Models solvable by Bethe Ansatz [1]

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Abstract

Diagonalization of integrable spin chain Hamiltonians by the quantum inverse scattering method gives rise to the connection with representation theory of different (quantum) algebras. Extending the Schur-Weyl duality between sl_2 and the symmetric group S_N from the case of the isotropic spin 1/2 chain (XXX-model) to a general spin chains related to the Temperley-Lieb algebra $TL_N(q)$ one finds a new quantum algebra $\mathcal{U}_q(n)$ with the representation ring equivalent to the sl_2 one.

2000 MSC: 17B37, 16D90

1 Introduction

The development of the quantum inverse scattering method (QISM) [1, 2, 3, 4] as an approach to construction and solution of quantum integrable systems has lead to the foundations of the theory of quantum groups [5, 6, 7, 8, 9].

The theory of representations of quantum groups is naturally connected to the spectral theory of the integrals of motion of quantum systems. In particular, this connection appeared in the combinatorial approach to the question of completeness of the eigenvectors of the XXX Heisenberg spin chain [10] with the Hamiltonian

$$H_{XXX} = \sum_{n=1}^{N-1} \left(\sigma_n^x \sigma_{n+1}^x + \sigma_n^y \sigma_{n+1}^y + \sigma_n^z \sigma_{n+1}^z\right) \tag{1.1}$$

where σ_n^α ($\alpha = x, y, z$) are the Pauli matrices.

Three algebras are connected to this system: the Lie algebra $\mathfrak{sl}_2$ of rotations, the group algebra $\mathbb{C}[S_N]$ of the symmetric group S_N and the infinite dimensional algebra $\mathcal{Y}(\mathfrak{sl}_2)$ – the Yangian [7], with the corresponding R-matrix $R(\lambda) = \lambda I + \eta\mathcal{P}$, where $\mathcal{P}$ is the 4×4 permutation matrix flipping the two factors of $\mathbb{C}^2 \otimes \mathbb{C}^2$.

The Yangian is the dynamical symmetry algebra which contains all the dynamical observables of the system. It is important to note that the algebras $\mathfrak{sl}_2$ and $\mathbb{C}[S_N]$ are related by the Schur-Weyl duality in the representation space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}} \mathbb{C}^2$. This follows from the fact that $\mathfrak{sl}_2$ and $\mathbb{C}[S_N]$ are each other's centralizers in this representation space. As a consequence, since the Hamiltonian commutes with the global generators of $\mathfrak{sl}_2 : S^\alpha = 1/2\sum_{n=1}^N \sigma_n^\alpha, \alpha = x, y, z$, it is an element of $\mathbb{C}[S_N]$. This can also be seen from the expression of H_{XXX} in terms of the permutation operators, which are the generators of the symmetric group S_N

$$\sum_\alpha \sigma_n^\alpha \sigma_{n+1}^\alpha = 2\mathcal{P}_{nn+1} - \mathbb{I}_{nn+1}$$

[1]Presented at the 3rd Baltic-Nordic Workshop "Algebra, Geometry, and Mathematical Physics", Göteborg, Sweden, October 11–13, 2007.

An analogous situation arises in the anisotropic XXZ chain

$$H_{XXZ} = \sum_{n=1}^{N-1} \left(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z\right) + \frac{\left(q-q^{-1}\right)}{2}\left(\sigma_1^z - \sigma_N^z\right) \tag{1.2}$$

which commutes [11] with the global generators of the quantum algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ [5]. Here the role of the second algebra is played by the Temperley-Lieb algebra $TL_N(q)$, whose generators $\check{R}_n := \check{R}_{n,n+1}(q)$ in the space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}}\mathbb{C}^2$ coincide with the constant R-matrix ($\omega(q) = q - 1/q$)

$$\check{R}_{XXZ}(q) = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \omega(q) & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{1.3}$$

As in the case of the XXX spin chain, the Hamiltonian (1.2) can be expressed in terms of the generator (1.3) of the algebra $TL_N(q)$ ($\Delta = (q+q^{-1})/2$)

$$\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z + \frac{\omega(q)}{2}\left(\sigma_n^z - \sigma_{n+1}^z\right) = \check{R}_{XXZ}(q) - \left(\frac{\omega(q)}{2} + q\right)\mathbb{I}_{nn+1}$$

The dynamical symmetry algebra of the XXZ chain is the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}_2})$.

The eigenvectors for both models can be constructed by the coordinate Bethe Ansatz (see [24]) or by an algebraic Bethe Ansatz [1, 2, 3]. The latter one follows from the main relation of the QISM for the auxiliary L-operator (see Sec. 2)

$$L_{aj}(\lambda) = \lambda I + \frac{\eta}{2}\sum_\alpha \sigma_a^\alpha \otimes \sigma_j^\alpha \tag{1.4}$$

where the indices a and j refer to the corresponding auxiliary and quantum spaces $\mathbb{C}_a^2$, $\mathbb{C}_j^2$.

The Temperley-Lieb algebra is a quotient of the Hecke algebra (see section 3) and allows for an R-matrix representation in the space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}}\mathbb{C}^n$ for any $n = 2, 3, \ldots$. There is corresponding spectral parameter depending R-matrix obtained by the Yang - Baxterization process. Consequently, it is possible to construct an integrable spin chain [12]. The open spin chain Hamiltonian is the sum of the $TL_N(q)$ generators $X_j = \check{R} - qI$

$$H_{TL} = \sum_{j=1}^{N-1} X_j \tag{1.5}$$

where X_j act nontrivially on $\mathbb{C}_j^n \otimes \mathbb{C}_{j+1}^n$ and as the identity matrix on the other factors of $\mathcal{H}$. The aim of this work is to describe the quantum algebra $\mathcal{U}_q(n)$ which is the symmetry algebra of such spin system and to show that the structures (categories) of finite dimensional representations of these algebras $\mathcal{U}_q(n)$ and $\mathfrak{sl}_2$ coincide. In this case $\mathcal{U}_q(n)$ and $TL_N(q)$ are each other's centralizers in the space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}}\mathbb{C}^n$. We consider the general case when the complex parameter $q \in \mathbb{C}^*$ is not a root of unity.

Let us note that the relation between $TL_N(q)$ and integrable spin chains was actively used in many works and monographs (see for example [13, 14, 15, 16, 17, 18, 19] and the references within). However, the authors used particular realizations of the generators X_j, related to some Lie algebras (or quantum algerbas). Characteristic property of the latter ones was the existence of one-dimensional representation in the decomposition of the tensor product of two fundamental representations $V_j \otimes V_{j+1}$. Then X_j was proportional to the rank one projector on this subspace, and the symmetry algerba was identified with the choosen algebra. We point out that the symmetry algerba $\mathcal{U}_q(n)$ is bigger and its Clebsch - Gordan decomposition of $V_j \otimes V_{j+1}$ has only two summands similar to the sl(2) case $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^3 \oplus \mathbb{C}^1$.

The dual Hopf algebra $\mathcal{U}_q(n)^*$ was introduced as the quantum group of nondegenerate bilinear form in [20, 21]. The categories of co-modules of $\mathcal{U}_q(n)^*$ and their generalisations were studied in [22, 23] where it was shown that the categories of co-modules of $\mathcal{U}_q(n)^*$ are equivalent to the category of co-modules of the quantum group $SL_q(2)$.

2 Bethe Ansatze

Using the L-operator (1.4) a new set of variables (operators in the space $\mathcal{H}$ depending on the parameter λ) is introduced by an ordered product of $L_{aj}(\lambda)$ as 2×2 matrices on the auxiliary space $\mathbb{C}^2_a$ according to the QISM [1]-[4]

$$T(\lambda) := L_{aN}(\lambda) L_{aN-1}(\lambda) \dots L_{a1}(\lambda)$$
$$T(\lambda) = \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix} \tag{2.1}$$

The entries of the monodromy matrix $T(\lambda)$ are new variables. The commutation relations of the new operators $(A(\lambda), \dots, D(\lambda))$ can be obtained from the local relaltion for the L-operator at one site:

$$R_{12}(\lambda-\mu) L_{1j}(\lambda) L_{2j}(\mu) = L_{2j}(\mu) L_{1j}(\lambda) R_{12}(\lambda-\mu) \tag{2.2}$$

where R-matrix is $R_{12}(\lambda) = \lambda I + \eta \mathcal{P}_{12} \in \mathrm{End}(\mathbb{C}^2_1 \otimes \mathbb{C}^2_2)$ and it acts on the tensor product of two auxiliary spaces $\mathbb{C}^2_1 \otimes \mathbb{C}^2_2$, while the index j refers to the space of spin quantum states $\mathbb{C}^2_j$. The relation for $T(\lambda)$ is of the same form [1]-[4]

$$R_{12}(\lambda-\mu) T_1(\lambda) T_2(\mu) = T_2(\mu) T_1(\lambda) R_{12}(\lambda-\mu)$$

where $T_1(\lambda) := T(\lambda) \otimes I$, $T_2(\mu) = I \otimes T(\mu)$. One can extract 16 relations for the entries of $T(\lambda)$.

We will use only few of them to construct algebraically eigenvectors of the Hamiltonian H_{XXX}:

$$A(\lambda)B(\mu) = f(\lambda-\mu)B(\mu)A(\lambda) + g(\lambda-\mu)B(\lambda)A(\mu)$$
$$D(\lambda)B(\mu) = f(\mu-\lambda)B(\mu)D(\lambda) + g(\mu-\lambda)B(\lambda)D(\mu)$$
$$B(\lambda)B(\mu) = B(\mu)B(\lambda)$$

where $f(\lambda-\mu) = (\lambda-\mu-\eta)/(\lambda-\mu)$, $g(\lambda-\mu) = \eta/(\lambda-\mu)$. Multiplying the RTT-relation by $R^{-1}_{12}(\lambda-\mu)$ and taking the trace over two auxiliary spaces one gets commutativity property of transfer matrix $t(\lambda)$:

$$t(\lambda)t(\mu) = t(\mu)t(\lambda), \quad t(\lambda) := \mathrm{tr}\, T(\lambda) = A(\lambda) + D(\lambda) \tag{2.3}$$

The operator $B(\mu)$ is a creation operator of the eigenvectors we are looking for. These operators act on a vacuum state (a highest weight vector) Ω. The Hamiltonian is extracted from the transfer matrix $t(\lambda)$ which is a generating function of mutually commuting integrals of motion. The vector Ω is the tensor product of states corresponding to spin up at each site of the chain:

$$\Omega = \bigotimes_1^N e_m^{(+)}, \quad \sigma_m^z e_m^{(\pm)} = \pm e_m^{(\pm)}, \quad \sigma_m^+ e_m^{(+)} = 0, \quad \sigma_m^- e_m^{(+)} = e_m^{(-)}$$

Using the explicit form of the L-operator and the definition of the monodromy matrix $T(\lambda)$ it is easy to get the relations

$$C(\lambda)\Omega = 0, \quad A(\lambda)\Omega = a(\lambda)\Omega, \quad D(\lambda)\Omega = d(\lambda)\Omega$$

where $a(\lambda) = (\lambda + \frac{\eta}{2})^N$, $d(\lambda) = (\lambda - \frac{\eta}{2})^N$. It follows also from quadratic relation of $A(\lambda)$, $B(\mu)$ that

$$A(\lambda)\prod_{j=1}^{M} B(\mu_j) = \prod_{j=1}^{M} f(\lambda-\mu_j)B(\mu_j)A(\lambda) + \sum_{k=1}^{M} g(\lambda-\mu_k)B(\lambda)\prod_{j\neq k} f(\mu_k-\mu_j)B(\mu_j)A(\mu_k)$$

and a similar relation for $D(\lambda)$ and the product of $B(\mu_j)$. Sum of these relations acting on the vacuum Ω gives the eigenvector of the transfer matrix $t(\lambda)$

$$t(\lambda)\prod_{j=1}^{M} B(\mu_j)\Omega = \Lambda\left(\lambda|\{\mu_k\}_1^M\right)\prod_{j=1}^{M} B(m_j)\Omega$$

under the condition that the parameters μ_k satisfy the Bethe equations $(k = 1, 2, \ldots, M)$

$$\frac{a(\mu_k)}{d(\mu_k)} = \prod_{j\neq k}^{M} \frac{f(\mu_j-\mu_k)}{f(\mu_k-\mu_j)}$$

The eigenvalue is

$$\Lambda(\lambda|\{\mu_k\}_1^M) = a(\lambda)\prod_{j=1}^{M} f(\lambda-\mu_j) + d(\lambda)\prod_{j=1}^{M} f(\mu_j-\lambda)$$

This construction of the eigenvectors of quantum integrable models was coined as algebraic Bethe Ansatz (ABA) [1]-[3].

Originally these eigenvectors of the XXX spin chain were found by H. Bethe at 1931 as a linear combination of one magnon eigenstates using the local operators σ_j^α

$$\Psi(z) = \sum_{k=1}^{N} z^k \sigma_k^- \Omega$$

It is easy to see that $\Psi(z)$ is an eigenvector of H_{XXX} with the eigenvalue $2(z+z^{-1}-2)$. However, the condition of periodicity i.e. the requirement that $\Psi(z)$ is also an eigenvector of the shift operator: $U\sigma_j^\alpha = \sigma_{j-1}^\alpha U$, results in the quantization of z

$$U\Psi(z) = z\Psi(z), \quad z^N = 1, \quad \text{or} \quad z = \exp(2\pi i m/N)$$

These yields $N-1$ states, $m = 1, 2, \ldots, N-1$ (because the state with $z = 1$ belongs to the vacuum multiplet: $\Psi(1) = S^-\Omega$). Multimagnon states are given by a Bethe sum or (coordinate Bethe Ansatz)

$$\Psi(\{z_j\}_1^M) = \sum_{\{n_k\}}\sum_{P\in S_m} A(P)\prod_{j=1}^{M} z_{P_j}^{n_k}\sigma_k^-\Omega$$

where the coefficients (amplitudes) $A(P)$ are defined by the elements P of permutation group S_M, quasimomenta z_{P_j} and the two magnon S-matrix [24]

$$(1 + z_1 z_2 - 2z_2)/(2z_1 - 1 - z_1 z_2), \quad z_j = \left(\mu_j + \frac{\eta}{2}\right)/\left(\mu_j - \frac{\eta}{2}\right)$$

Constructed eigenstates $\Psi = \prod_{j=1}^{M} B(\mu_j)\Omega$, $M \le [N/2]$ are highest weight vectors for the global symmetry algebra sl_2 with generators

$$S^\alpha = \frac{1}{2}\sum_{n-1}^{N}\sigma_n^\alpha, \quad S^+\Psi(\mu_1,\ldots,\mu_M) = 0$$

The proof is purely algebraic and it follows from the RTT-relation and the asymptotic of the monodromy matrix [10]

$$T(\lambda) = \lambda^N I + 2\eta\lambda^{N-1}\sum_\alpha \sigma_a^\alpha \otimes S^\alpha + O\left(\lambda^{N-2}\right)$$

3 Hecke and Temperley-Lieb Algebras

Both algebras $\mathcal{H}_N(q)$ and $TL_N(q)$ are quotients of the group algebra of the braid group $\mathcal{B}_N$ generated by $(N-1)$ generators $\check{R}_j$, $j = 1, 2, \ldots, N-1$, their inverses $\check{R}_j^{-1}$ and subject to the relations (see [25])

$$\begin{aligned} \check{R}_j\check{R}_k &= \check{R}_k\check{R}_j, & |j-k| > 1 \\ \check{R}_j\check{R}_k\check{R}_j &= \check{R}_k\check{R}_j\check{R}_k, & |j-k| = 1 \end{aligned} \tag{3.1}$$

The Hecke algebra $\mathcal{H}_N(q)$ is obtained by adding to these relations the following characteristic equations obeyed by generators

$$\left(\check{R}_j - q\right)\left(\check{R}_j + 1/q\right) = 0. \tag{3.2}$$

It is known that $\mathcal{H}_N(q)$ is isomorphic to the group algebra $\mathbb{C}[S_N]$. Consequently, irreducible representations of the Hecke algebra, as that of S_N, are parametrized by Young diagrams. By virtue of (3.2) we can write $\check{R}$ using the idempotents P_+ and P_- ($P_+ + P_- = 1$):

$$\check{R} = qP_+ - \frac{1}{q}P_- = q\mathbb{I} - \left(q + \frac{1}{q}\right)P_- := q\mathbb{I} + X \tag{3.3}$$

Substituting the expression (3.3) for $\check{R}$ in terms of X, into the braid group relations (3.1) one gets relations for X_j, X_k, $|j-k| = 1$

$$X_jX_kX_j - X_j = X_kX_jX_k - X_k \tag{3.4}$$

Requiring that each side of (3.4) is zero we obtain the quotient algebra of the Hecke algebra, the Temperley-Lieb algebra $TL_N(q)$. It is defined by the generators X_j, $j = 1, 2, \ldots, N-1$ and the relations ($\nu(q) = q + 1/q$):

$$\begin{aligned} X_j^2 &= -\left(q + \frac{1}{q}\right)X_j = -\nu(q)X_j \\ X_jX_kX_j &= X_j, \quad |j-k| = 1 \end{aligned} \tag{3.5}$$

The dimension of the Hecke algebra is $N!$, the same as the dimension of the symmetric group S_N, the dimension of $TL_N(q)$ is equal to the Catalan number $C_N = (2N)!/N!(N+1)!$. In connection with integrable spin systems we will be interested in representations of $TL_N(q)$ on the tensor product space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}}\mathbb{C}^n$. One representation is defined by an invertible $n \times n$ matrix $b \in GL(n,\mathbb{C})$ which can also be seen as an n^2 dimensional vector $\{b_{cd}\} \in \mathbb{C}^n \otimes \mathbb{C}^n$ [17]. We use the notation $\bar{b} := b^{-1}$ and view this matrix also as an n^2 dimensional vector $\{\bar{b}_{cd}\} \in \mathbb{C}^n \otimes \mathbb{C}^n$. The generators X_j can be expressed as

$$(X_j)_{cd,xy} = b_{cd}\bar{b}_{xy} \in \mathrm{Mat}\left(\mathbb{C}^n_j \otimes \mathbb{C}^n_{j+1}\right) \tag{3.6}$$

where we explicitely write the indices corresponding to the factors in the tensor product space $\mathcal{H}$. It is easy to see, that the second relation (3.5) is automatically satisfied and the first one determines the parameter q ($\nu(q) = q + 1/q$):

$$X_j^2 = X_j \,\mathrm{tr}\, b^t\bar{b}, \quad \mathrm{tr}\, b^t\bar{b} = -\left(q + \frac{1}{q}\right) = -\nu(q) \tag{3.7}$$

An obvious invariance of the braid group relations (the Yang - Baxter equation) (3.1) in this representation with respect to the transformation of the R-matrix

$$\check{R} \to \mathrm{AdM} \otimes \mathrm{AdM}\left(\check{R}\right), \quad \mathrm{M} \in \mathrm{GL}(n, \mathbb{C})$$

results in the following transformation of the matrix $b \to \mathrm{M}b\mathrm{M}^t$. If one uses an R-matrix depending on a spectral parameter (Yang - Baxterization of $\check{R}(q)$)

$$\check{R}(u;q) = u\check{R}(q) - \frac{1}{u}\left(\check{R}(q)\right)^{-1} = \omega(uq)\mathbb{I} + \omega(u)X \tag{3.8}$$

where $\check{R}(q)^{-1} = (1/q)\mathbb{I} + X$, then relation (3.5) can be written as

$$\check{R}(q^{-1};q)\check{R}(q^{-2};q)\check{R}(q^{-1};q) = 0 \tag{3.9}$$

In terms of constant R-matrices (generators of $TL_N(q)$) this relation has the form ($|i-k| = 1$, $\check{R}_i = \check{R}_{i,i+1}$)

$$\left(\check{R}_i - q\mathbb{I}\right)\left(\nu(q)\check{R}_k - q^2\mathbb{I}\right)\left(\check{R}_i - q\mathbb{I}\right) = 0 \tag{3.10}$$

Replacing in (3.9) the expression $\check{R}(u;q) = \omega(u)\check{R}(q) + u^{-1}\omega(q)\mathbb{I}$, or in (3.5) substituting $X = \check{R} - qI$ yields the vanishing of the q-antisymmetriser

$$\mathbb{I} - q^{-1}\left(\check{R}_{12} + \check{R}_{23}\right) + q^{-2}\left(\check{R}_{12}\check{R}_{23} + \check{R}_{23}\check{R}_{12}\right) - q^{-1}\check{R}_{12}\check{R}_{23}\check{R}_{12} = 0 \tag{3.11}$$

Thus the irreducible representations of $TL_N(q)$ are parametrized by Young diagrams containing only two rows with N boxes.

The constructed representation (3.3), (3.6) is reducible. The decomposition of this representation into the irreducible ones will be discussed in the next section.

4 Quantum Algebra $\mathcal{U}_q(n)$

According to the R-matrix approach to the theory of quantum groups [26], the R-matrix defines relations between the generators of the quantum algebra $\mathcal{U}_q$ and its dual Hopf algebra, the quantum group $\mathcal{A}(R)$. In this paper the emphasis will be on the quantum algebra $\mathcal{U}_q$ and its finite dimensional representations V_k, $k = 0, 1, 2, \ldots$. The generators of $\mathcal{U}_q$ can be identified with the L-operator (L-matrix) entries and their exchange relations (commutation relations) follow from the analogue of the Yang-Baxter relations (2.2) (withouht spectral parameter)

$$\check{R}_{12}L_{a_1q}L_{a_2q} = L_{a_1q}L_{a_2q}\check{R}_{12} \tag{4.1}$$

where the indices a_1 and a_2 refer to the representation spaces V_{a_1} and V_{a_2}, respectively, and index q refers to the algebra $\mathcal{U}_q$. Hence the equation (4.1) is given in $\mathrm{End}\,(V_{a_1} \otimes V_{a_2}) \otimes \mathcal{U}_q$.

In general the L-operator is defined through the universal R-matrix, where a finite dimensional representation is applied to one of the factors of the universal R-matrix

$$\mathcal{R}_{univ} = \sum_j \mathcal{R}_1^{(j)} \otimes \mathcal{R}_2^{(j)} := \mathcal{R}_1 \otimes \mathcal{R}_2 \in \mathcal{U}_q \otimes \mathcal{U}_q \tag{4.2}$$

$$L_{aq} = (\rho \otimes \mathrm{id})\, \mathcal{R}_{univ} = \rho(\mathcal{R}_1) \otimes \mathcal{R}_2 \tag{4.3}$$

where $\rho : \mathcal{U}_q \to End(V_a)$.

Furthermore, the universal R-matrix satisfies Drinfeld's axioms of the quasi-triangular Hopf algebras [7, 25]. In particular,

$$(\mathrm{id} \otimes \Delta)\, \mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} \tag{4.4}$$

Thus, choosing the appropriate representation space as the first space, one obtains the co-product of the generators of $\mathcal{U}_q$ from the following matrix equation

$$(\mathrm{id} \otimes \Delta)\, L_{aq} = L_{aq_2} L_{aq_1} \in \mathrm{End}(V_a) \otimes \mathcal{U}_q \otimes \mathcal{U}_q \tag{4.5}$$

The case when $V_a = \mathbb{C}^3$ is of particular interest and it will be presented below in detail. To this end the generators of $\mathcal{U}_q$ are denoted by $\{A_i, B_i, C_i, i = 1, 2, 3\}$ and the L-matrix is given by

$$L_{aq} = \begin{pmatrix} A_1 & B_1 & B_3 \\ C_1 & A_2 & B_2 \\ C_3 & C_2 & A_3 \end{pmatrix} \tag{4.6}$$

Multiplying two L-matrices with entries in the corresponding factors $\mathcal{U}_q(n) \otimes \mathcal{U}_q(n)$ we obtain

$$\Delta\,(L_{ab}) = \sum_{k=1}^{3} L_{kb} \otimes L_{ak} = \sum_{k=1}^{3} (\mathbb{I} \otimes L_{ak})\,(L_{kb} \otimes \mathbb{I}) \tag{4.7}$$

or explicitly for the generators

$$\Delta(B_1) = B_1 \otimes A_1 + A_2 \otimes B_1 + C_2 \otimes B_3, \tag{4.8}$$
$$\Delta(B_2) = B_3 \otimes C_1 + B_2 \otimes A_2 + A_3 \otimes B_2 \tag{4.9}$$
$$\Delta(B_3) = B_3 \otimes A_1 + B_2 \otimes B_1 + A_3 \otimes B_3 \tag{4.10}$$

etc. The central element in $\mathcal{U}_q$ is obtained from the defining relation (4.1)

$$b^{-1} L_{aq} b L_{aq}^t = c_2 \mathbb{I} \tag{4.11}$$

$$c_2 = \sum_{jkl} \left(b^{-1}\right)_{1j} L_{jk} b_{kl} L_{1l}$$

However this central element is group-like: $\Delta c_2 = c_2 \otimes c_2$. It is proportional to the identity in the tensor product of representations. The analogue of the $\mathcal{U}_q(2)$ Casimir operator can be obtained according to [26] using $L_+ := L$ and $L_- := (\rho \otimes \mathrm{id})\,(\mathcal{R}_{21})^{-1}$ as $\mathrm{tr}_q L_+ L_-^{-1} = \mathrm{tr}\, b\bar{b}^t L_+ L_-^{-1}$.

In the case when V_a, V_q are the three dimensional space $V_a \simeq V_q \simeq \mathbb{C}^3$ and the b matrix is taken from the references [15, 16]

$$b_{ij} = p^{2-i} \delta_{i4-j} = \left(b^{-1}\right)_{ij} \tag{4.12}$$

c_2 is written as

$$c_2 = p \left(\frac{1}{p} A_3 A_1 + C_2 B_1 + p C_3 B_3 \right)$$

Parameters p and q are related: $p^2 + 1 + p^{-2} = -(q + q^{-1})$. For the explicit L-operator and its 3×3 blocks we get $c_2 = qI_q$ where I_q is the identity operator on $V_q = \mathbb{C}^3$. The form of the generators $\{A_i, B_i, C_i, i = 1, 2, 3\}$ which corresponds to the choice of the b matrix (4.12), follows from the expression for the $\check{R}$ and L-matrices $L = \mathcal{P}\check{R} = \mathcal{P}\,(qI + X)$, where $\mathcal{P}$ is the permutation matrix.

For example, we have

$$B_1 = \begin{pmatrix} 0 & 0 & 0 \\ q & 0 & 0 \\ 0 & p^{-1} & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 \\ p^{-1} & 0 & 0 \\ 0 & q & 0 \end{pmatrix} \tag{4.13}$$

If we choose b as in equation (4.12) the R-matrix commutes with $h \otimes 1 + 1 \otimes h$ where $h = \mathrm{diag}(1, 0, -1)$. As a highest spin vector (pseudovacuum of the corresponding integrable spin chain [3]) we choose $\theta = (1, 0, 0)^t \in \mathbb{C}^3$ [12, 16]. If we act on the tensor product of these vectors $\theta \otimes \theta \in \mathbb{C}^3 \otimes \mathbb{C}^3$ with the coproduct of the lowering operators $\Delta(B_j)$, $j = 1, 2, 3$ we obtain new vectors. By looking at the explicit forms of the operators A_j,B_j,C_j in the space $\mathbb{C}^3$ we can convince ourselves that the vectors $(\Delta(B_i))^k \theta \otimes \theta$, $i = 1, 2; k = 1, 2, 3$ are linearly independent. Together with the vectors $\theta \otimes \theta$ and

$$(\Delta(B_i))^4 \theta \otimes \theta \simeq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

they span an 8 dimensional subspace. The vector $\Delta(B_3)\theta \otimes \theta$ is a linear combination of the vectors $(\Delta(B_i))^2 \theta \otimes \theta$, $i = 1, 2$. The vector $|b\rangle = (00p|010|p^{-1}00)^t$ spans a one dimensional invariant subspace. Thus we have the following decomposition

$$\mathbb{C}^3 \otimes \mathbb{C}^3 = \mathbb{C}^8 \oplus \mathbb{C} \tag{4.14}$$

This decomposition can also be obtained using the projectors P_+, P_- (3.3), (3.6) expressed in terms of b matrix (vector) (4.12). Due to the commutativity of the R-matrix $\check{R}_{q_1 q_2}$ with the co-product (4.5), (4.7) the corresponding subspaces $P_\pm(\mathbb{C}^3 \otimes \mathbb{C}^3)$ are invariant.

Similarly, using $\Delta^3(Y), Y \in \mathcal{U}_q(3)$ one can get the decomposition of

$$\mathbb{C}^3 \otimes \mathbb{C}^3 \otimes \mathbb{C}^3 = \mathbb{C}^{21} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \tag{4.15}$$

This type of decomposition is valid for any n.

The result of applying the co-product Δ, given by (4.5), on the generators of the quantum algebra $\mathcal{U}_q$ several times can also be presented in the matrix form

$$\left(\mathrm{id} \otimes \Delta^N\right) L_{aq} = L_{aq_N} \ldots L_{aq_2} L_{aq_1} := T^{(N)} \tag{4.16}$$

where $\Delta^N : \mathcal{U}_q \to (\mathcal{U}_q)^{\otimes N}$, $\Delta^1 := \mathrm{id}$, $\Delta^2 := \Delta$, $\Delta^3 := (\mathrm{id} \otimes \Delta) \circ \Delta$, etc. In general case of the tensor representation of $TL_N(q)$, with the space $\mathbb{C}^n$ at each site, the generators of the algebra $\check{R}_{q_k q_{k+1}} := \check{R}_{k,k+1}$ commute with the generators (4.16) of the global (diagonal) action of the quantum algebra $\mathcal{U}_q(n)$ in the space $\mathcal{H} = \overset{N}{\underset{1}{\otimes}} \mathbb{C}^n$. This follows from the relation

$$\check{R}_{k,k+1} L_{aq_{k+1}} L_{aq_k} = L_{aq_{k+1}} L_{aq_k} \check{R}_{k,k+1}$$

and the possibility due to the co-associativity of the coproduct to write the product of of L_{aq_j} as

$$T^{(N)} = L_{aq_N} \ldots L_{aq_{k+2}} \Delta_k(L_{aq_k}) L_{aq_{k-1}} \ldots L_{aq_1}$$

Hence,

$$\check{R}_{k,k+1}T^{(N)} = T^{(N)}\check{R}_{k,k+1}$$

Thus, the algebras $\mathcal{U}_q(n)$ and $TL_N(q)$ are each other's centralizers in the space $\mathcal{H}$. The tensor representation of $TL_N(q)$ in $\mathcal{H}$ decomposes into irreducible factors whose multiplicities are given by the dimensions of the irreducible representations of the algebra $\mathcal{U}_q(n)$, corresponding to the same Young diagrams

$$\mathcal{H} = \overset{N}{\underset{k}{\otimes}}\mathbb{C}^n = \overset{N}{\underset{k}{\oplus}} p_k(n)W_k(N) = \overset{N}{\underset{k}{\oplus}} \nu_k(N)V_k(n) \tag{4.17}$$

In this decomposition the index k parametrizes the Young diagrams with two rows and N boxes and multiplicities are given by the dimensions of the corresponding irreducible representations

$$p_k(n) = \dim V_k(n), \quad \nu_k(N) = \dim W_k(N) \tag{4.18}$$

of the algebras $\mathcal{U}_q(n)$ and $TL_N(q)$, respectively. As for the finite dimensional irreducible representations of the Lie algebra $\mathfrak{sl}_2$, $V_0(n) = \mathbb{C}$ is the one-dimensional (scalar) representation and the fundamental representation of the algebra $\mathcal{U}_q(n)$ is n dimensional, $V_1(n) \simeq \mathbb{C}^n$. The dimensions of other representations follow from the trivial multiplicities of the factors in the decomposition of the tensor product of the $V_k(n)$ and the fundamental representation $V_1(n)$ into two irreducible factors, as for the $\mathfrak{sl}_2$,

$$V_1(n) \otimes V_k(n) = V_{k+1}(n) \oplus V_{k-1}(n) \tag{4.19}$$

Thus, for the dimensions $p_k(n) = \dim V_k(n)$ the following recurrence relation is valid

$$n \cdot p_k(n) = p_{k+1}(n) + p_{k-1}(n) \tag{4.20}$$

with the initial conditions $p_{-1}(n) = 0$, $p_0(n) = 1$, whose solutions are Chebyshev polynomials of the second kind

$$p_k(n) = \frac{\sin(k+1)\theta}{\sin\theta}, \quad n = 2\cos\theta \tag{4.21}$$

The multiplicity $\nu_k(N)$, or the dimensions of the subspaces $W_k(N)$ in (4.17) is the number of paths that go from the top of the Bratteli diagram to the Young diagram corresponding to the representation $W_k(N)$. If $\lambda \vdash N$ is the partition of N, $\lambda = (\lambda_1 \geq \lambda_2 | \lambda_1 + \lambda_2 = N)$, then $k = \lambda_1 - \lambda_2$ and

$$\nu_k(N) = \nu_{k+1}(N-1) + \nu_{k-1}(N-1) \tag{4.22}$$

The subspaces invariant under the diagonal action of the quantum algebra $\mathcal{U}_q(n)$ on the space $\mathcal{H}$, can be obtained using the projectors (idempotents), which can be expressed in terms of the elements of the Temperley-Lieb algebra $TL_N(q)$. Using the R-matrix depending on a spectral parameter, the projector $P_N^{(+)}$ on the symmetric subspace can be written in the following way [4, 27]

$$P_N^{(+)} \simeq P_{N-1}^{(+)}\check{R}_{N-1N}\left(q^{N-1};q\right)P_{N-1}^{(+)} \tag{4.23}$$

Similar construction can be done with the underlying Lie algebra $sl(3)$. Then the corresponding q-antisymmetrizer $P_-^{(4)}$ which defines a quotient of the Hecke algebra is [4, 27]

$$P_-^{(4)} \simeq P_-^{(123)}\check{R}_{3\,4}(q^3;q)P_-^{(123)} \tag{4.24}$$

This form follows from the intertwiner of four monodromy matrices $T_1(u_1)T_2(u_2)T_3(u_3)T_4(u_4)$

$$J^{(4)} = R_{1\,2}R_{1\,3}R_{2\,3}R_{1\,4}R_{2\,4}R_{3\,4} \tag{4.25}$$

where $R_{i\,j} := R_{i\,j}(u_i/u_j)$. Multiplying by appropriate product of the permutation operators $\mathcal{P}_{k\,k+1}$ one can get the expression in terms of the baxterized Hecke generators

$$\check{R}_{1\,1}(u_3/u_4)\check{R}_{2\,3}(u_2/u_4)\check{R}_{1\,2}(u_2/u_3)\check{R}_{3\,4}(u_1/u_4)\check{R}_{2\,3}(u_1/u_3)\check{R}_{1\,2}(u_1/u_2)$$

The q-antisymmetrizer (4.24) is obtained by fixing shifts of the spectral parameters $u_k = uq^{1-k}$ [4, 27].

Theorem 4.1. *Consider the quotient of the Hecke algebra $\mathcal{H}_N(q)$ $(N > 3)$ by the ideal $\mathcal{I}$ generated by the q-antisymmetrizers $P_-^{(4)}$,*

$$\mathcal{H}_N^{(4)}(q) = \mathcal{H}_N(q)\Big/\mathcal{I}\left(P_-^{(4)}\right) \tag{4.26}$$

The tensor product representation of $\mathcal{H}_N^{(4)}(q)$ in the space $\mathcal{H}_N = \otimes_1^N \mathbb{C}^n$ $(n \geq 3)$ with the q-antisymmetrizers $P_-^{(3)}$ of rank 1 *define the quantum algebra $\mathcal{U}_q(\mathfrak{sl}(3); n)$ as the centralizer algebra of $\mathcal{H}_N^{(4)}(q)$.*

Let us mention that although the spectrum of the spin chains related to the general Temperley-Lieb R-matrix was found by the fusion procedure and a functional Bethe Ansatz [12], it would be nice to get the corresponding eigenvectors. Also the subject of reconstructing algebras from their representation ring structure is actively discussed in the literature (see e.g [28]).

Acknowledgement

It is a pleasure to thank the organizers for having arranged this nice Baltic-Nordic Workshop. The useful discussions with P. Etingof, A. Mudrov and A. Stolin are highly appreciated. This research was partially supported by RFBR grant 06-01-00451.

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Received December 20, 2007
Revised March 10, 2008