

Matrix Bosonic realizations of a Lie colour algebra with three generators and five relations of Heisenberg Lie type

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Abstract

We describe realizations of a Lie colour algebra with three generators and five relations by matrices of power series in noncommuting indeterminates satisfying Heisenberg's canonical commutation relation of quantum mechanics. The obtained formulas are used to construct new operator representations of this Lie colour algebra using canonical representation of the Heisenberg commutation relation and creation and annihilation operators of the quantum mechanical harmonic oscillator.

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1 Introduction

The main object studied in this paper is the unital associative algebra with three generators A_1 , A_2 , and A_3 satisfying defining commutation relations

$$A_1 A_2 + A_2 A_1 = A_3 \tag{1.1}$$

$$A_1 A_3 + A_3 A_1 = 0 \tag{1.2}$$

$$A_2 A_3 - A_3 A_2 = 0 \tag{1.3}$$

$$A_2^2 = 0 \tag{1.4}$$

$$A_3^2 = 0 \tag{1.5}$$

The main goal is to show how A_1 , A_2 , and A_3 can be expressed, using elements A and B , obeying Heisenberg's canonical commutation relation

$$AB - BA = I \tag{1.6}$$

The canonical representation of the commutation relation (1.6) is given by choosing A as the usual differentiation operator and B as multiplication by x acting on differentiable functions of one real variable x , on polynomials in one variable, or on some other suitable linear space of functions invariant under these operators. In quantum mechanics, these operators, when considered on the Hilbert space of square integrable functions, are essentially the same as the canonical Heisenberg-Schrödinger observables of momentum and coordinate, differing just by a complex scaling factor. The Heisenberg canonical commutation relation (1.6) is also satisfied by the annihilation and creation operators in a quantum harmonic oscillator.

Since the 1970s, generalized (colour) Lie algebras have been an object of constant interest in both mathematics and physics [1, 3, 4, 5, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21, 25, 26, 27, 23]. Description of representations of these algebras is an important and interesting general problem. It is well known that representations of three-dimensional Lie algebras play an important role in the representation theory of general Lie algebras and groups, both as test examples and building blocks. Similarly, one would expect the same to be true for three-dimensional Lie colour algebras and superalgebras with respect to general Lie colour algebras and superalgebras. The representations of nonisomorphic algebras have different structure. In [26, 27], three-dimensional Lie colour algebras are classified in terms of their structure constants, that is, in terms of commutation relations between generators. In [11, 17, 23], quadratic central elements and involutions on these algebras are calculated. In [16, 25], Hilbert space $*$ -representations are described for the graded analogues of the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$ and of the Lie algebra of the group of plane motions, two of the nontrivial algebras from the classification. The classification of $*$ -representations in [16, 25] is achieved, using the method of dynamical systems based on generalized Mackey imprimitivity systems.

The colour Heisenberg Lie algebra is another important nontrivial algebra in the classification of three-dimensional Lie colour algebras obtained in [26, 27]. In the paper [24] we approached representations of this algebra in a totally different way than it was done in [16, 25]. Namely, we studied those representations which can be obtained as power series in operator representations of Heisenberg's canonical commutation relations by first obtaining in general realizations of the colour Heisenberg Lie algebra generators in terms of power series in elements of an associative algebra obeying the Heisenberg's canonical commutation relations and then combining these realizations with canonical representations of Heisenberg's canonical commutation relations.

In this paper we extend these investigations of realizations via Heisenberg's canonical commutation relations to another colour Lie algebra with three generators and five relations. This algebra can be considered as another colour analogue of the Heisenberg Lie algebra. However, we show that a structure of this algebra is quite different from that for the algebra considered in [24] as far as realization via Heisenberg's canonical commutation relations is concerned. In Section 2 we show that, with a natural choice for A_1 as the first generator of the Heisenberg algebra corresponding to differentiation, there are no nonzero power series in Heisenberg generators which can be taken as A_2 and A_3 so that the three relations (1.1)–(1.3) are satisfied. In Lemma 2.2, we describe all such formal power series solutions A_2 and A_3 for the first two relations (1.1)–(1.2) as in [24]. In Theorem 2.4, we present all such formal power series solutions for A_2 and A_3 satisfying the three relations (1.1)–(1.3) showing that $A_3 = 0$ is the only possibility. Using this result we get in Corollary 2.7 that $A_2 = A_3 = 0$ must hold for such realizations of the five relations (1.1)–(1.5). However, by considering 2×2 matrices with entries chosen as formal power series in the noncommuting indeterminates A and B satisfying Heisenberg's canonical commutation relations, we demonstrate how it is possible to construct nontrivial realizations of (1.1)–(1.5). We also construct concrete operator representations by applying this construction to the canonical representation of Heisenberg canonical commutation relations and to the simple quantum mechanical harmonic oscillator.

2 Matrix power series realizations

Throughout this article \mathbb{C} denotes the field of complex numbers and \mathbb{N} the set of nonnegative integers. By $\mathbb{C}[x]$ and $\mathbb{C}[[x]]$ we mean the ring of polynomials and formal power series over \mathbb{C} , respectively.

Consider a set $\{A_1, A_2, A_3\}$ in some associative algebra over \mathbb{C} with unit element I satisfying commutation relations (1.1)–(1.5). From relation (1.1) we obtain $A_1A_2A_3 + A_2A_1A_3 = A_3^2$, and hence by (1.2) it follows that

$$A_1(A_2A_3) - (A_2A_3)A_1 = A_3^2 \quad (2.1)$$

Using only (1.2) and (1.3) we may conclude that A_3^2 commutes with both A_1 and A_2 , that is, $[A_1, A_3^2] = [A_2, A_3^2] = 0$. If merely (1.1) and (1.2) are satisfied and if $A_3^2 = \alpha^2 I$ for some nonzero $\alpha \in \mathbb{C}$, then by (2.1) we have

$$A_1(A_2A_3) - (A_2A_3)A_1 = \alpha^2 I \quad (2.2)$$

Applying the famous Wintner-Wielandt theorem [18, 28, 29], we have that no elements in any unital normed algebra can satisfy the Heisenberg canonical commutation relation

$$AB - BA = \lambda I, \quad \lambda \neq 0$$

So, we obtain from (2.2) the following result.

Proposition 2.1. *The commutation relations*

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

together with $A_3^2 = \alpha^2 I$, $\alpha \neq 0$, cannot be satisfied by bounded operators on a Hilbert space or even generally by elements in any unital normed algebra.

When computing with power series in noncommuting elements A and B of an associative unital algebra \mathfrak{A} , we use the usual addition and multiplication rules of the Magnus algebra of noncommutative formal power series in two indeterminates (see [2]). However, we assume that A and B are not free, but satisfy at least the Heisenberg commutation relation as elements in \mathfrak{A} . We denote the algebra that we are working with by $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$. In addition to the subalgebra of \mathfrak{A} generated by A and B , consisting of noncommutative polynomials in A and B , the algebra $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ may contain other elements which are infinite noncommutative power series in A and B not belonging to \mathfrak{A} . The problem of equality of two elements in $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ is a very complex matter in itself, deeply connected both to the properties of noncommutative power series and Heisenberg's relation and to the structure of the algebra \mathfrak{A} , and properties of A and B in \mathfrak{A} . We say that an element of $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ is in the (B, A) -normal form (resp., (A, B) -normal form) if it is a noncommutative power series built of only ordered monomials $\{B^j A^k \mid j, k \in \mathbb{N}\}$ (resp., $\{A^j B^k \mid j, k \in \mathbb{N}\}$). In order to be able to enjoy the equality properties in a similar way with formal power series as in the polynomial case, we assume throughout this article that two formal power series in A and B , written in the (B, A) -normal form (resp., in the (A, B) -normal form), are equal if and only if their coefficients are the same and in particular such a series is zero if and only if all coefficients are zero. This important equality assumption is actually an assumption on $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$, on the algebra \mathfrak{A} as well as on A and B as elements in \mathfrak{A} . In the particular case of polynomials in A and B , that is, for the subalgebra of \mathfrak{A} generated by A and B , the assumption yields the same property as in $\mathcal{H}_1(A, B)$, namely that $\{B^j A^k \mid j, k \in \mathbb{N}\}$ (resp., $\{A^j B^k \mid j, k \in \mathbb{N}\}$) are linearly independent as subsets of \mathfrak{A} .

With the assumption above we may claim the equality of two elements of $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ if they are equal to the same element in (B, A) -normal form (resp., in (A, B) -normal form). However, it is important to observe that $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ may well contain elements which

cannot be represented on (B, A) -normal form, or (A, B) -normal form or even on either of them. In most of the statements in this article we will adhere to the (B, A) -normal forms and the corresponding equality assumptions. But, we will also comment and at some instances will formulate the corresponding results when instead the (A, B) -normal forms and corresponding equality of series is used. Which of these assumptions is used will be clear from the context. We refer to [6, 7] for further discussion on power series extensions of the Heisenberg algebra, Diamond lemma, and normal forms.

In the paper [24] we have proved the following useful result.

Lemma 2.2. *Let $A_1 = A$ and assume that A_2 and A_3 are elements of the algebra $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ written in the (B, A) -normal form, that is,*

$$A_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} B^j A^k, \quad A_3 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{jk} B^j A^k, \quad a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$$

Then A_1 , A_2 , and A_3 satisfy the commutation relations

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0$$

if and only if

$$A_2 = T(B, A)V(A) + BT(B, A)W(A), \quad A_3 = T(B, A)W(A)$$

where $T(B, A) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k$ and $V(A), W(A) \in \mathbb{C}[[A]]$.

Remark 2.3. It follows by this lemma that the commutation relations

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0$$

can be satisfied by polynomial A_2 and A_3 (finite sums) only if $A_2 = A_3 = 0$.

Theorem 2.4. *Suppose that A_1 , A_2 , and A_3 are elements of the algebra $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ such that $A_1 = A$ and A_2 , A_3 are formal power series in the (B, A) -normal form given as*

$$A_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} B^j A^k, \quad A_3 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{jk} B^j A^k, \quad a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$$

Then A_1 , A_2 , and A_3 will satisfy the commutation relations

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0, \quad A_2 A_3 - A_3 A_2 = 0$$

if and only if

$$A_2 = T(B, A)V(A), \quad A_3 = 0$$

where $T(B, A) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k$ and $V(A) \in \mathbb{C}[[A]]$.

Proof. By Lemma 2.2 we have, considering only the two anticommutation relations, a general solution of the form

$$\begin{aligned} A_2 &= T(B, A)V(A) + BT(B, A)W(A) \\ A_3 &= T(B, A)W(A) \end{aligned}$$

Applying the rules of Lemma 4 in [24] yields

$$\begin{aligned} A_2 A_3 &= V(-A)W(A) + BW(-A)W(A) \\ A_3 A_2 &= W(-A)V(A) - BW(-A)W(A) + W'(-A)W(A) \end{aligned}$$

and hence

$$\begin{aligned} A_2 A_3 - A_3 A_2 &= V(-A)W(A) - W(-A)V(A) - W'(-A)W(A) \\ &\quad + 2BW(-A)W(A) = 0 \end{aligned}$$

which is a functional-differential equation for V and W . Thus $A_2 A_3 - A_3 A_2 = 0$ implies $2BW(-A)W(A) = 0$, which for a formal power series $W(A)$ yields $W(-A)W(A) = 0$. The set of complex formal power series $\mathbb{C}[[x]]$ is an integral domain and hence $W(A) = 0$. This means that $A_2 = T(B, A)V(A)$ and $A_3 = 0$. \square

Remark 2.5. Note that in Theorem 2.4, $A_1 A_2 + A_2 A_1 = A_3 = 0$ is the only nontrivial relation left. The other two relations are trivially satisfied when $A_3 = 0$, independently of A_1 and A_2 . So, under conditions of Theorem 2.4, one can say that

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0, \quad A_2 A_3 - A_3 A_2 = 0$$

is equivalent to

$$A_1 A_2 + A_2 A_1 = 0, \quad A_3 = 0$$

Remark 2.6. In Lemma 2.2, the formal series A_2 and A_3 are expressed in the (B, A) -normal form. This is a natural ordering when we think of A as the usual differentiation operator ∂ and B as a multiplication operator M acting on differentiable functions on the real line, given by $\partial f = f'$ and $Mf(t) = tf(t)$. In other situations, it may be more appropriate to consider the reversed order.

Taking A_2 and A_3 in Lemma 2.2 to be in the (A, B) -normal form, but keeping $A_1 = A$, the general solution will be changed to the following form:

$$A_2 = V(A)U(A, B) - W(A)U(A, B)B \quad A_3 = W(A)U(A, B)$$

where

$$U(A, B) = \sum_{k=0}^{\infty} \frac{2^k}{k!} A^k B^k$$

If we consider a solution in the (A, B) -normal form satisfying the three relations in Theorem 2.4, a completely similar proof shows that also in this case $A_2 = V(A)U(A, B)$ and $A_3 = 0$.

Corollary 2.7. Suppose that A_1 , A_2 , and A_3 are elements of $\mathcal{H}_1 \langle\langle A, B; \mathfrak{A} \rangle\rangle$ such that $A_1 = A$, and A_2 , A_3 are formal power series in the (B, A) -normal form given as

$$A_2 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} a_{jk} B^j A^k, \quad A_3 = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \tilde{a}_{jk} B^j A^k, \quad a_{jk}, \tilde{a}_{jk} \in \mathbb{C}$$

Then A_1 , A_2 , and A_3 satisfy

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0, \quad A_2 A_3 - A_3 A_2 = 0, \quad A_2^2 = A_3^2 = 0$$

if and only if $A_2 = A_3 = 0$.

Proof. By Theorem 2.4, we know that $A_2 = T(B, A)V(A)$ and $A_3 = 0$. Since

$$A_2^2 = T(B, A)V(A)T(B, A)V(A) = T(B, A)^2V(-A)V(A) = V(-A)V(A)$$

the relation $A_2^2 = 0$ holds only if $V(-A)V(A) = 0$, which, for a power series $V(A)$, yields $V(A) = 0$ and $A_2 = 0$. \square

Now let A_1 , A_2 , and A_3 be 2×2 matrices on the form

$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$$

where A_{11} , A_{12} , A_{21} , A_{22} , L , and M are elements of some associative algebra or ring \mathcal{A} . Then

$$\begin{aligned} A_1A_2 + A_2A_1 &= \begin{pmatrix} 0 & A_{11}L \\ 0 & A_{21}L \end{pmatrix} + \begin{pmatrix} LA_{21} & LA_{22} \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} LA_{21} & A_{11}L + LA_{22} \\ 0 & A_{21}L \end{pmatrix} \\ A_1A_3 + A_3A_1 &= \begin{pmatrix} MA_{21} & A_{11}M + MA_{22} \\ 0 & A_{21}M \end{pmatrix} \end{aligned}$$

Therefore, $A_1A_2 + A_2A_1 = A_3$ is equivalent to the conditions

$$LA_{21} = A_{21}L = 0, \quad A_{11}L + LA_{22} = M \quad (2.3)$$

and similarly we have $A_1A_3 + A_3A_1 = 0$ if and only if

$$MA_{21} = A_{21}M = 0, \quad A_{11}M + MA_{22} = 0 \quad (2.4)$$

Remark 2.8. Observe that for any elements L and M in \mathcal{A} the relations

$$A_2A_3 - A_3A_2 = 0, \quad A_2^2 = 0, \quad A_3^2 = 0$$

are always satisfied, since moreover we have $A_2A_3 = A_3A_2 = 0$. Thus, under the conditions (2.3) and (2.4), elements A_1 , A_2 , and A_3 satisfy commutation relations (1.1)–(1.5). This gives a method of construction for realizations of (1.1)–(1.5).

Let us assume that either L or M are left or right invertible. Then $A_{21} = 0$, and hence A_1 is upper triangular, that is,

$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$

So, we have the following useful statement.

Lemma 2.9. *Let L and M be elements of an associative algebra or ring \mathcal{A} , such that at least one of them is left or right invertible in \mathcal{A} . Then the elements*

$$A_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & L \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \quad (2.5)$$

of the algebra (ring) $\mathcal{M}_2(\mathcal{A})$ of 2×2 matrices over \mathcal{A} satisfy the commutation relations

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0 \quad (2.6)$$

if and only if

$$A_{21} = 0, \quad \text{i.e.,} \quad A_1 = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \quad (2.7)$$

together with

$$A_{11}L + LA_{22} = M, \quad A_{11}M + MA_{22} = 0 \quad (2.8)$$

Given A_1 and A_2 of the form in (2.7) and (2.5), respectively, there exists an M so that A_3 of the form in (2.5) satisfies (2.6) if and only if

$$A_{11}^2L + 2A_{11}LA_{22} + LA_{22}^2 = 0 \quad (2.9)$$

Proof. The first part was proved before the statement. The last relation follows by eliminating M using (2.8). On the other hand, if (2.9) holds, then $M = A_{11}L + LA_{22}$ satisfies (2.8). \square

Lemma 2.10. *Let the matrices*

$$A_1 = \begin{pmatrix} B_1 & A_{12} \\ 0 & B_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & B_2 \\ 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & B_3 \\ 0 & 0 \end{pmatrix}$$

be elements of $\mathcal{M}_2(\mathcal{A})$, the algebra (ring) of 2×2 matrices over an associative algebra (ring) \mathcal{A} , and assume $B_1, B_2, B_3, A_{12} \in \mathcal{A}$. Then

$$A_1A_2 + A_2A_1 = A_3, \quad A_1A_3 + A_3A_1 = 0$$

are satisfied if and only if B_1, B_2 , and B_3 satisfy

$$B_1B_2 + B_2B_1 = B_3, \quad B_1B_3 + B_3B_1 = 0$$

Moreover, $A_2^2 = A_3^2 = 0$ and $A_2A_3 = A_3A_2 = A_2A_3 - vA_3A_2 = 0$, independently on $B_2, B_3 \in \mathcal{A}$.

Proof. The proof follows from the following easily checked equalities:

$$A_1A_2 + A_2A_1 = \begin{pmatrix} 0 & B_1B_2 + B_2B_1 \\ 0 & 0 \end{pmatrix}, \quad A_1A_3 + A_3A_1 = \begin{pmatrix} 0 & B_1B_3 + B_3B_1 \\ 0 & 0 \end{pmatrix} \quad \square$$

By Lemma 2.2 we know that if $AB - BA = I$ for A and B in some unital associative algebra \mathfrak{A} , then

$$B_1 = A, \quad B_2 = T(B, A)V(A) + BT(B, A)W(A), \quad B_3 = T(B, A)W(A) \quad (2.10)$$

satisfy commutation relations

$$B_1B_2 + B_2B_1 = B_3, \quad B_1B_3 + B_3B_1 = 0 \quad (2.11)$$

and also if $B_1 = A$ and B_2 and B_3 are (B, A) -normally ordered power series in A and B , then B_1, B_2 , and B_3 satisfy (2.11) if and only if B_2 and B_3 are of the form (2.10). A similar result is obtained if we choose to write B_2 and B_3 in the (A, B) -normal form.

Note also that we can apply Lemma 2.2 and its version in the (A, B) -normal form on a new set of generators $\tilde{A} = B$ and $\tilde{B} = -A$ satisfying $\tilde{A}\tilde{B} - \tilde{B}\tilde{A} = I$. Observing that $T(-A, B) = U(A, B)$ and $U(B, -A) = T(B, A)$, the resulting general solutions are

$$\begin{aligned} B_1 &= B, & B_2 &= U(A, B)V(B) - AU(A, B)W(B), & B_3 &= U(A, B)W(B) \\ B_1 &= B, & B_2 &= V(B)T(B, A) + W(B)T(B, A)A, & B_3 &= W(B)T(B, A) \end{aligned}$$

Combining this with Lemma 2.10, we can construct a realization of (1.1)–(1.5) using A and B satisfying the Heisenberg commutation relation $AB - BA = I$.

Theorem 2.11. Assume A and B are two elements of an associative unital algebra or a ring (with identity) \mathfrak{A} , satisfying the Heisenberg commutation relation $AB - BA = I$. Let $T(B, A) = \sum_{k=0}^{\infty} \frac{(-2)^k}{k!} B^k A^k$ and define the following 2×2 matrices:

$$\begin{aligned} A_1 &= \begin{pmatrix} A & A_{12} \\ 0 & A \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & T(B, A)V(A) + BT(B, A)W(A) \\ 0 & 0 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 0 & T(B, A)W(A) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where $A_{12} \in \mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ and $V(A), W(A) \in \mathbb{C}[[A]]$. Then A_1 , A_2 , and A_3 satisfy the commutation relations

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0, \quad A_2 A_3 - A_3 A_2 = 0, \quad A_2^2 = A_3^2 = 0$$

Moreover, $A_2 A_3 = A_3 A_2 = 0$.

When the (A, B) -normal forms and the corresponding equality assumption in $\mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ is used, the following analogue of Theorem 2.11 holds.

Theorem 2.12. Assume A and B are two elements of an associative unital algebra or a ring (with identity) \mathfrak{A} , satisfying the Heisenberg commutation relation $AB - BA = I$. Let $U(A, B) = \sum_{k=0}^{\infty} \frac{2^k}{k!} A^k B^k$, and define the following 2×2 matrices:

$$\begin{aligned} A_1 &= \begin{pmatrix} A & A_{12} \\ 0 & A \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & V(A)U(A, B) - W(A)U(A, B)B \\ 0 & 0 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 0 & W(A)U(A, B) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

where $A_{12} \in \mathcal{H}_1\langle\langle A, B; \mathfrak{A} \rangle\rangle$ and $V(A), W(A) \in \mathbb{C}[[A]]$. Then A_1 , A_2 , and A_3 satisfy the commutation relations

$$A_1 A_2 + A_2 A_1 = A_3, \quad A_1 A_3 + A_3 A_1 = 0, \quad A_2 A_3 - A_3 A_2 = 0, \quad A_2^2 = A_3^2 = 0$$

Moreover, $A_2 A_3 = A_3 A_2 = 0$.

Example 2.13. Introduce the operators

$$A = M : f(x) \mapsto x f(x), \quad B = -\partial : f(x) \mapsto -f'(x)$$

acting on $\mathbb{C}[x]$. These operators satisfy $AB - BA = I$. Note also that $U(A, B) = U(M, -\partial) = T(M, \partial)$. Let $A_{12} : \mathbb{C}[x] \rightarrow \mathbb{C}[x]$ be any operator. Then A_1 , A_2 , and A_3 defined in Theorem 2.12 are realized on $\mathbb{C}[x]$ by the operators

$$\begin{aligned} C_1 &= \begin{pmatrix} M & C_{12} \\ 0 & M \end{pmatrix} \\ C_2 &= \begin{pmatrix} 0 & V(M)T(M, \partial) + W(M)T(M, \partial)\partial \\ 0 & 0 \end{pmatrix} \\ C_3 &= \begin{pmatrix} 0 & W(M)T(M, \partial) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

acting on $\mathbb{C}[x] \oplus \mathbb{C}[x] = \left\{ \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \right\}$. By the same theorem they satisfy the five relations for any power series $V(M)$ and $W(M)$ in M . Note that here $T(M, \partial) : f(x) \mapsto f(-x)$ is the parity operator (see [24]). For example, if $V(t) = 1$ and $W(t) = t^2$, then

$$\begin{aligned} A_1 : \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} &\mapsto \begin{pmatrix} xf(x) + A_{12}g(x) \\ xg(x) \end{pmatrix} \\ A_2 : \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} &\mapsto \begin{pmatrix} g(-x) + x^2g'(-x) \\ 0 \end{pmatrix} \\ A_3 : \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} &\mapsto \begin{pmatrix} x^2g(-x) \\ 0 \end{pmatrix} \end{aligned}$$

Example 2.14. Referring to [22] and [24, Example 5], we consider a sequence of functions $\{\phi_\nu\}_{\nu=0}^\infty$ defined by

$$\sqrt{\nu+1}\phi_{\nu+1}(x) = a^*\phi_\nu(x), \quad \nu \geq 0; \quad \phi_0(x) = \frac{1}{\sqrt{\sqrt{\pi}x_0}} \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right)$$

where a^* is the linear operator

$$a^* = -\frac{x_0}{\sqrt{2}}\partial + \frac{1}{\sqrt{2}x_0}M$$

defined for any positive real constant x_0 . It follows that

$$\begin{aligned} \phi_n(x) &= \frac{1}{\sqrt{n!}}(a^*)^n\phi_0(x) = \frac{1}{\sqrt{n!\sqrt{\pi}x_0}}(a^*)^n \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right) \\ &= \frac{1}{\sqrt{2^n n! \sqrt{\pi}x_0}} \exp\left(-\frac{1}{2}\left(\frac{x}{x_0}\right)^2\right) H_n\left(\frac{x}{x_0}\right) \end{aligned}$$

for $n = 0, 1, 2, \dots$, where H_n are the Hermite polynomials. Defining the linear operator a as

$$a = \frac{x_0}{\sqrt{2}}\partial + \frac{1}{\sqrt{2}x_0}M$$

it can be shown that $a\phi_0 = 0$ and $a\phi_\nu = \sqrt{\nu}\phi_{\nu-1}$ for $\nu \geq 1$. Moreover,

$$(a^*)^k a^k \phi_\nu = \begin{cases} \frac{\nu!}{(\nu-k)!} \phi_\nu & \text{if } k \leq \nu \\ 0 & \text{if } k > \nu \end{cases}$$

for any $k \in \mathbb{N}$. The sequence of functions $\{\phi_\nu\}_{\nu=0}^\infty$ describes the energy eigenstates of the simple quantum mechanical harmonic oscillator. Consider now the two differential operators a^* (“creation” operator) and a (“annihilation” operator) defined on the linear space $\Omega = \text{linspan}_{\mathbb{C}}(\{\phi_\nu\}_{\nu=0}^\infty)$ consisting of all complex linear combinations of functions from the set of eigenfunctions $\{\phi_\nu\}_{\nu=0}^\infty$. Since now $aa^* - a^*a = 1$, we can define A and B in Theorem 2.11 as $A = a$ and $B = a^*$ acting on Ω . Suppose $A_{12} : \Omega \rightarrow \Omega$ is any operator. Then A_1 , A_2 , and

A_3 introduced in Theorem 2.11 are realized as the operators

$$\begin{aligned} A_1 &= \begin{pmatrix} a & A_{12} \\ 0 & a \end{pmatrix} \\ A_2 &= \begin{pmatrix} 0 & T(a^*, a)V(a) + a^*T(a^*, a)W(a) \\ 0 & 0 \end{pmatrix} \\ A_3 &= \begin{pmatrix} 0 & T(a^*, a)W(a) \\ 0 & 0 \end{pmatrix} \end{aligned}$$

acting on $\Omega \oplus \Omega = \left\{ \begin{pmatrix} f \\ g \end{pmatrix} \mid f, g \in \Omega \right\}$.

Recall that $a\phi_0 = 0$ and $a\phi_\nu = \sqrt{\nu}\phi_{\nu-1}$ for $\nu \geq 1$. Hence

$$\begin{aligned} A_1 \begin{pmatrix} \phi_0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A_1 \begin{pmatrix} \phi_\nu \\ 0 \end{pmatrix} = \sqrt{\nu} \begin{pmatrix} \phi_{\nu-1} \\ 0 \end{pmatrix}, \quad \nu \geq 1 \\ A_2 \begin{pmatrix} \phi_\nu \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad A_3 \begin{pmatrix} \phi_\nu \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \nu \geq 0 \end{aligned}$$

Moreover, for $\nu \geq 1$,

$$A_1 \begin{pmatrix} 0 \\ \phi_\nu \end{pmatrix} = \sqrt{\nu} \begin{pmatrix} 0 \\ \phi_{\nu-1} \end{pmatrix} + \begin{pmatrix} A_{12} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_\nu \\ 0 \end{pmatrix}$$

Now let $V(a) = \sum_{k=0}^{\infty} v_k a^k$ and $W(a) = \sum_{k=0}^{\infty} w_k a^k$, where $v_k, w_k \in \mathbb{C}$. Since

$$a^k \phi_\nu = \sqrt{\frac{\nu!}{(\nu-k)!}} \phi_{\nu-k}, \quad 0 \leq k \leq \nu$$

it readily follows that

$$T(a^*, a)V(a)\phi_\nu = \sum_{i=0}^{\nu} (-1)^i v_{\nu-i} \sqrt{\frac{\nu!}{i!}} \phi_i$$

keeping in mind that $T(a^*, a)\phi_\nu = (-1)^\nu \phi_\nu$ for all $\nu \geq 0$.

Similarly, using the fact that $a^* \phi_i = \sqrt{i+1} \phi_{i+1}$, one obtains

$$a^* T(a^*, a)W(a)\phi_\nu = \sum_{i=0}^{\nu} (-1)^i w_{\nu-i} \sqrt{i+1} \sqrt{\frac{\nu!}{i!}} \phi_{i+1}$$

This means that

$$\begin{aligned} A_2 \begin{pmatrix} 0 \\ \phi_\nu \end{pmatrix} &= \sum_{i=0}^{\nu+1} (-1)^i (v_{\nu-i} - i w_{\nu-i+1}) \sqrt{\frac{\nu!}{i!}} \begin{pmatrix} \phi_i \\ 0 \end{pmatrix} \\ A_3 \begin{pmatrix} 0 \\ \phi_\nu \end{pmatrix} &= \sum_{i=0}^{\nu} (-1)^i w_{\nu-i} \sqrt{\frac{\nu!}{i!}} \begin{pmatrix} \phi_i \\ 0 \end{pmatrix} \end{aligned}$$

In the first expression, we have to assume that $v_{-1} = 0$. Introduce a basis $\{\Phi_\nu\}_{\nu=0}^{\infty}$ of $\Omega \oplus \Omega$ given by

$$\Phi_{2\nu} = \begin{pmatrix} \phi_\nu \\ 0 \end{pmatrix}, \quad \Phi_{2\nu+1} = \begin{pmatrix} 0 \\ \phi_\nu \end{pmatrix}, \quad \nu \geq 0$$

For $\nu \geq 0$, it follows that

$$A_1\Phi_0 = A_2\Phi_{2\nu} = A_3\Phi_{2\nu} = 0$$

And, for $\nu \geq 1$,

$$A_1\Phi_{2\nu} = \sqrt{\nu}\Phi_{2\nu-2}, \quad A_1\Phi_{2\nu+1} = \sqrt{\nu}\Phi_{2\nu-1} + \begin{pmatrix} A_{12} & 0 \\ 0 & 0 \end{pmatrix} \Phi_{2\nu}$$

Moreover, for all $\nu \geq 0$,

$$A_2\Phi_{2\nu+1} = \sum_{i=0}^{\nu+1} (-1)^i (v_{\nu-i} - iw_{\nu-i+1}) \sqrt{\frac{\nu!}{i!}} \Phi_{2i}, \quad A_3\Phi_{2\nu+1} = \sum_{i=0}^{\nu} (-1)^i w_{\nu-i} \sqrt{\frac{\nu!}{i!}} \Phi_{2i}$$

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