Mathematical Model of Complete Shallow Water Problem with Source Terms, Stability Analysis of Lax-Wendroff Scheme

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Abstract

The most effective simulations of complete physical problems consist of the evaluation of maximum water levels and discharges that may be attained at particular locations during the development of an exceptional meteorological event. There is also the provision of the scenario subsequent to the almost instantaneous release of a great volume of liquid. The situation is that of the breaking of a man made dam. There is therefore a necessity to develop a model capable of reproducing solutions of the complete equations despite the irregularities of a non-prismatic bed. This requires the development of efficient and effective numerical schemes able to predict water levels and discharges in hydraulic systems. The use of mathematical models as a predictive tool in the simulation of free surface flows represents a good candidate for the application of many of the techniques developed in fluid dynamics. In this paper we develop a 1-D complete model of shallow water equations with source terms using both conservation of water mass and conservation of the momentum content of the water. We describe the Lax-Wendroff scheme for these nonlinear partial differential equations (PDEs) and we analyze the stability restriction of the method. This extends the nonstationary shallow water problems without source terms which are deeply studied in literature. Some numerical experiments are considered and critically discussed.

Keywords: 1-D shallow water; Saint-Venant equations; Source terms; Lax-Wendroff scheme; Von Neumann or Fourier stability analysis; Stability restriction

Introduction

Most open-channel flows of interest in the physical, hydrological, biological, engineering and social sciences are unsteady and can be considered to be one-dimensional (1-D). In unsteady flow, some aspects of the flow (velocity, depth, pressure, or another characteristic) is changing with time. In 1-D flow, longitudinal acceleration is significant, whereas transverse and vertical accelerations are negligible. Many interesting problems involving 1-D nonstationary flows have been approximated by assumption of steady flows (for example, constant peak discharges in flood plain delineation studies) or piecewise steady flows, where in storage outflow relations are derived for channel reaches from a steady flow hydraulic model and used in simple hydrologic routing methods. Piecewise steady flow analysis typically does not consider all the forces acting on the flow and only partially accounts for channel storage effects. The approximate solutions for steady flow and piecewise steady flow analysis are adequate for certain simplified planning or design problems but are inadequate for many others (for example, streams with rapidly rising and falling stage, flat slopes, and broad flood plains where storage and acceleration effects could be substantial). No criteria are available to guide researchers especially, when steady flow methods are acceptable and when a complete unsteady flow analysis is necessary. Further, problems such as tidally affected flows and sudden releases from power generation stations require 1-D unsteady flow analysis. In general only nonuniform unsteady flow is of interest in hydraulic analysis.

Three conservation principles: conservation of water mass, conservation of the mechanical energy content of the water and conservation of the momentum content of the water are available for analysis of 1-D unsteady flow. Conservation of thermal energy is not considered because temperature change and heat transfer effects do not affect flow depth and discharge. In some works [1-5] researchers provide a detailed list of differences between the energy and momentum approaches and argue for combined application of the conservation of mass and conservation of momentum principles as the equations of motion because this combination gives the correct wave speed and height should abrupt waves (hydraulic bores) form during the modeling of rapidly increasing or decreasing flow. In contrast, application of the conservation of energy principle provides no simple approximation that can be applied to yield the correct wave speed and height. The applicability of the conservation of momentum principle to the solution of lateral inflow problems has been demonstrated in modeling of side channel spillways and wash water troughs (for example, [6]), both of which cause much greater turbulence than normally results in unsteady flow. In order to take care of this problem we can follow different approaches: for instance in [7] the authors give further evidence for the choice of the momentum conservation principle. Further, because the use of Manning’s equation for resistance losses yields a better estimate of the resistance coefficient for the momentum principle than for the energy principle, methods based on momentum conservation yield better estimates of the water surface profile than do methods based on energy conservation, especially if Manning’s number n is calibrated to measured water-surface profiles or historic high water marks. In addition, the resistance coefficient estimated from the momentum principle was insensitive to variations in the velocity of lateral inflow (many applications of unsteady flow involve a wide range of lateral

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inflow rates). Finally, the equation obtained with the conservation of momentum principle is simpler than the equation obtained with the conservation of energy principle.

In the analysis of unsteady flow in open channels, using suitable assumptions [8], formal statements of the conservation of water volume (mass) and conservation of water momentum can be developed. No forces of any kind are considered in the conservation of mass. Forces, momentum fluxes and the momentum of water in storage are related in the conservation of momentum principle. If all the factors are included in the analysis, the equations are referred to as the complete Saint-Venant or shallow water equations with source terms.

The 1-D complete Saint-Venant equations with source terms solved by analytical method is too complex that’s why in this note, we apply a numerical scheme known as Lax-Wendroff method. This algorithm is super-convergent when applied to some test examples to detect possible deterioration of accuracy due to strong oscillations in the parameters that determine the stencil [9-12]. So this scheme is compared to many numerical methods of high order of accuracy, such as, the linear Central Weighted Essential-Non-Oscillatory (CWENO) scheme which is superior to full nonlinear CWENO method [13-15], to high-resolution TVD conservative schemes along with high order Central Schemes for hyperbolic systems of conservative Laws [9,16] and to Central-upwind schemes for the shallow water system [17]. In a search for stable and more accurate shock capturing numerical approach, the authors [16,18,19] have compared some numerical schemes, namely, Leapfrog, Lax-Wendroff, Lax-Friedrichs, and so on, for shallow water equations without source terms. Their results have shown that the Lax-Wendroff is an explicit second order method, is more efficient and effective than the others and the stability restriction of this scheme is given by the famous Courant-Friedrichs-Lewy (CFL) condition. Furthermore, Lax-Wendroff’s approach is one of the most frequently encountered in the literature related to classical Shock-capturing schemes. However, difficulties have been reported when trying to include source terms in the discretization and to keep the second order accuracy at the same time. For more detail we refer the readers to [14,18]. In this report the attention is focused on the complete Saint-Venant equations with source terms and, more specifically, we are interested in the following four items.

1. Mathematical modeling of 1-D complete shallow water equations with source terms;
2. Complete description of the Lax-Wendroff method for these complex nonlinear PDEs;
3. Stability requirement of this algorithm: this item together with item 2 are our original contributions and they represent a generalization of [16,18], where r considered as the lateral inflow per unit length along the channel (the so-called source term) is assumed to be identically equal to zero;
4. A wide set of numerical evidences concerning the simulations of the Lax-Wendroff approach for 1-D complete shallow water equations with source terms, and regarding the effectiveness of this scheme according to the theoretical indications given in the first three items.

In particular, we consider the case where the channel is prismatic and the interesting result is that the algorithm seems to be second order accurate while the stability limitation is not the same as the CFL condition widely studied in the literature for hyperbolic partial differential equations (for example: linear advection equation, wave equation, inviscid burgers equations, etc.). However, while the stability requirement is highly unusual, the result has a potential positive implication since the stability restriction obtained in this work controls the famous CFL condition. Indeed the nice feature is that, as required in a stability context, we normally find the stability condition from a Fourier stability analysis. On the other hand, it follows from this analysis that an instability occurs when $\Delta t \geq \Delta x / c$, where $c$ is the velocity of the wave.

\[ p \rho g y(x,t) = \int_{y}^{0} T(x,z) \, dz \]

\[ A(x,y(x,t)) = \int_{y}^{0} T(x,z) \, dz \]

\[ F = \rho g \int_{y}^{0} (y(x,y) - z)T(x,z) \, dz \]

\[ (x,y(x,t)) \]
\[ J[x,y(x,t)] = \int_{y(x)}^{y(x,t)} \{ y(x,y) - z \} T(x,z) \, dz \]  

(3)

Expansion of relation (3) and integration by parts yields
\[ J[x,y(x,t)] = \int_{0}^{y(x,t)} A(X,Z) \, dz \]

The qualifier that the first moment of area should be about the water surface is now dropped, because this is the only axis where moments are determined.

**Definition 4.5.** The wetted perimeter \( P(x,y(x,t)) \) is the length of the boundary of the cross section that is under water for a given height of water \( y \). It can be defined in terms of an integral involving derivatives of the boundary shape (the mathematics will not be discussed here because the characteristic can be simply described).

**Remark 4.1.** The wetted perimeter is never less than the top width and is often nearly equal to the top width. However, there are cross sections for which the difference between top width and wetted perimeter is substantial. Therefore, the conveyance which includes the wetted perimeter implicitly, is used in FEQ and FEQUTL (Franz and Melching, [20]) simulations of a channel.

**Definition 4.6.** The conveyance is the simplest of the dynamic elements, at least if the Manning friction loss relation is applied. A compact channel is shaped such that the ratio of the flow area to the wetted perimeter (that is, the hydraulic radius) adequately describes the effect of channel shape on the friction losses. The conveyance for a compact channel is
\[ K(x,y) = \frac{1}{n_{h}} A(x,y) R(x,y)^{2/3}, \]  

where \( R(x,y) \) is the hydraulic radius, which equals \( A(x,y)/P(x,y) \), and \( n_{h} \) is the Manning’s roughness coefficient. If the cross section is noncompact, it must be subdivided. The subdivision of compound and composite cross sections is discussed in Franz and Melching [20].

**Definition 4.7.** The effects of nonuniform velocity distributions are corrected with momentum and kinetic energy flux coefficients. In 1-D flow analysis, the average velocity is used to compute the flux of momentum and kinetic energy. However, these fluxes involve powers of the velocity at each point of the cross section (local velocities) so that an error results if the average velocity is used. The square of the velocity at each point of the cross section (local velocities) so that an error results if the average velocity is used. Thus, is negative if the lateral flow is out of the channel. The left-hand side of equation (10) is the change in volume of water contained in the control volume during the time interval \( t \), while the right-hand side of (10) is the net volume of inflow to the control volume (inflow minus outflow) during the time interval. Thus, equation (10) indicates that the change in volume of the water in the control volume during any time interval is equal to the difference between the volume of inflow and the volume of outflow during that time interval. On the other hand,
\[ \int_{0}^{t} \{ A(x,s) - A(x,t) \} \, dx = \int_{0}^{t} (Q(x,s) - Q(x,t)) \, dt. \]  

(10)

Where \( x_{i} \), downstream boundary of the control volume; \( x_{i} \), initial time; \( t \), one time step later than \( t \), that is, \( t > t_{i} \). The term \( \dot{Q}(t) \) denotes the inflow of water that enters the control volume through the sides of the channel and thus, is negative if the lateral flow is out of the channel. The left-hand side of equation (10) is the change in volume of water contained in the control volume during the time interval \( t_{i},t \) while the right-hand side of (10) is the net volume of inflow to the control volume (inflow minus outflow) during the time interval. Thus, equation (10) indicates that the change in volume of the water in the control volume during any time interval is equal to the difference between the volume of inflow and the volume of outflow during that time interval. On the other hand,
the principle of conservation of momentum includes the momentum flux and various forces on the boundaries of the control volume. In most basic fluid mechanics texts (for example, [24]), the conservation of momentum for a control volume in one dimension results in

\[
F_i = \frac{\partial}{\partial t} \int_{V_i} \rho \mathbf{u} dV + \int_{S_i} \rho \mathbf{u} \mathbf{n} dA,
\]

where \( F_i \) are the forces acting on the control volume (CV), \( v_i \) is the velocity in the \( x \)-direction, \( dV \) is the volume differential, \( \mu \) is the frictional stress, and \( dA \) is the differential area taken as a vector normal to the control surface of the control volume. The first term on the right-hand side of equation (11) is the rate of change in momentum stored in the control volume and the second term is the momentum flux through the control volume. By moving the momentum stored in the control volume to the left-hand side and the sum of forces to the right-hand side and expanding the sum of forces, the conservation of momentum for the control volume becomes

\[
\rho \int_0^t \{Q(x, t) - Q(x, t_0)\} dt + \rho g \int_0^t \frac{d}{dt} \{\rho \mathbf{u}(x, t) - \rho \mathbf{u}(x, t_0)\} dt + \rho g \int_0^t \int_S S_p A \mathbf{n} \cdot dA - \int_0^t \int_S \tau F \cdot dA + \int_0^t \int_S C_D(w) \rho \bar{U}^2 \cos \phi \cdot dA,
\]

where \( S_p \) is the bottom slope of the channel, \( F \) is the average shear stress on the water from the channel boundary, \( C_D(w) \) is the dimensionless drag coefficient for wind shear stress, \( \phi \) is the angle between the downstream flow direction in the channel and the wind velocity. Although complicated, equation (12) is a precise mathematical statement of the conservation of momentum principle. The friction force term simplifies if it is assumed that the relation between slope and boundary friction from steady uniform flow,

\[
S_b = \frac{\tau F}{\rho g A},
\]

(13)

can be generalized to unsteady flow by replacing the bottom slope \( S_b \) with the friction slope \( S_f \) given by relation (15). Applying this definition of the friction slope and dividing equation (12) by \( \rho \) yields the integral form of the conservation of momentum equation for open-channel flow. That is,

\[
\int_0^t \{Q(x, t) - Q(x, t_0)\} dt - \int_0^t \int_S \{\rho \mathbf{u}(x, t) - \rho \mathbf{u}(x, t_0)\} \cdot dA + \frac{\rho g}{\int_0^t \int_S S_p A \mathbf{n} \cdot dA + \int_0^t \int_S C_D(w) \rho \bar{U}^2 \cos \phi \cdot dA + \int_0^t \int_S S_p A \mathbf{n} \cdot dA - \int_0^t \int_S \tau F \cdot dA + \int_0^t \int_S C_D(w) \rho \bar{U}^2 \cos \phi \cdot dA,
\]

In equation (14), the momentum contribution from the lateral inflow is ignored. The friction slope must be estimated from sectional characteristics and the flow. In terms of the total channel conveyance \( K \), the friction slope is computed from

\[
S_f = \frac{Q^2}{K^2}.
\]

(15)

The use of the product \( Q^2 \) instead of \( Q^2 \) as normally seen in steady flow analysis gives the result that the friction is a retarding force on the water in the control volume for either direction of flow.

The differential form of equations derived by manipulating the integral form or an approximation of it by taking limits as the time and distance intervals approach zero. Approximating the integrals in equations (10) and (14) by finite differences and taking limits gives the following conservative system of time dependent partial differential equations

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = R,
\]

(16)

where the wind stress terms are omitted in these developments to simplify the equations because these terms are not necessary for the general development of the differential equations of motion. In addition, the momentum flux correction coefficients \( \beta \) are assumed to be 1. Here \( r \) is the lateral inflow per unit length along the channel, defined as a function of distance and time such that

\[
I(t) = \int_r^\infty r(x,t)dx.
\]

(17)

The system of partial differential equations given by relation (16) is often called a 1-D complete shallow water problem with source terms. All the quantities in system (16) are algebraic expressions and can be positive or negative. Therefore, a negative outflow is an inflow. The first equation of system (16) is a statement of the conservation of mass principle (with \( \rho \) constant) on a per unit length basis. Similarly, the second equation of (16) is a statement of the principle of conservation of momentum per unit length. The terms involving derivatives of \( J \) on the right-hand side of the equal sign represent the net downstream pressure force per unit length. The derivative of \( \rho Q \) then moved to the right of the equal sign, represents the net efflux of momentum per unit length. Finally, the term \( gA(S,S_b) \) is the net downstream force per unit length from gravity and friction forces.

**Lemma 4.2.**

The system of partial differential equations (16) is equivalent to the following conservative system

\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial x} = R,
\]

(18)

where \( R = gA(Q_x^{\prime} + g\frac{\partial U}{\partial x})^2 + gA(S,S_b) \). Equation (18) emphasizes the conservative character of the system (16). Here, \( I_1 \) represents a hydrostatic pressure force term as defined in [23]

\[
I_1 = \int_0^\infty \{y(x,t)-\theta\} T(x,\eta) d\eta,
\]

(19)

in terms of the surface water level \( Y(x,t) \) and the breadth

\[
T(x,\eta) = \frac{\partial A}{\partial \eta}.
\]

(20)

\( I_1 \) accounts for the pressure forces in a volume of constant depth \( y \) due to longitudinal width variations

\[
I_2 = \int_0^\infty \{y(x,t)-\theta\} \frac{\partial T}{\partial x} d\eta.
\]

(21)

In the particular case of prismatic channels of constant breadth (or top width), they reduce to the original equations

\[
\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = S,
\]

(22)
Lax-Wendroff scheme for full shallow-water equations

In this section, we describe the Lax-Wendroff numerical scheme for 1-D complete surface water equations with source terms given by (16). This method seems to be the more convenient since it is one of finite difference schemes of second order accuracy for hyperbolic partial differential equations [26]. The development of this scheme for nonlinear PDEs follows from the Taylor series

\[ G(x,t+\Delta t) = G(x,t) + \Delta t \frac{\partial G(x,t)}{\partial t} + \frac{(\Delta t)^2}{2} \frac{\partial^2 G(x,t)}{\partial t^2} + \cdots \]

To approximate solutions of (22)-(23) (see appendix A) we discretize in both space and time assuming uniform mesh spacings of \( \Delta x \) and \( \Delta t \), respectively. We denote the spatial grid-points by \( x_j = j\Delta x \) and the time steps by \( t_n = n\Delta t \)

**Lemma 5.1.** The Lax-Wendroff numerical scheme for 1-D complete Saint-Venant equations with source terms (22)-(23) is given by

\[
A^{n+1} = A^n + \Delta t \cdot \left\{ \frac{\partial A^n}{\partial t} + \frac{A^n}{\Delta x} \right\} + \frac{(\Delta t)^2}{2\Delta x^2} \left\{ \frac{\partial^2 A^n}{\partial x^2} \right\} + \frac{\Delta t}{2} \left\{ \frac{\partial A^n}{\partial x} + \frac{A^n}{\Delta x} \right\} + \frac{(\Delta t)^2}{2} \left\{ \frac{\partial^2 A^n}{\partial x^2} \right\} + \cdots
\]

and

\[
G^{n+1} = G^n + \Delta t \cdot \left\{ \frac{\partial G^n}{\partial t} + \frac{G^n}{\Delta x} \right\} + \frac{(\Delta t)^2}{2\Delta x^2} \left\{ \frac{\partial^2 G^n}{\partial x^2} \right\} + \frac{\Delta t}{2} \left\{ \frac{\partial G^n}{\partial x} + \frac{G^n}{\Delta x} \right\} + \frac{(\Delta t)^2}{2} \left\{ \frac{\partial^2 G^n}{\partial x^2} \right\} + \cdots
\]

Proof. The proof of this Lemma is given in Appendix A.

Using Lemma 5.1 we are ready to analyze the stability restriction of the Lax-Wendroff scheme.

**Stability Analysis**

This section deals with the stability analysis of the Lax-Wendroff
numerical scheme for 1-D complete shallow water equations with source terms in the case where the channel is prismatic. First, we present a rainfall hydrograph test, based on experimental measurements realized thanks to the SATREPS project METHOD in a flume at the rain simulation facility at Benoué-Garoua (Cameroon). The flume is 1150m long with a slope of 4%. The simulation duration is 40s. The rainfall intensity $I(x,t)$ is described by

$$I(x,t) = \begin{cases} 
1.18 \times 10^{-4} \text{m/s} & \text{if } (x,t) \in [0, 0.1] \times [200, 240], \\
0 & \text{otherwise},
\end{cases}$$

(32)

For this test, as there is no rain on the last 150m, we have a wet/dry transition. The measured output is an outputgraph, that is a plot of the discharge versus time. The mathematical model for this ideal overland flow is the following: we consider a uniform plane catchment whose overall length in the direction of flow is $L$. The surface roughness and slope are assumed to be invariant in space and time. We consider a constant rainfall excess such that

$$r(x,t) = \begin{cases} 
1 & \text{if } t \leq t_s \leq t_f, \\
0 & \text{otherwise},
\end{cases}$$

(33)

where $I$ is the rainfall intensity and $t_f$ is the final time of the rainfall excess. According to relations (32) and (33) we assume in the following that $r$ is more less that $I$ and $Q$, i.e., $r \leq A$. Furthermore, Lemma 4.1 gives the "temporary" stability limitation of the Lax-Wendroff algorithm described in section 3.

**Lemma 6.1.** The numerical scheme (30) is stable if estimate (34) holds.

$$\left\| \frac{\Delta t}{\Delta x} \right\| \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq 1,$$

(34)

with the restriction: $\left| k \Delta t \right| \leq \frac{1}{2}$. Here, $\nu = I$, $\nu = Q$ and $\mu = Q/A$.

Proof. Regarding the proof of this result we refer the readers in Appendix B.

In way similar, the following result gives the stability restriction of the numerical scheme (31).

**Lemma 6.2.** The numerical scheme (31) is stable if estimate (35) holds.

$$\left\| \frac{\Delta t}{\Delta x} \right\| \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq 1,$$

(35)

with the requirement: $\left| k \Delta t \right| \leq \frac{1}{2}$. Here, $\nu = I$, $\nu = Q$ and $\mu = Q/A$.

$$\frac{N_\nu}{\Delta x} = 1 + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} + \left| \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \right| \Delta x,$$

(36)

Obviously, $N_\nu > 1 + \sqrt{\frac{\Delta t}{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} > 1$.

**Proof.** The detail of the proof is given in Appendix B.

Now, using the above results (namely, Lemmas 4.1 and 4.2) we are ready to give the stability requirement of the Lax-Wendroff scheme (30)-(31) and to compare it with what is available in the literature (for example, Courant-Friedrichs-Lewy condition for linear hyperbolic partial differential equations).

**Theorem 6.1.** The Lax-Wendroff scheme for 1-D complete shallow water equations with source terms (16) is stable if

$$\left| \frac{\Delta t}{\Delta x} \right| \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq 1,$$

(37)

with the requirement: $\left| k \Delta t \right| \leq \frac{1}{2}$. In relation (37): $\nu = Q$, $\mu = Q/A$ and $N_\nu$ is given by relation (36).

**Proof.** The proof follows from both estimates (34) and (35). That is,

$$\left| \frac{\Delta t}{\Delta x} \right| \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq 1,$$

(38)

with the requirement: $\left| k \Delta t \right| \leq \frac{1}{2}$. System of estimates (38) is equivalent to relation

$$\left| \frac{\Delta t}{\Delta x} \right| \max \left( 1 + \frac{1}{2} \left| \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \right| \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \right) \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq 1.$$

In addition, it is obvious to see that estimate (39) holds

$$\left( 1 + \frac{1}{2} \left| \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \right| \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \right) \left| \mu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \leq N_\nu.$$

(39)

**Some important remarks on stability analysis.**

In the subsequent paragraphs we give some useful remarks on the stability restrictions obtained in this note and we compare it with what is known in the literature, for example, the Courant-Friedrichs-Lewy condition.

The stability restriction (37) shows that a small space step $\Delta x$ forces the time step $\Delta t$ to be more potentially small. This makes the Lax-Wendroff scheme extremely slow. For example, let us consider a spatial domain $[0, 1]$ with space step $\Delta x=5.10^{-2}$ Then, the required time step $(\Delta t)_{req}$ must be less than the maximum solution (in modulus) of equation: $20\Delta t \left| \nu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} = 1$. More especially, since $N_\nu > 1 + \sqrt{\frac{\Delta t}{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)}$, the required time step $(\Delta t)_{req}$ must be less or equal than $\left[ 20 \left| \nu \right| + \frac{\sqrt{\Delta t}}{\sqrt{\rho}} \left| \nu \right| e^{\left(\frac{\Delta t}{\nu}\right)} \right]$. The Lax-Wendroff scheme (30)-(31) for 1-D complete surface water equations has stability restrictions (34)-(35) that limit the maximum time step. The stability requirement given by estimate (37)
does not coincide with the famous Courant-Friedrichs-Lewy (CFL) condition obtained for simple hyperbolic partial differential equations (for example: linear advection equation, wave equation, Burgers equations, etc...) because the Lax-Wendroff scheme is applied to a more 1-D complex unsteady partial differential equations. As discussion on the stability restrictions one can refer to the stability analysis of the two step Lax-Wendroff method and the MacCormack scheme applied to complete burgers equations [26]. However, it is easy to show that the greatest eigenvalue (in modulus) $\frac{\Delta x}{\sqrt{1+4\Delta t}}$ of the Jacobian matrix $J$ of conservative system (18) is bounded by the positive quantities

$$\frac{|\Delta t|}{\sqrt{T}}$$

and $|\Delta t+\frac{\sqrt{T}}{1+4\Delta t}|^{\gamma}$. Thus it is obvious that inequality (40) holds

$$\frac{\Delta t}{\sqrt{T}} \lambda_{\| \mathbf{e} \|} \leq \frac{\Delta t}{\sqrt{T}} \lambda_{\| \mathbf{e} \|} \cdot N_{\lambda}^\ast,$$  (40)

which means that the stability limitation given by (37) controls the CFL condition, and so it is more restrictive. The stability restriction (37) is highly unusual. Since we normally find condition (37) from a Fourier stability analysis, it follows from estimate (40) that an instability occurs when $|\Delta t|$ is greater than some $|\Delta t|$ which can be viewed as $|\Delta t}_{\gamma}$. As observed in proving Lemmas 4.1 and 4.2, it was not easy to obtain the stability criterion for the Lax-Wendroff scheme applied to 1-D complete Saint-Venant equations (16). However, it follows from conditions given by relations (36) and (37) that the empirical formula

$$|\Delta | \leq \frac{|\Delta |}{\sqrt{T}}$$

(41)
can be used with an appropriate safety factor. The latter formula (41) reduces to the usual inviscid condition $|\Delta t |/\sqrt{T}$ which is the case where the right-hand side of equation (16) is assumed equals zero) when $N_{\lambda}^\ast$ is set equal to 1. It should be remembered that the "heuristic" stability analysis, i.e., equation (37), can only provide a necessary condition for stability. Thus, for some finite difference algorithms, only partial information about the complete stability bound is obtained and for others (such as algorithms for the heat equation) a more complete theory must be employed.

- Once the stability is assumed the Lax-Wendroff scheme is both convergent and an explicit one step two time level method.
- Relation (37) illustrates the effect that the choice in both space step and time step have on the stability of the Lax-Wendroff scheme.

**Numerical Evidences**

In this section we simulate the Lax-Wendroff scheme described in section 4 for 1-D complete shallow water equations with source terms. We focus on a practical application of a shallow water flow based on the Benoué river. This river is a 7000m long reach of the upstream part (altitude=174.22 m) and it is located in Cameroon. Being a mountain river, it is characterized by strong irregularities in the cross section, by a rather steep part in the first kilometers and by a low base discharge (708m3/s) which, altogether, produce a high velocity basic flow, transcritical in some parts. More specifically, we consider the problem of floods observed in this river in 2012 because it is a classical example of time dependent nonlinear flow with shocks to expect floods and to test conservation in numerical schemes. Furthermore, we assume that this model is generated by the 1-D complete shallow water equations with source terms for the ideal case of a flat and frictionless channel with prismatic cross section, i.e., constants top width ($T=348m$) and wetted perimeter. ($P=366.4m$) Using the initial data provided by the river: $Q(0,t_1)=6290.6$, $Q(0,t_2=200)=708$, $A(0,t_1)=2364$ and $A(0,t_2)=635.8$ straightforward computations show that the initial conditions are defined as follows

$$A(x,t)=\begin{cases} \frac{708\exp(2x), \quad 0 \leq x \leq 1/2} {708\exp(2(2-x)), \quad 2 \leq x \leq 3/2} \end{cases}, \quad A(x,t_2)=\begin{cases} \frac{708\exp(2x), \quad 0 \leq x \leq 1/2} {635.8\exp(2(2-x)), \quad 1/2 \leq x \leq L} \end{cases}$$  (42)

where $t_1$ is the initial time ($t_1=200s$), $A(x,t)$ is the area of cross section and $Q(x,t)$ is the discharge.

The calculation times used are so as to avoid the interaction with the boundaries of the channel. So the boundary conditions are given by

$$g_{i}(t)=Q(L,t)=Q(x_0=0,t)=e^{\frac{-2.912}{t_i}} \exp(i\pi x)$$  (43)

and

$$h_{i}(t)=A(L,t)=A(x_0=0,t)=e^{\frac{-0.0328}{t_i}} \exp(i\pi x).$$  (44)

Indeed, the study is done in the channel on 4 October 2012 and whose the purpose is to expect floods in the next years. Although the problem is defined by a system of shallow water equations with source terms, it is considered as a system of hyperbolic partial differential equations and can serve as a standard test case for validation of schemes whenever an analytical solution is known. Starting from initial and boundary conditions given by still water, the theory of characteristics can supply an exact evolution solution [27] that can be used as a reference. In the example presented, when using the initial and boundary conditions given by relations (42), (43), and (44), simple calculations yield the values of parameters $a_i, b_i, c_i, t_i, L, K_i$, defined in section 4, i.e.,

$$a_i = \frac{\ln(635.8)}{t_i} \cong 0.0328, \quad \beta_i \cong \pi = 3.1416, \quad b_i = \frac{\ln(708)}{t_i} \cong 0.0328, \quad k_i = \pi = 3.1416$$

and

$$t_i = \frac{\ln(2690.6) - \ln(2364)}{\ln(708) - \ln(635.8)} \cong 240 \text{ s},$$

where $t_2=200s$ is the initial time, $t_f$ is the final time, $K_i$ is the wave number, and $T_i=t_i=\frac{200}{240}=0.833$ is the time interval length. Using the definition of $Q(L,t_i)$ together with the boundary conditions we have $Q(L,t_i)=\exp(i\beta_i + i\pi x)\exp(i\pi x)$, so we can take $L=1000m$ where $L$ is the rod interval length for the ideal case of a flat and frictionless channel with prismatic cross section. In addition, the averaged shear stress is assumed equals zero, i.e., $r=0$, the Manning’s number ($\beta_i$) equals $0.025/s/m^{1/3}$, and the rain density $I(x,t)$ is described according to relation (32), i.e.,

$$r(x,t) = \begin{cases} I(x,t), & \text{if } (x,t) \in [0,L] \times [t_0,t_f]; \\ 0, & \text{otherwise,} \end{cases}$$

where $I$ is the rainfall intensity defined by relation (45), $t_0 \sim 200s$ and $t_f \sim 240s$ are initial and final time, respectively, of the rainfall excess computed above, and $L=1000m$ is the rod interval length. The approximate solutions given by numerical schemes (30) and (31) obtained from 20 to 10850 iterations, respectively, are displayed in Figures 1 and 2. Using the experimental values of parameters $a_i, b_i, c_i, P, T, K_i$ and $g_i$ computed above one easily shows, according to relation (36) that $N_{\lambda}^\ast < 10.5$ for all values of $\Delta t$ and $\Delta x$ satisfying

$$\Delta t \leq \frac{1}{7} \text{ and } |\Delta t+\frac{\sqrt{T}}{1+4\Delta t}|^{\gamma} \leq 7.7,$$  for all $t_i \in [200,240]$. Different values of $k=\Delta t=8.2 \times 10^{-5}$, $k=\Delta t=8.2 \times 10^{-4}$ and $k=\Delta t=8.2 \times 10^{-3}$ numbers obtained from (37) as the steady flow cases and both space steps of $\Delta x=5m, \Delta x=2\times10^3m$ and $\Delta x=10^4m$ in the mesh are used. Before 200
iterations are encountered, the discharge wave propagates with almost a perfectly constant value at different times (Figures 1 and 2). Further, after 200 iterations are encountered, the discharge wave also destroys at different times (Figures 1 and 2). So, the graphs show that the solution of the difference equations may grow with time (for example, Figure 1 (test 5) and Figure 1 (test 3)) and still satisfy the Von Neumann necessary condition. On the other hand, we obtain similar observations for the cross section. Furthermore, the figures indicate that the cross section starts to destroy after a fixed time and can become negative. Moreover, combining the different values of $\Delta x$ and $\Delta t$ we observe from the figures that the cross section also can become negative if the ratio $\Delta t / \Delta x$ is less than $1.64 \times 10^{-3}$. Thus, it is not hard to see that good solutions are obtained for a small time step $\Delta t$ and a mesh size $\Delta x$ satisfying the stability limitation (37) along with the estimate $\frac{\Delta t}{\Delta x} \geq 4.1 \times 10^{-2}$.

Physical insight must be used when the stability limitation (37) of the Lax-Wendroff method is investigated. Finally, the figures show that the solutions do not increase exponentially with time. More specifically, they indicate that stability for the Lax-Wendroff scheme is subtle. It is not unconditionally unstable, but stability depends on the parameters $\Delta x$ and $\Delta t$ as shown Figures 1 and 2. We conclude that the numerical examples indicate the crucial role played by the ratio $\Delta t / \Delta x$.

Similarly, the MacCormack method which is a predictor-corrector version of the Lax-Wendroff scheme provides a reasonably good result at discontinuities. This method is much easier to apply than the Lax-Wendroff scheme because the Jacobian does not appear. The amplification factor and stability requirement almost are the same as presented for the Lax-Wendroff method [26] case of inviscid burgers equation. It is important to note that the solutions obtained for the same problem at the same Courant number are different from those.
obtained using the Lax-Wendroff scheme. This is due both to the switched differencing in the predictor and the corrector and the nonlinear nature of the governing PDE. One should expect results that show some differences, even though both methods are equivalent for linear problems. In addition, it should be noted that reversing the differencing in the predictor and corrector steps leads to quite different results. The best resolution of discontinuities occurs when the difference in the predictor is in the direction of propagation of the discontinuity [26].

General conclusion and future works

In this paper, we have presented a mathematical model of 1-D complete shallow water equations with source terms and we have described the Lax-Wendroff scheme for these hyperbolic partial differential equations in the case of a prismatic channel. The stability analysis of the method is also considered and deeply studied together with some numerical experiments. From this analysis it follows that while the stability limitation is highly unusual, the result has a potential positive implication since the stability requirement presented in this work controls the famous Courant-Friedrich-Lewy condition which is well known in the literature. In the future, the following problems will be subject of our investigations.

1. Stability analysis and second order accuracy of the Lax-Wendroff scheme for 1-D complete shallow water problems in an open channel;
2. Stability analysis of two steps explicit MacCormack scheme for 1-D complete Saint-Venant equations with source terms in the case of a prismatic channel;
3. Analysis of stability and second order accuracy of two steps explicit MacCormack method for 1-D complete shallow water problems with source terms in the case of an open channel.

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References