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Meander Graphs and Frobenius Seaweed Lie Algebras III

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Abstract

We investigate properties of a Type-A meander, here considered to be a certain planar graph associated to seaweed subalgebra of the special linear Lie algebra. Meanders are designed in such a way that the index of the seaweed may be computed by counting the number and type of connected components of the meander. Specifically, the simplicial homotopy types of Type-A meanders are determined in the cases where there exist linear greatest common divisor index formulas for the associate seaweed. For Type-A seaweeds, the homotopy type of the algebra, defined as the homotopy type of its associated meander, is recognized as a conjugation invariant which is more granular than the Lie algebra's index.

Keywords: Lie algebra; Seaweed; Biparabolic; Meander; Rank; Index; Frobenius

Introduction

The *index* of a Lie algebra is an important invariant of the Lie algebra and is bounded by the algebra's rank: ind $\mathfrak{g} \leq$rk $\mathfrak{g}$, with equality when $\mathfrak{g}$ is reductive. More formally, the index of a Lie algebra $\mathfrak{g}$ is given by

$$\text{ind}\,\mathfrak{g} = \min_{f\in\mathfrak{g}^*} \dim\left(\ker\left(B_f\right)\right),$$

where f is a linear functional on $\mathfrak{g}$ and B_f is the associated skew-symmetric *Kirillov form* defined by $B_f(x, y) = f([x, y])$ for all $x, y \in \mathfrak{g}$. Of particular interest are those Lie algebras which have index zero. Such algebras are called *Frobenius* and have been studied extensively from the point of view of invariant theory [1] and are of special interest in deformation and quantum group theory stemming from their connection with the classical Yang-Baxter equation [2, 3]. Simple Lie algebras can never be Frobenius but there always exits subalgebras that must be.

In [4] Dergachev and A. Kirillov introduced a combinatorial method for computing the index of certain seaweed (biparabolic) subalgebras of $A_{n-1} = \mathfrak{sl}(n)$, based on counting the number and type of connected components of a planar graph representation of the seaweed algebra, called a (Type-A) *meander*. A Type-C, or *symplectic* meander and an attendant combinatorial index formula was developed by Coll et al in [2] for seaweed subalgebras of $\mathfrak{sp}(n)$. (See also [5]).

Using a collection of deterministic graph theoretic moves, a given meander can be "wound down" to reveal its (simplicial) *homotopy type*. The sequence of moves used in this winding-down procedure is called the *signature* of the meander and may be regarding as a graph theoretic rendering of Panyushev's well-known reduction [6]. In [7,8], Coll et al used the signature to develop closed form formulas for the index of a seaweed algebra in terms of the block sizes of the defining flags in the Type-A and Type-C cases when the number of blocks in the flags is small. Subsequently, Karnauhova and Liebsher [10] used signature type moves and complexity arguments to establish that the index formulas developed in these papers are the only linear greatest common divisor formulas for the index based on the flags defining the seaweed. One finds that in the Type-A case, a seaweed is Frobenius precisely when its associated meander consists of a single path [4, 7, 8]. For a Type-C seaweed to be Frobenius, its associated meander must reduce to a certain collection of paths [5, 10].

Since the homotopy type is not defined in terms of the algebraic structure of the original Lie algebra, it is not *a priori* clear that it is an algebraic invariant of the original algebra. In fact, this remains an open question for an arbitrary seaweed. However, it is implicit in the work of Moreau and Yakimova [11] that for Type-A seaweeds the homotopy type is a conjugation invariant which is more granular than the index. The example at the end of this paper provides two seaweed algebras which have the same dimension, rank, and index – but different homotopy types. So, are not conjugate.

In Theorems 5.1 and 5.2, we classify those homotopy types of seaweeds where there exist linear greatest common divisor index formulas developed in the prequels to this article.

Seaweed Algebras

Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two parabolic subalgebras of a reductive Lie algebra g. If $\mathfrak{q} + \mathfrak{q}' = $ g then $\mathfrak{q}\cap\mathfrak{q}'$ is called a *seaweed*, or in the terminology of A. Joseph [13] *biparabolic*, subalgebra of $\mathfrak{g}$. In what follows, we further assume that $\mathfrak{g}$ is simple and comes equipped with a triangular decomposition

$$\mathfrak{g} = \mathfrak{u}_+ \oplus \mathfrak{h} \oplus \mathfrak{u}_-$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{u}_+$ and $\mathfrak{u}_-$ are the subalgebras consisting of the upper and lower triangular matrices, respectively. Let Π be the set of $\mathfrak{g}$'s simple roots and for $\beta\in\Pi$, let $\mathfrak{g}_\beta$ denote the root space corresponding to β. A seaweed subalgebra $\mathfrak{q}\cap\mathfrak{q}'$ is called standard if $\mathfrak{q}\supseteq\mathfrak{h}\oplus\mathfrak{u}_+$ and $\mathfrak{q}'\supseteq\mathfrak{h}\oplus\mathfrak{u}_-$. We tacitly assume that the ground field is an algebraically closed field of characteristic zero, so that any seaweed is conjugate to a standard one. Note that while two standard parabolic subalgebras cannot be conjugate, two standard seaweeds can be.

In the case that $\mathfrak{q}\cap\mathfrak{q}$ is standard, let $\Psi = \{\in\Pi : \mathfrak{g}_{-\beta} \not\in \mathfrak{q}\}$, $\Psi' = \{\beta\in\Pi\colon \mathfrak{g}_\beta \not\in \mathfrak{q}'\}$, and denote the seaweed by $\mathfrak{p}(\Psi|\Psi')$. Such a seaweed is parabolic if one of Ψ or Ψ' or is the empty set, and called maximal parabolic if it is of the form $\mathfrak{p}(\{\beta\}|\emptyset)$ or $\mathfrak{p}(\emptyset|\{\beta\})$, respectively.

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Type-A Seaweeds

Let $A_{n-1} = \mathfrak{sl}(n)$ be the algebra of $n\times n$ matrices with trace zero and consider the triangular decomposition of $\mathfrak{sl}(n)$ as above. Let = $\{\beta_1,\ldots, \beta_{n-1}\}$ be the set of simple roots of $\mathfrak{sl}(n)$ with the standard ordering and let $\mathfrak{p}_n^{\mathrm{A}}(\Psi \mid \Psi')$ denote a seaweed subalgebra of $\mathfrak{sl}(n)$ where Ψ and Ψ' are subsets of Π. Let comp$_n$ denote the set of sequences of positive integers whose sum is n (i.e., comp$_n$ is the set of compositions of n). It will be convenient to index seaweeds of $\mathfrak{sl}(n)$ by pairs of elements of comp$_n$. Let $\mathcal{P}(X)$ denote the power set of a set X. Let φ_{A} be the usual bijection from comp$_n$ to a set of cardinality $n-1$. That is, given $\underline{a}=(a_1, a_2, \ldots, a_m) \in$ comp$_n$, define φ_{A}: comp$_n \to \mathcal{P}(\Pi)$ by

$$\varphi_{\mathrm{A}}(\underline{a}) = \left\{\beta_{a_1}, \beta_{a_1+a_2}, \ldots, \beta_{a_1+a_2+\cdots+a_{m-1}}\right\}.$$

Now, following the notational conventions established in [3], define the type of the seaweed $\mathfrak{p}_n^{\mathrm{A}}(\varphi_{\mathrm{A}}(\underline{a}) \mid \varphi_{\mathrm{A}}(\underline{b}))$ to be the symbol

$$\frac{a_1 \mid a_2 \mid \ldots \mid a_m}{b_1 \mid b_2 \mid \ldots \mid b_t}, \text{ where } \sum_{i=1}^{m} a_i = \sum_{j=1}^{t} b_j = n.$$

By construction, the sequence of numbers in $\underline{a}$ determines the heights of triangles below the main diagonal in $\mathfrak{p}_n^{\mathrm{A}}(\varphi_{\mathrm{A}}(\underline{a}) \mid \varphi_{\mathrm{A}}(\underline{b}))$, which may have nonzero entries, and the sequence of numbers in $\underline{b}$ determines the heights of triangles above the main diagonal. For example, the seaweed $\mathfrak{p}_{20}^{\mathrm{A}}(\{\beta_{17}\} \mid \{\beta_{10}, \beta_{14}\})$ of type $\frac{17|3}{10|4|6}$ has the following shape, where * indicates the possible nonzero entries from the ground field. See Figure 1, where we have chosen such a large example to fully illustrate the winding-down moves of Section 4, but also to provide a seaweed with an interesting homotopy type.

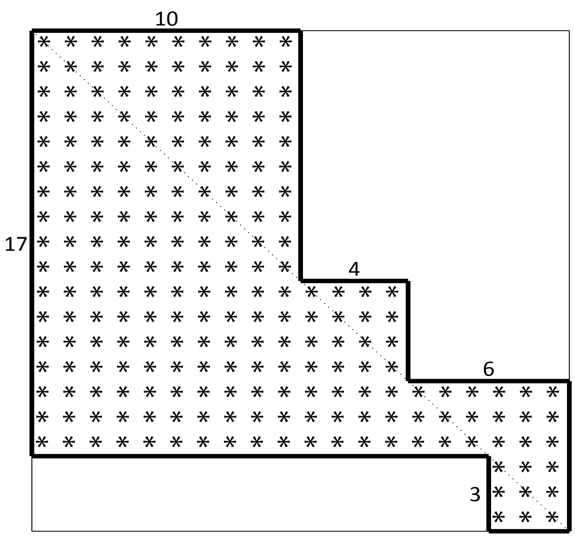

Figure 1: A seaweed of type $\frac{17|3}{10|4|6}$.

Meanders, Index Formulas, and Homotopy Type

Type-A meanders

Following Dergechev and A. Kirillov [5], we associate a planar graph to each seaweed of type $\frac{a_1|a_2|\ldots|a_m}{b_1|b_2|\ldots|b_t}$ as follows. Line up n vertices horizontally and label them $v_1, v_2, \ldots, v_n$. Partition the set of vertices into two set partitions, called top and bottom. The top partition groups together the first a_1 vertices, then the next a_2 vertices, and so on, lastly grouping together the last a_m vertices. In a similar way, the bottom partition is determined by the sequence $b_1,\ldots,b_t$. We call each set within a set partition a block. For each block in the top (likewise bottom) partition we build up the graph by adding edges in the same way. First, add an edge from the first vertex of a block to the last vertex of the same block drawn concave down (respectively concave up in the bottom part case). The edge addition is then repeated between the second vertex and the second to last and so on within each block of both partitions. More explicitly, given vertices v_j, v_k in a top block of size a_i, there is an edge between them if and only if $j + k = 2(a_1 + a_2 + \ldots + a_{i-1}) + a_i + 1$. If v_j, v_k are in a bottom block of size b_i, there is an edge between them if and only if $j + k = 2(b_1 + b_2 + \ldots + b_{i-1}) + b_i + 1$. The resulting undirected planar graph is called the *meander* associated to the given seaweed. We say that the meander has the same type as its associated seaweed See Figure 2.

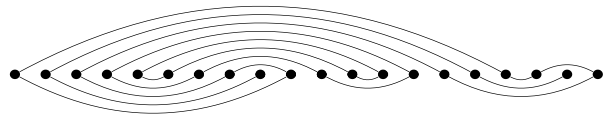

Figure 2: A meander of type $\frac{17|3}{10|4|6}$.

Type-A Index Formulas

Evidently, every meander consists of a disjoint union of cycles, paths, and points (degenerate paths). The main result of [5] is that the index of the meander can be computed by counting the number and type of each of these components.

Theorem 3.1 (Theorem 5.1, [5]): If $\mathfrak{p}$ is a seaweed subalgebra of $\mathfrak{sl}(n)$, then

$$\operatorname{ind} \mathfrak{p} = 2C + P - 1,$$

where C is the number of cycles and P is the number of paths in the associated meander.

This elegant result, and the Type-C analogue ([Theorem 4.5) are difficult to apply in practice. However, in certain cases, the following index formulas allow us to ascertain the index directly from the block sizes of the flags that define the seaweed. The following formulas were developed in the first two articles in this series. We hasten to add that the formula in Theorem 3.2, which follows as a corollary to Theorem 3.3, was known early on to Elashvilli [14].

Theorem 3.2 (Theorem 7, [3]): A seaweed of type $\frac{a|b}{n}$ has index gcd(a; b)–1

Theorem 3.3 (Theorem 8, [3]): A seaweed of type $\frac{a|b|c}{n}$, or type $\frac{a|b}{c|n-c}$, has index gcd($a + b$; $b + c$) – 1.

The following result establishes that the formulas in Theorems 3.2 and 3.3 the only nontrivial linear ones that are available in the parabolic case.

Theorem 3.4 (Theorem 5.3, [11]): If $m \geq 4$ and $\mathfrak{p}$ is a seaweed of

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type $\frac{a_1|a_2|\dots|a_m}{n}$, then there do not exist homogeneous polynomials f_1, $f_2 \in \mathbb{Z}[x_1,\dots, x_m]$ of arbitrary degree, such that the index of $\mathfrak{p}$ is given by $\gcd(f_1(a_1,\dots, a_m), f_2(a_1,\dots, a_m))$.

Homotopy type

Definition 3.5: We say that a planar graph has *homotopy type* $H(a_1, a_2,\dots,a_m)$ if its homotopy type is equivalent to the meander of type $\frac{a_1|a_2|\dots|a_m}{a_1|a_2|\dots|a_m}$. That is, a union of m non-concentric subgraphs, where each subgraph has homotopy type $\frac{a_i}{2}$ concentric circles if a_i is even, and $\lfloor a_i/2 \rfloor$ concentric circles with a point in the center if a_i is odd.

Example: A planar graph with homotopy type $H(5, 1, 2)$ is homotopically equivalent to the graph in the following Figure 3.

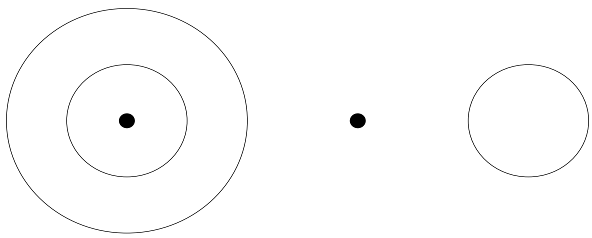

Figure 3: A planar graph with homotopy type $H(5; 1; 2)$.

When it makes sense, we define the *homotopy type of a seaweed* to be the homotopy type of its corresponding meander. More pointedly, we note that, unlike the index, the homotopy type of a Lie algebra $\mathfrak{g}$ is not defined directly in terms of $\mathfrak{g}$'s Lie Theory. It is therefore not *a priori* clear to what extent the homotopy type is an algebraic invariant. However, the following theorem follows from Theorem 5.3 in the recent paper by Moreau and Yakimova [11].

Theorem 3.6: Conjugate seaweed subalgebras of $\mathfrak{sl}(n)$ have the same homotopy type.

Winding Down and the Signature

In this section, we recall two technical Lemmas See [8,9]. The first Lemma (Winding Down) can be used to discern the homotopy type of a meander. The second Lemma (Winding Up), in cooperation with the Winding-Down Lemma, will be used in the proof of the classification Theorems 5.1 and 5.2. In these theorems, we classify the homotopy types in the cases where there exist linear greatest common divisor index formulas for the associated seaweed.

The Winding-Down Lemma establishes that, through a deterministic sequence of graph theoretic moves, each meander can be contracted or "wound down" to reveal its homotopy type. Since the sequence of moves is uniquely determined by the meander, we refer to this sequence as the meander's signature. Essentially, the winding-down moves, and the attendant signature, are a graph-theoretic recasting of Panyushev's reduction algorithm which was used in [14] to develop inductive formulas for the index of seaweeds in $\mathfrak{gl}(n)$.

Lemma 4.1 (Winding Down): Given a meander M of type $\frac{a_1|a_2|\dots|a_m}{b_1|b_2|\dots|b_t}$, create a meander M' by one of the following moves. For all moves except the Component Elimination, M and M' have the same homotopy type.

1. Flip (F): If $a_1 < b_1$, then $M \mapsto M'$ of type $\frac{b_1|b_2|\dots|b_t}{a_1|a_2|\dots|a_m}$

2. Component Elimination ($C(c)$): If $a_1 = b_1 = c$, then $M \mapsto M'$ of type $\frac{a_2|a_3|\cdots|a_m}{b_2|b_3|\dots|b_t}$.

3. Rotation Contraction (R): If $b_1 < a_1 < 2b_1$, then $M \mapsto M'$ of type $\frac{b_1|a_1|a_2|\dots|a_m}{(2b_1 - a_1)|b_2|\dots|b_t}$.

4. Block Elimination (B): If $a_1 = 2b_1$, then $M \mapsto M'$ of type $\frac{b_1|a_2|..|a_m}{b_2|b_3|\dots|b_t}$.

5. Pure Contraction (P): If $a_1 > 2b_1$, then $M \mapsto M'$ of type $\frac{(a_1 - 2b_1)|b_1|a_1|a_2|\dots|a_m}{b_2|b_3|\dots|b_t}$.

Example: We continue with our running example, and wind down the meander of Figure 2 in Figure 4 below.

Note that each of the winding-down moves can be reversed to yield a winding-up move. The winding-up moves, which we record in the following Lemma, can be used to build up any meander, of any size and block configuration.

Lemma 4.2 (Winding Up): Every meander is the result of a sequence of the following moves applied to the empty meander. For all moves, except Component Creation, M and M' have the same homotopy type.

Given a meander M of type $\frac{a_1|a_2|\dots|a_m}{b_1|b_2|\dots|b_t}$, create a meander M' by one of the following moves:

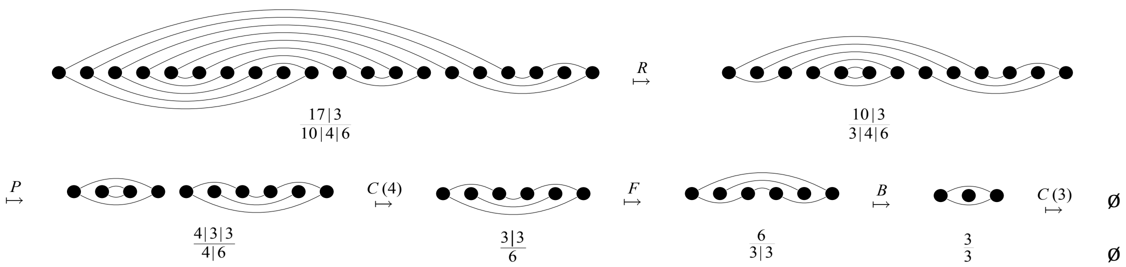

Figure 4: Winding down the meander $\frac{17|3}{10|4|6}$, with signature $RPC(4)FBC(3)$.

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1. Flip $(\tilde{F}): M \mapsto M'$ of type $\frac{b_1|b_2|\dots|b_t}{a_1|a_2|\dots|a_m}$.

2. Component Creation $(\tilde{C}(c))$: If $a_1 = b_1 = c$, then $M \mapsto M'$ of type $\frac{c|a_1|a_2|\cdots|a_m}{c|b_1|b_2|\dots|b_t}$.

3. Rotation Expansion $(\tilde{R})$: If $b_1 < a_1 < 2b_1$, then $M \mapsto M'$ of type $\frac{(2a_1 - b_1)|a_2|\dots|a_m}{|a_1|b_2|\dots|b_t}$.

4. Block Creation $(\tilde{B}): M \mapsto M'$ of type $\frac{2a_1|a_2|\cdots|a_m}{a_1|b_1|b_2|\dots|b_t}$.

5. Pure Expansion $(\tilde{P}): M \mapsto M'$ of type $\frac{a_1 + 2a_2|a_3|a_4|\dots|a_m}{a_2|b_1|b_2|\dots|b_t}$.

Classification Theorems

In this section, we present the promised homotopy type classification theorems. The proofs are inductive in nature and based on the Winding-Up lemma.

Theorem 5.1. If $\mathfrak{p}$ is a seaweed subalgebra of $\mathfrak{sl}(n)$ of type $\frac{a|b}{n}$ or $\frac{2a}{a|a}$, with $\gcd(a,b) = k$, then its homotopy type is $H(k)$.

Proof. Given any integer $m \geq 2$, we prove by induction that the theorem holds for all such seaweeds created using m winding-up moves. For the base of induction, the only seaweeds of this type that can be created by two moves are those created by Component Creation followed by Block Creation. Such a meander has type $\frac{2a}{a|a}$. The homotopy type is $H(a)$, and $\gcd(a,a) = a$.

Now, let $m > 2$ and assume the theorem holds for all such seaweeds created using m–1 winding-up moves. Let $\mathfrak{p}$ be a seaweed of type $\frac{a|b}{n}$ or $\frac{a|b}{n}$ created by m-1 winding-up moves. Also let $k = \gcd(a, b)$. Let $\mathfrak{p}'$ be the seaweed obtained from $\mathfrak{p}$ by applying the appropriate winding-down move, so that $\mathfrak{p}'$ is created using m-1 winding-up moves. We break the proof into cases depending on which winding-down move is applied. For all cases except Component Elimination, $\mathfrak{p}$ and $\mathfrak{p}'$ have the same homotopy type.

Case 1: A Flip is applied to $\mathfrak{p}$.

This case is trivial.

Case 2: A Component Elimination is applied to $\mathfrak{p}$.

This case is vacuously true – a Component Elimination cannot be applied to a seaweed of type $\frac{a|b}{n}$ or $\frac{n}{a|b}$.

Case 3: A Rotation Contraction is applied to $\mathfrak{p}$.

In order to apply this move, $\mathfrak{p}$ must have type $\frac{n}{a|b}$ with $n < 2a$, thus $\mathfrak{p}'$ has type $\frac{a}{2a - n\,|\,b}$. For any positive integers x and y we have $\gcd(x, y) = \gcd(x, x + y)$. Therefore,

$$\gcd(b, 2a - n) = \gcd(b, b + 2a - n) = \gcd(b, a) = k.$$

The claim then follows from the inductive hypothesis.

Case 4: A Block Elimination is applied to $\mathfrak{p}$.

In order to apply this move, $\mathfrak{p}$ must have type $\frac{n}{a|b}$ with $n=2a$, thus $b = a$ and $k = \gcd(a; b) = a$. The claim now follows from the fact that $\mathfrak{p}'$ has type $\frac{a}{a}$.

Case 5: A Pure Contraction is applied to $\mathfrak{p}$.

In order to apply this move, $\mathfrak{p}$ must have type $\frac{n}{a|b}$ with $n>2a$, thus $\mathfrak{p}'$ has type $\frac{n-2a\,|\,a}{b}$. Now

$$\gcd(a, n-2a) = \gcd(a, n-2a + a) = \gcd(a, b) = k.$$

So, the claim follows from the inductive hypothesis.

Theorem 5.2. Let $\mathfrak{p}$ seaweed subalgebra of $\mathfrak{sl}(n)$ of type $\frac{a|b|c}{n}$, $\frac{n}{a|b|c}$, or $\frac{a|b}{c|d}$. Let $k = \gcd(a + b, b + c)$, and let $r \equiv a \bmod k$, and $s \equiv b \bmod k$, where $0 \leq r < k$ and $0 \leq s < k$.

1. If $r = 0$ or $s = 0$, then the homotopy type is $H(k)$.

2. If r and s are nonzero, then the homotopy type is $H(r, s)$.

Proof. Note that k divides $(a + b)-(a + c) = a-c$, so $r \equiv a \equiv c \bmod k$. In the case that a seaweed has type $\frac{a|b}{c|d}$, we have $2n = (b + c) + (a + d)$ and so, $k = \gcd(n, b + c) = \gcd(n, a + d)$. Also, k divides $(a + b)-(a + d) = b-d$, so $s \equiv b \equiv d \bmod k$.

Given any integer $m \geq 2$, we prove, by induction, that the theorem holds for all such seaweeds created using m winding-up moves. For the base of induction, the only seaweeds of this type that can be created by two moves are those created by two Component Creation moves. Such a meander has type $\frac{a|b}{a|b}$ and the homotopy type is $H(a, b)$. Since $k = \gcd(a + b, b + a) = a + b$, we have $r = a$ and $s = b$, as desired.

Now, let $m > 2$ and assume the theorem holds for all such seaweeds created using m–1 winding-up moves. Let $\mathfrak{p}$ be a seaweed of type $\frac{a|b|c}{n}$, $\frac{n}{a|b|c}$, or $\frac{a|b}{c|d}$ created by m winding-up moves. Also, let $k = \gcd(a + b, b + c)$ and $\mathfrak{p}'$ be the seaweed obtained from $\mathfrak{p}$ by applying the appropriate winding-down move, so that $\mathfrak{p}'$ is created using m–1 winding-up moves. We break the proof into cases depending on which winding-down move is applied. For all cases except Component Elimination, $\mathfrak{p}$ and $\mathfrak{p}'$ have the same homotopy type.

Case 1: A Flip is applied to $\mathfrak{p}$.

This case is trivial.

Case 2: A Component Elimination is applied to $\mathfrak{p}$.

To apply this move, $\mathfrak{p}$ must have type $\frac{a|b}{a|b}$. As noted above, the homotopy type is $H(r, s)$, as desired.

Case 3: A Rotation Contraction is applied to $\mathfrak{p}$.

If $\mathfrak{p}$ has type $\frac{n}{a|b|c}$, then this move cannot be applied.

If $\mathfrak{p}$ has type $\frac{n}{a|b|c}$, then we must have $n < 2a$ in order to apply this move, and $\mathfrak{p}'$ has type $\frac{a}{2a - n|b|c}$. Note that

$\gcd(c+b, b+2a-n) = \gcd(b+c, a-c) = \gcd(b+c, b+c+a-c) = k.$

So, by induction, $\mathfrak{p}$ has homotopy type $H(k)$ or $H(r, s)$ depending on the values of $r \equiv c \bmod k$ and $s \equiv b \bmod k$, respectively.

If $\mathfrak{p}$ has type $\frac{a|b}{c|d}$, then we must have $c < a < 2c$ in order to apply this move, and $\mathfrak{p}'$ has type $\frac{c|b}{2c-a|d}$.

Now, $\gcd(c+b, c+d) = \gcd(c+b, n) = k$, so, by induction, $\mathfrak{p}'$ has homotopy type $H(k)$ or $H(r; s)$ depending on the values of $r \equiv c \bmod k$ and $s \equiv d \bmod k$, respectively.

Case 4: A Block Elimination is applied to $\mathfrak{p}$.

If $\mathfrak{p}$ has type $\frac{a|b|c}{n}$, then this move cannot be applied.

If $\mathfrak{p}$ has type $\frac{n}{a|b|c}$, then we must have $n = 2a$ in order to apply this move. Therefore, $b + c = a = n/2$. Since $\mathfrak{p}'$ has type $\frac{a}{b|c}$, we know by Theorem 5.1 that the homotopy type is $H(j)$ where $j = \gcd(c,b)$, but since j divides $b+c = a$, we also see that j divides $a+b$, so $j \le k$. Now, since $k = \gcd(a+b, b+c) = \gcd(n/2+b, n/2)$, we have that k divides b, and so k also divides a and c. This implies $k \le j$. We conclude that $k = j$. Note that $r = s = 0$ and the homotopy type of $\mathfrak{p}$ is $H(k)$, as desired.

If $\mathfrak{p}$ has type $\frac{a|b}{c|d}$, then we must have $a = 2c$ in order to apply this move, and $\mathfrak{p}'$ has type $\frac{c|b}{d}$. Again, by Theorem 5.1, the homotopy type is $H(j)$ where $j = \gcd(c, b)$. Clearly, j divides $b + c$, and j also divides $a + b = 2c + b$, so $j \le k$. Now, k divides $(a + b)-(b + c) = a-c = c$, so k also divides b and a. This implies that $k \le j$. We conclude that $k = j$. Once again $r = s = 0$ and the homotopy type of $\mathfrak{p}$ is $H(k)$, as desired.

Case 5: A Pure Contraction is applied to $\mathfrak{p}$.

If $\mathfrak{p}$ has type $\frac{a|b|c}{n}$, then this move cannot be applied.

If $\mathfrak{p}$ has type $\frac{n}{a|b|c}$, then we must have $n > 2a$ in order to apply this move, and $\mathfrak{p}'$ has type $\frac{n-2a|a}{b|c}$. Now $\gcd(b + c, a + b) = \gcd(b + c, n) = k$, so, by induction, $\mathfrak{p}'$ has homotopy type $H(k)$ or $H(r, s)$ depending on the values of $r \equiv c \bmod k$ and $s \equiv b \bmod k$, respectively.

If $\mathfrak{p}$ has type $\frac{a|b}{c|d}$, then we must have $a > 2c$ in order to apply this move, and $\mathfrak{p}'$ has type $\frac{a-2c|c|b}{d}$. Note that

$\gcd(a-2c+c, c+b) = \gcd(a-c, b+c) = \gcd(a-c+b+c, b+c) = k.$

So, by induction, $\mathfrak{p}'$ has homotopy type $H(k)$ or $H(r, s)$ depending on the values of $r \equiv c \bmod k$ and $s \equiv b \bmod k$, respectively.

The following example demonstrates that the homotopy type can sometimes distinguish between algebras when grosser invariants are unable to detect differences.

Example: The seaweeds of type $\frac{5|3}{3|3|2}$ and $\frac{4|4}{2|4|2}$ both have dimension (27), rank (7), and index (1), but have homotopy types $H(1, 1)$ and $H(2)$, respectively.

Summary and Looking Ahead

We define a meander to be a planar graph associated to a seaweed algebra which is designed in such a way that the index of the seaweed may be computed by counting the number and type of connected components of the graph. Such combinatorial formulas have been established in the Type-A case by Dergachev and A. Kirillov [4] and in the Type-C case by Coll et al. [11] (and independently by Panyushev and Yakimova [5]). These elegant formulas are difficult to apply in practice, but in certain cases can be replaced by closed form linear greatest common divisor formulas based on the flags that define the seaweed in its standard representation. For maximal parabolics in the Type-A case, this was done early on by Elashvili, and later, as a corollary to their combinatorial result, by Dergachev and A. Kirillov, and still later by Coll et al., as a consequence of a more generalized formula. The Type-C case was addressed by Coll et al. in [10]. Investigations by Karnauhova and Liebsher [13] show that these formulas are the full complement of linear greatest common divisor index formulas available in the Type-A and Type-C cases.

The simplicial homotopy type of a meander is evidently well-defined. Recent work of Moreau and Yakimova [12] establishes that the homotopy type of a Type-A seaweed is also well-defined, up to conjugation. In a forthcoming note, and after the fashion of this article, we will consider the Type-C case. In particular, we will establish that the homotopy type of a Type-C seaweed is a conjugation invariant. Furthermore, we will classify the homotopy types in all cases where linear greatest common divisor formulas for the index exist.

Finally, it follows from Joseph ([13], Theorem 8.4) that the index of a Type-B seaweed is the same as the index of a Type-C seaweed having the same flags. Consequently, the construction of Type-C meanders and related index formulas for Type-C seaweeds [10, 5], as well as results on the homotopy type of Type-C meanders, carry over *Mutatas Mutandis* to the Type-B case.

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