factorization which is the limit of the triangular factorizations of the finite products $\tilde{g}_2^{(n)}$. By Theorem 3.1 and continuity, this factorization will have the special form in (II.3). Thus (II.2) implies (II.1) and (II.3).

Remark 3.3. We now want to give a naive argument for Conjecture 3.2. Suppose that we are given g_2 as in (II.1) and (II.3). Recall that l_2 has values in $\dot{N}^+$. We can therefore write

$$l_2 = \exp(\sum_{j=1}^{\infty} x_j^* f_{\tau_j}), \quad x_j^* \in \mathbb{C}. \tag{3.20}$$

(the use of x^* for the coefficients is consistent with our notation in the $SU(1,1)$ case (II.3) of Theorem 0.1).

As a temporary notation, let X denote the set of g_2 as in (II.1) and (II.3); x^* is a global linear coordinate for this space. We consider the map

$$c^\infty \to X \text{given by} \zeta \mapsto g_2. \tag{3.21}$$

This map induces bijective correspondences among finite sequences ζ, $g_2 \in X \cap L_{fin}\dot{K}$ and finite sequences x^*, and the maps ζ to x^* and $x^* \to \zeta$ are given by rational maps (i.e. rational in ζ and $\bar{\zeta}$); however (although it seems likely) it is not known that the limits of these rational maps actually make sense even for rapidly decreasing sequences (Appendix of [11] for the $SU(2)$ case). We will use an inverse function argument to show that the map (3.21) has a global inverse (technically, to apply the inverse function theorem, we should consider the maps of Sobolev spaces $w^n \to X^n$ where X^n is the W^n completion of X, but we will suppress this).

Given a variation of ζ, denoted ζ′, we can formally calculate the derivative of this map,

$$g_2^{-1}g_{2'} = \sum_{n=1} Ad(g_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n)\begin{pmatrix} 1 & \bar{\zeta}_n \\ -\zeta_n & 1 \end{pmatrix}\{\mathbf{a}(\zeta_n)'\begin{pmatrix} 1 & \bar{\zeta}_n \\ \zeta_n & 1 \end{pmatrix} + \mathbf{a}(\zeta_n)\begin{pmatrix} 0 & \bar{\zeta}_{n'} \\ \zeta_{n'} & 0 \end{pmatrix}\})) \tag{3.22}$$

$$= \sum_{n=1} Ad(g_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n)^{-1}\mathbf{a}(\zeta_n)'\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \mathbf{a}(\zeta_n)^2\begin{pmatrix} \bar{\zeta}_n\zeta_{n'} & \bar{\zeta}_{n'} \\ \zeta_{n'} & \zeta_n\zeta_{n'} \end{pmatrix}))$$

$$= \sum_{n=1} Ad(g_2^{(n-1)})^{-1}(i_{\tau_n}(\mathbf{a}(\zeta_n)^2\begin{pmatrix} \frac{1}{2}(\bar{\zeta}_n\zeta_{n'} - \zeta_n\bar{\zeta}_{n'}) & \bar{\zeta}_{n'} \\ \zeta_{n'} & -\frac{1}{2}(\bar{\zeta}_n\zeta_{n'} - \zeta_n\bar{\zeta}_{n'}) \end{pmatrix}))$$

As before it is clear that this is convergent, so that (3.21) is smooth. At ζ = 0 this is clearly injective with closed image, so that there is a local inverse. Consider more generally a fixed $g_2 \in X \cap L_{fin}\dot{G}_0$, so that $g_2^{(n-1)} = g_2$ for large n. Recall that the root spaces for the τ_n are independent and fill out $\dot{n}^-(z\mathbb{C}[z])$. Given a variation such that $g_2^{-1}g_2' = 0$, the terms in the last sum in the derivative formula (3.22) must be zero for large n. But we know that the map (3.21) is a bijection on finite ζ. Thus for a variation of a finite number of ζ_j which maps to zero, the variation vanishes. It is clear that the image of the derivative (3.22) is closed. The image is therefore the tangent space to X (because we know that finite variations will fill out a dense subspace of the tangent space). This implies there is a local inverse. This local inverse is determined by its values on finite x^*, and hence there is a uniquely determined global inverse. This shows that (II.1) and (II.3) imply (II.2).

Finally, (3.16) follows by continuity from (3.3).

Generalization of Theorem 3.3

Theorem 3.3. *Suppose* $\tilde{g} \in \tilde{L}\dot{G}_0$ *and* $\Pi(\tilde{g}) = g$.

(a) The following are equivalent:

(i) $\tilde{g}$ *has a triangular factorization* $\tilde{g} = lmau$, *where l and u have* C^∞ *boundary values, and satisfy the conditions* $l(z), u^{-1}(z) \in \dot{G}_0\dot{B}^+$ *for all* $z \in S^1$.

(ii) $\tilde{g}$ *has a (partial root subgroup) factorization of the form*

$$\tilde{g} = \Theta(\tilde{g}_1^*)\exp(\chi)\tilde{g}_2,$$

where $\chi \in \tilde{L}\mathfrak{t}$, *and* $\tilde{g}_1$ *and* $\tilde{g}_2$ *are as in (I.3) and (II.3) of Theorem 3.2, respectively.*

(b) *In reference to (ii) of part (a),*

$$a(\tilde{g}) = a(g) = a(g_1)a(\exp(\chi))a(g_2), \quad \Pi(a(g)) = \Pi(a(g_1))\Pi(a(g_2)) \tag{3.23}$$

and

$$a(\exp(\chi)) = |\sigma_0|(\exp(\chi))^{h_0}\prod_{j=1}^{r}|\sigma_0|(\exp(\chi))^{\check{a}_j h_j}. \tag{3.24}$$

Remarks. Suppose that $\dot{G}_0 = SU(1,1)$. In this case the last condition in (i) in Theorem 3.3, that $l(z), u^{-1}(z) \in \dot{G}_0\dot{B}^+$, is equivalent to the condition in Theorem 0.3 that the boundary values l_{21}/l_{11} and u_{21}/u_{22} are <1 in magnitude on S^1, and part (b) specializes to the statement of Theorem 0.4.

Proof. Our strategy of proof is the following. We will first show that in part (a), (ii) implies (i). In the process we will prove part (b). We will then show that (i) implies (ii).

Suppose that we are given $\tilde{g}$ as in (ii). Both $\tilde{g}_1$ and $\tilde{g}_2$ have triangular factorizations by Theorem 3.2. In the notation of Theorem 3.2,

$$\tilde{g} = \Theta((l_1a_1u_1)^*)\exp(\chi)(l_2a_2u_2) = \Theta(u_1^*)a_1(\Theta(l_1^*)\exp(\chi)l_2)a_2u_2 \tag{3.25}$$

since Θ preserves the A factor. The basic observation is that

$$b = \Theta(l_1^*)\exp(\chi)l_2 \in (\tilde{L}\dot{B}^+)_0 \tag{3.26}$$

(the inverse image in the affine extension for the identity component of loops in $\dot{B}^+$), and b will have a triangular factorization which we can compute. To do this requires some care with the central extension, and this involves some preparation.

Because $\dot{B}^+$ is the semidirect product of $\dot{H}$ and $\dot{N}^+$, there is an isomorphism of loop groups

$$L\dot{B}^+ = L\dot{H} \ltimes L\dot{N}^+.$$

The central extension is trivial for $L\dot{N}^+$, and hence there is an isomorphism

$$\tilde{L}\dot{B}^+ = \tilde{L}\dot{H} \ltimes L\dot{N}^+$$

where the action of $\tilde{L}\dot{H}$ on $L\dot{N}^+$ is the same as the conjugation action

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of $L\dot{H}$ on $L\dot{N}^+$, and $\widetilde{L\dot{H}}$ is a Heisenberg extension determined by the bracket (1.3).

Given $\chi \in \widetilde{L}\mathfrak{t}$ as above, let $\chi = \chi_-^* + \chi_0 + \chi_+$ denote the linear triangular decomposition, where $\chi_0 \in \mathfrak{t}$, $\chi_+ \in H^0(\Delta, 0; \dot{\mathfrak{h}}, 0)$ and $\chi_- = -\chi_+^*$. Then (calculating in terms of the Heisenberg extension)

$$\exp(\chi) = \exp(\chi_-)\exp(\chi_0)\exp(-[\chi_-, \chi_+])\exp(\chi_+)$$

$$= \exp(\chi_-)\exp(\chi_0)\exp(\sum_{j=1}^{\infty} j\langle \chi_j, \chi_j \rangle c)\exp(\chi_+).$$

Substituting this into (3.26) we find

$$b = \exp(\chi_-) b_1 \exp(\chi_+)$$

where

$$b_1 = \exp(-\chi_-)\Theta(l_1^*)\exp(\chi_-)\exp(\chi_0)\exp(\sum_{j=1}^{\infty} j\langle \chi_j, \chi_j \rangle c)\exp(\chi_+) l_2 \exp(-\chi_+).$$

Thus, b has a triangular factorization

$$b = \left(\exp(\chi_-)L\right)\left(m(b)a(b)\right)\left(U\exp(\chi_+)\right),$$

where $m(b) = m(b_1) = \exp(\chi_0)$, $a(b) = a(b_1) = \exp(\sum_{j=1}^{\infty} j\langle \chi_j, \chi_j \rangle c)$,

$$L = l(\exp(-\chi_-)\Theta(l_1^*)\exp(\chi_-)\exp(\chi_0)\exp(\chi_+) l_2 \exp(-\chi_+)) \in H^0(\Delta^*, \infty; \dot{N}^+, 1),$$

and

$$U = u(\exp(-\chi_-)\Theta(l_1^*)\exp(\chi_-)\exp(\chi_0)\exp(\chi_+) l_2 \exp(-\chi_+)) \in H^0(\Delta; \dot{N}^+).$$

Thus, from (3.25), $\tilde{\ }$ will have a triangular factorization $l(\tilde{g})m(\tilde{g})a(\tilde{g})u(\tilde{g})$ with

$$l(\tilde{g}) = \Theta(u_1^*)\exp(\chi_-) a_1 L a_1^{-1}, \quad m(\tilde{g}) = m(b) = \exp(\chi_0), \tag{3.27}$$
$$a(\tilde{g}) = a_1 a_2 \exp(\sum_{j=1}^{\infty} j\langle \chi_j, \chi_j \rangle c), \quad u(\tilde{g}) = a_2^{-1} U a_2 \exp(\chi_+) u_2.$$

Thus, (ii) implies (i) in part (a). At the same time this also implies part (b).

Now we need to show that (i) implies (ii). For this direction, there is not any need to consider the central extension, so we will no longer use tildes for group elements.

Suppose $g = lmau$, as in (i). At each point of the circle there exist $\dot{N}^+ \dot{A} \dot{G}_0$ decompositions

$$l^{-1} = \dot{n}_1 \dot{a}_1 \dot{g}_1, \quad u = \dot{n}_2 \dot{a}_2 \dot{g}_2. \tag{3.28}$$

This is a consequence of the somewhat bizarre hypotheses in (i). Then $\dot{g}_1 = \Theta(\dot{g}_1^{-1})^* = \dot{a}_1^{-1}\Theta(\dot{n}_1^*)\Theta(\ell^*)$ since $\dot{g}_2 \mapsto \Theta(\dot{g}_2^{-1})^*$ is the involution fixing $\dot{G}_0$ in $\dot{G}$, and Θ acts as the inverse on $\dot{A}$ under the Hermitian type assumption.

In turn, there are Birkhoff decompositions

$$\dot{a}_i^{-1} = \exp(\chi_i^* + \chi_{i,0} + \chi_i), \quad \chi_i \in H^0(\Delta, \dot{\mathfrak{h}}), \quad \chi_{i,0} \in \dot{\mathfrak{h}}_{\mathbb{R}}$$

for i = 1,2. Define

$$g_i = \exp(-\chi_i^* + \chi_i)\dot{g}_i$$

for i = 1,2. Then

$$g_1 = \exp(-\chi_{1,0} - 2\chi_1^*)\Theta(\dot{n}_1^*)\Theta(\ell^*)$$

has triangular factorization with

$$l(g_1) = l(\exp(-\chi_{1,0} - 2\chi_1^*)\Theta(\dot{n}_1^*)\exp(\chi_{1,0} + 2\chi_1^*)) \in H^0(\Delta^*, \infty; \dot{N}^-, 1),$$

$$\Pi(a(g_1)) = \exp(\chi_{1,0}),$$

and similarly

$$g_2 = \exp(2\chi_2 + \chi_{2,0})\dot{n}_2^{-1} u$$

has triangular factorization with

$$l(g_2) = l(\exp(2\chi_2 + \chi_{2,0})\dot{n}_2^{-1}\exp(-2\chi_2 - \chi_{2,0})) \in H^0(\Delta^*, \infty; \dot{N}^+, 1),$$

$$\Pi(a(g_2)) = \exp(\chi_{2,0}).$$

The conclusion is somewhat miraculous. On the one hand $(g_1^*)^{-1} g g_2^{-1}$ has values in $\dot{G}_0$ because $g_1 \mapsto \Theta(g_1^*)^{-1}$ is the pointwise involution fixing $\dot{G}_0$ in $\dot{G}$. On the other hand

$$\Theta(g_1^*)^{-1} g g_2^{-1} = \Theta(g_1^*)^{-1} lma(\exp(2\chi_2 + \chi_{2,0})\dot{n}_2^{-1} u)^{-1}$$

$$= \Theta(g_1^*)^{-1} lma\dot{n}_2 \exp(-2\chi_2 - \chi_{2,0})$$

$$= \exp(-\chi_{1,0} - 2\chi_1^*)\dot{n}_1 \Pi(ma)\dot{n}_2 \exp(-2\chi_2 - \chi_{2,0}) \tag{3.29}$$

has values in $\dot{B}^+$. Therefore $\Theta(g_1^*)^{-1} g g_2^{-1}$ has values in $\dot{G}_0 \cap \dot{B}^+ = \dot{T}$. It is also clear that (3.29) is connected to the identity, and hence $\Theta(g_1^*)^{-1} g g_2^{-1} \in (L\dot{T})_0$ and thus equals $\exp(\quad)$. Hence, $g = \Theta(g_1^*)\exp(\chi) g_2$. Thus (i) implies (ii). This completes the proof.

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