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Modified Variational Iteration Method for the Numerical Solutions of some Non-Linear Fredholm Integro-Differential Equations of the Second Kind

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Abstract

This paper provides approximate solutions to some nonlinear Fredholm-Integro differential equations of the second kind by using a Modified Variational Iteration Method. Comparison of the approximate solutions of this method with other known methods shows that the Modified Variational Iteration scheme is more accurate, reliable and readily implemented.

Keywords: Modified variational iteration method; Nonlinear Fredholm-integro-differential equations of the second kind

Introduction

Many researchers in engineering and physical sciences have used different numerical methods to solve Fredholms Integro-differential equations. Many of these numerical methods gave reliable and accurate solutions [1] applied multi-wavelet direct method for solving integro -differential equations; Ghasemi et al. [2] used Homotopy perturbation method to solve integrodifferential equations [3-6]. Maleknejad et al. [7] use integral mean value theorem II [8-10] adopted Bernstein collocation method find approximate solution in Fredholm Integro-differential equation, and Jianhua et al. [4] used Hybrid Function Operational Matrix techniques in Solving Fredholm Integro-differential Equations. Lakestani et al. used spline wavelets method to solve the integro-differential equations, Rashidinia and Tahmasebi. Used modified Taylor expansion Method in solving Fredholm intgro-differential equations and Shahooth et al. use Bernstein Polynomials Method. In this paper, nonlinear Fredholm integro-differential equations of the second kind where solved by modified variational iteration method which uses few numbers of iterations. Numerical examples and graphical results will demonstrate the efficiency of the method and will be shown that the method is accurate and readily implemented compared to some existing methods.

Non-Linear Fredholm Integro-Differential Equations

Consider the general non-linear, second kind Fredholm Integro-Differential equations of the form:

$$u^{(n)}(x)=f_i(x)+\sum^{m}\lambda_1'\left(K_l(x,t)F\left(u(t)\right)\right)dt,u^k(0)=c_k,0\le k\le(n-1) \quad (1)$$

$u^{(n)}(x)$ indicate the n-th derivatives of $u(x)$, c_k are constants that represent the initial conditions and $F(u(t))$ is non-linear. $u(x)$, $fi(x)$, assumed to be real and, λ_1' is real finite constants F, f_i and k_i are continuous functions and is the unknown function to be determined.

Derivation of Modified Variational Iteration Method

To illustrate the basic concepts of Modified Variational Iteration Method, we consider the differential equation:

$$Lu+Nu=g(x) \quad (2)$$

L, N are linear, nonlinear operators respectively and are the inhomogeneous term. The variational iteration method presents a correction functional for eqn. (2) in the form:

$$u_{n+1}(x)=u_n(x)+\int_a^x\lambda(\xi)\left(Lu_n(\xi)+N\bar{u}_n(\xi)-g(\xi)\right)d\xi \quad (3)$$

λ a general Lagrange's multiplier, which can be identified optimally via variational theory, that is, integration by parts and by using a restricted variation.

Determining the Lagrange's multiplier, this can be identified optimally via integration by parts and by using a restricted variation.

Setting $Lu_n(\xi)=u'(\xi)$ (4)

$$\int_a^x\lambda(\xi)\left(u_n'(\xi)\right)d\xi=\lambda(\xi)u_n(\xi)-\int_a^x(\xi)\left(u_n(\xi)\right)d\xi \quad (5)$$

$$\left.\begin{aligned}&\int_a^x\lambda(\xi)\left(u_n''(\xi)\right)d\xi=\lambda(\xi)u_n'(\xi)-\lambda'(\xi)\left(u_n(\xi)\right)+\int_a^x\lambda''(\xi)\left(u_n(\xi)\right)d\xi\\&\int_a^x\lambda(\xi)\left(u_n'''(\xi)\right)d\xi=\lambda(\xi)u_n''(\xi)-\lambda'(\xi)\left(u_n(\xi)+\lambda''(\xi)u_n(\xi)\right)-\int_a^x\lambda'''(\xi)\left(u_n(\xi)\right)d\xi\\&\int_a^x\lambda(\xi)\left(u_n^{iv}(\xi)\right)d\xi=\lambda'(\xi)u_n'''(\xi)-\lambda'(\xi)\left(u_n''(\xi)+\lambda''(\xi)u_n'(\xi)-\lambda'''(\xi)u_n(\xi)\right)+\int_a^x\lambda^{iv}(\xi)\left(u_n(\xi)\right)d\xi\end{aligned}\right\} \quad (6)$$

The generalized integration by parts is

$$\int_a^x\lambda(\xi)\left(u_n^{\,n}(\xi)\right)d\xi=\lambda(\xi)u_n^{\,n-1}(\xi)-\lambda'(\xi)u_n^{\,n-2}+\lambda'(\xi)u_n^{\,n-3}(\xi)-\ldots-$$

$$(-1)^n\int_a^x\lambda^n(\xi)u_n(\xi)d\xi$$

Noting that in this method may be a constant or a function, and is a restricted value that means it behaves as a constant, is considered as restricted variation, i.e., where is the variational derivative. The extremum condition of requires that and this yields the stationary conditions:

$$1+\lambda\big|_{\xi=x}=0,\lambda'\big|_{\xi=x}=0 \quad \text{hence } \lambda=-1 \quad (7)$$

The successive approximations u_{n+1},` $n\ge0$ of the solution u(x) will be readily obtained upon using selective function $u_n(x)$

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The Non-linear term is expressed in a unique way that gives a better approximation than the Adomian polynomial, Bell polynomial, Orthogonal polynomial just to mention a few. Considering a special case of eqn. (1) as:

$$u^{(n)}(x)+f(x)+\lambda\int_{g(x)}^{h(x)}k(x,t)u^{(k)}(t)u^{(m)}(t)dt=g(x) \quad (8)$$

Subject to the initial conditions $u^{(r)}(0)=c_r$ where c_r , r=0,1…(n-1) are real constant and k, m are integers with k ≤m≤n.

In solving the general nth-order nonlinear integro-differential equations, we consider the following general functional equation of the form:

Lu=f+N(u)

Where N is the Non-linear differential operator, f is a known analytical function, $N(u_n)$ is the nonlinear operator which is decomposed as

$$N\left(\sum_{i=0}^{\infty}u_i\right)=N(u_0)+\sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i}u_j\right)-N\left(\sum_{j=0}^{i}u_j\right)\right\} \quad (9)$$

u_j are polynomials of x,

$$N(u_0+u_1+u_2+.....+)=N(u_0)+\sum_{i=1}^{\infty}\left\{N(u_0+u_1+......+u_i)-N(u_0+u_1+......+u_{i-1})\right\}$$

The recurrence relations are defined as

$$\left.\begin{aligned}&u_0=f\\&u_1=N(u_o)\\&u_2=N(u_0+u_1)-N(u_0)\\&u_3=N(u_0+u_1+u_2)-N(u_0)\\&\quad ,\\&u_{n+1}=N(u_0+u_1+....+u_n)-N(u_0+u_1+....+u_{n-1})\,n=1,2....\end{aligned}\right\} \quad (10)$$

Assume a series solution of the form:

$$u=\sum_{j=0}^{n-1}u_i \quad (11)$$

The non-linear term in eqn. (3) can be written as $N\tilde{u}_n(\xi)=Nu_n(\xi)$

The n-th term approximate solution in eqn. (10) is $u_0+u_1+....+u_{n+1}=N(u_0+u_1+...+u_n)$

$$u=\sum_{n=0}^{n-1}u_n(x)$$

Apply L^{-1} to the recurrence relation for the determination of the components, the (n+1)th approximation of the exact solutions for the unknown functions u(x) is obtained as

$$u_{n+1}(x)=N(u_0+u_1+.....u_n)-N(u_0+u_1+.....u_{n-1})=L^{-1}N(u_0+u_1+.....u_n)-L^{-1}N(u_0+u_1+.....u_{n-1})$$

The solution is constructed as:

$$u(x)=L^{-1}\sum_{n=0}^{n-1}u_n(x),n\geq 0 \quad (12)$$

$$u_{n+1}(x)=u_n(x)+\int_a^x\lambda(\xi)\left(Lu_n(\xi)+N\tilde{u}(\xi)-g(\xi)\right)d\xi \quad (13)$$

The modified algorithms is formulated as

$$u_{n+1}(x)=u_n(x)+\int_a^x\lambda(\xi)\left(Lu_n(\xi)+L^{-1}\sum_{n=0}^{n-1}u_n(\xi)\right)d\xi \quad (14)$$

Numerical Examples

In this section, some numerical examples are given to illustrate the accuracy and effectiveness properties of the method and MAPLE 17 package is used to carry-out the calculation. The absolute errors used is defined as $\left|u(x)-u_n\right|;a\leq x\leq b,\ n=0,1,2......,u(x)$ is the exact solution and u_n is the approximate solution. The numerical solutions of this method will be compared with the numerical solutions of other known methods.

Example 1

Consider the first-order Fredholm integro-differential eqns. (2) and (15)

$$u^{(')}(x)=\frac{5}{4}-\frac{x^2}{2}+\int_0^1(x^2-t)u^2(t)dt \quad (15)$$

With the initial conditions u(0)=0 and exact solution u(x)=x, the correction functional for eqn. (15) is constructed as

$$u_{n+1}(x)=u_n(x)+\int_a^x\lambda(\xi)\left[Lu_n(\xi)-\frac{5}{4}+\frac{x^2}{3}-l^{-1}\sum_{j=0}^{i}(\xi^2-r)\tilde{u}_j^{\,2}(r))dr\right]d\xi$$

And making the functional stationary and noting that, $\tilde{u}_n$ is a restriction variation, $\delta\tilde{u}_n=0$. To find the optimal λ(ξ) and calculate variation with respect to u_n, we have the stationary Conditions by applying eqns. (4) and (5):

$\delta u_n:1+\lambda|_{\xi=x}=0$ and $\delta u_n:\lambda'|_{\xi=x}=0$

The Lagrange multiplier can be identified as λ=-1

$$u_{n+1}(x)=u_n(x)-\int_a^x\left[Lu_n(\xi)-\frac{5}{4}+\frac{x^2}{3}-l^{-1}\sum_{j=0}^{i}(\xi^2-r)\tilde{u}_j^{\,2}(r))dr\right]d\xi$$

$u_0(x)=0$

Consequently, we have the following approximations (Table 1)

$u_1(x)=1.250000000x-0.1111111111x^3$

$u_3(x)=1.031957657x-0.01481615250x^3$

$u_3(x)=1.031957657x-0.01481615250x^3$

$u_4(x)=0.9888349561x+0.00518701277x^3$

$u_5(x)=1.003838294x-0.001782108797x^3$

$u_6(x)=0.9986730892x+0.000616215870x^3$

Example 2

Consider the following nonlinear system of third-order Fredholm integrodifferential equation (Table 2)

$$u^{(''')}(x)=\sin(x)-x-\int_0^{\frac{\pi}{2}}xtu'(t)dt \quad (16)$$

With the initial conditions $u(0)=1,u'(0)=1,u''(0)=-1$ for $x\in\left[0,\frac{\pi}{2}\right]$. The exact solution u(x)=cos(x) the correction functional for (16) is constructed as

$$u_{n+1}(x)=u_n(x)+\int_a^x\lambda(\xi)\left[Lu_n(\xi)-\sin(\xi)+\xi+l^{-1}\sum_{j=0}^{i}(\xi r)\left(\tilde{u}_j'(r)\right)dr\right]d\xi$$

And making the functional stationary and noting that, $\tilde{u}_n$ is a restriction variation, $\delta\tilde{u}_n=0$. To find the optimal λ(ξ) and calculate variation with respect to u_n, we have the stationary conditions by applying in eqns. (4) and (5):

$\delta u_n:\lambda'''|_{\xi=x}=0\ \ \delta u_n:\lambda''|_{\xi=x}=0\ \ \delta u_n:1+\lambda'|_{\xi=x}=0$ and

$\delta u_n:\lambda|_{\xi=x}=0$

The Lagrange multiplier can be identified as

$$\lambda=-\frac{1}{2!}(\xi-x)^2$$

X	Exact	Bernstein Polynomials Method (BPM)		MVIM	
		Approximate Solution n=32	Absolute Error	Approximate Solution n=6	Absolute Error
0.0	0.0	0.0000	0.00E+00	0.00000	0.00E+00
0.1	0.1	0.08810	1.19E-02	0.09987	1.32E-04
0.2	0.2	0.17802	2.20E-02	0.19974	2.60E-04
0.3	0.3	0.26784	3.22E-02	0.29962	3.81E-04
0.4	0.4	0.35861	4.14E-02	0.39951	4.91E-04
0.5	0.5	0.45067	4.93E-02	0.49941	5.86E-04
0.6	0.6	0.54433	5.57E-02	0.59934	6.63E-04
0.7	0.7	0.63991	6.01E-02	0.69928	7.17E-04
0.8	0.8	0.73773	6.23E-02	0.79925	7.46E-04
0.9	0.9	0.83811	6.19E-02	0.89926	7.45E-04
1.0	1.0	0.99935	6.50E-04	0.99929	7.11E-04

Table 1: Computations showing comparison of results for example 1.

X	Direct Redial Basis Function Method (DRBFM) n=15	MVIM n=3
5.0	6.823E-01	2.60E+01
10.0	1.197E+04	4.17E+02
15.0	1.336E+03	2.11E+03

Table 2: Error for example 2.

$$u_{n+1}(x)=u_n(x)-\int_a^x \frac{1}{2!}(\xi-x)^2\left[Lu_n(\xi)-\sin(\xi)+\xi+L^{-1}\sum_{j=0}^{i}(\xi r)(\tilde{u}_j(r))dr\right]d\xi$$

$$u_0(x)=1-\frac{x^2}{2}$$

Consequently, we have the following approximations and errors presented in Table 2.

$$u_1(x)=\cos(x)-0.04166666667x^4+0.008971723581x^7$$

$$u_2(x)=\cos(x)-0.04166666667x^4+0.007137527979x^7$$

$$u_3(x)=\cos(x)-0.04166666667x^4+0.007550623256x^7$$

Example 3

We seek the solution of the third order Non-linear Fredholm integro differential equation of the second kind (Figure 1).

$$u'''(x)=xe^x+\frac{1079}{360}e^x+\frac{1}{120}\int_0^1 e^{(x-2t)}u^2(t)\,dt \quad (17)$$

for x∈ $x\in[0,1]$ with the initial condition u(0)=2,u'(0)=1,u''(0)=2

The correction functional for eqn. (17) is constructed as

$$u_{n+1}(x)=u_n(x)+\int_a^x \lambda(\xi)\left[Lu_n(\xi)-\xi e^{\xi}-\frac{1}{120}L^{-1}\sum_{j=0}^{i}e^{(\xi-2r)}\tilde{u}_j^2(r))dr\right]d\xi$$

and making the functional stationary and noting that, $\tilde{u}_n$ is a restriction variation, $\delta\tilde{u}_n=0$. To find the optimal $\lambda(\xi)$ and calculate variation with respect to u_n, we have the stationary conditions by applying in eqns. (4) and (5):

$$\lambda(\xi)(u_n''(\xi))d\xi=\lambda(\xi)u_n^{n-1}(\xi)-\lambda'(\xi)u_n^{n-2}(\xi)+\lambda''(\xi)u_n^{n-3}(\xi)(-1)^n\int_a^x\lambda^n(\xi)u_n(\xi)d\xi$$

For n=3

$$\delta u_{n+1}=\delta u_n+\delta\lambda(\xi)u_n''(\xi)-\lambda'(\xi)\delta u_n'+\delta\lambda''(\xi)u_n-\int_a^x\lambda'''\delta(u_n)d\xi$$

$$\delta u_{n+1}=\delta u_n(\xi)\left(1+\lambda''\big|_{\xi=x}\right)+\delta\lambda(\xi)u_n''(\xi)-\lambda'(\xi)\delta u_n'-\int_a^x\lambda'''\delta(u_n)d\xi$$

$$\delta u_n:\lambda'''=0,\delta u_n:1+\lambda''\big|_{\xi=x}=0,\lambda'\big|_{\xi=x}=0\,and\,\delta u_n:\lambda\big|_{\xi=x}=0$$

Using the natural conditions, we have $\lambda''\big|_{\xi=x}=-1$

Applying as a natural condition, the Lagrange multiplier can be identified as $\lambda=-\frac{1}{2!}(\xi-x)^2$

Using the initial condition to obtain the zeroth approximation, we have u_0=x+x^2 consequently, we have the following approximations (Figure 1):

$$\begin{aligned}u_1={}&5.997222222x+2.998611111x^2+5.997222222\\&-0.000315941756x^3-0.000315941756x^4\\&-0.000157970878x^5-0.01562500000x^3e^{-2+x}\\&+0.002430555556e^x x^3+xe^x-5.997222222e^x\end{aligned}$$

$$\begin{aligned}u_2={}&5.997222222x+2.998611111x^2+5.997222222\\&-0.0002156748356x^3-0.0002156748356x^4\\&-0.0001078374178x^5-0.4313917376x^3e^{-2+x}\\&-0.05112536359e^5x^3-1.000000000xe^x-5.997222222e^x\\&-6.00000000010^{-12}x^2e^x+0.2897948689x^3e^{-1+x}\\&+0.06254608315x^3e^{-3+x}+4.84406002010^{-8}x^3e^{-4+x}\end{aligned}$$

$$\begin{aligned}u_3={}&5.997222221e^x+4.000000000010^{-9}e^{-2+x}\\&-2.00000000010^{-9}e^{-1+x}-2.00000000010^{-10}e^{-3+x}\\&+5.997222221-2.40000000010^{-9}e^{-2+x}x+5.997222222x\\&-2.00000000010^{-10}e^{-3+x}x+0.99999999996e^x x\\&+6.000000000\,10^{-10}e^{-1+x}x+2.998611111x^2\\&+1.00000000010^{-10}e^{-3+x}x^2+6.00000000010^{-10}e^{-2+x}x^2\\&+7.761846675\,10^{-7}e^{-6+x}x^3-0.2515520023e^{-4+x}x^3\\&+0.9341450524e^{-1+x}x^3+1.827832614e^{-3+x}x^3\\&-0.00001090193898\,e^{-5+x}x^3-2.212800948e^{-2+x}x^3\\&+4.655737595\,10^{-19}e^{-8+x}x^3+1.202289606\,10^{-12}e^{-7+x}x^3\\&-0.1303620642\,e^x x^3-0.0002156728717x^3\\&-0.0002156728715x^4-0.000107836435714x^3\end{aligned}$$

Conclusion

In this paper, numerical methods for approximating solution of

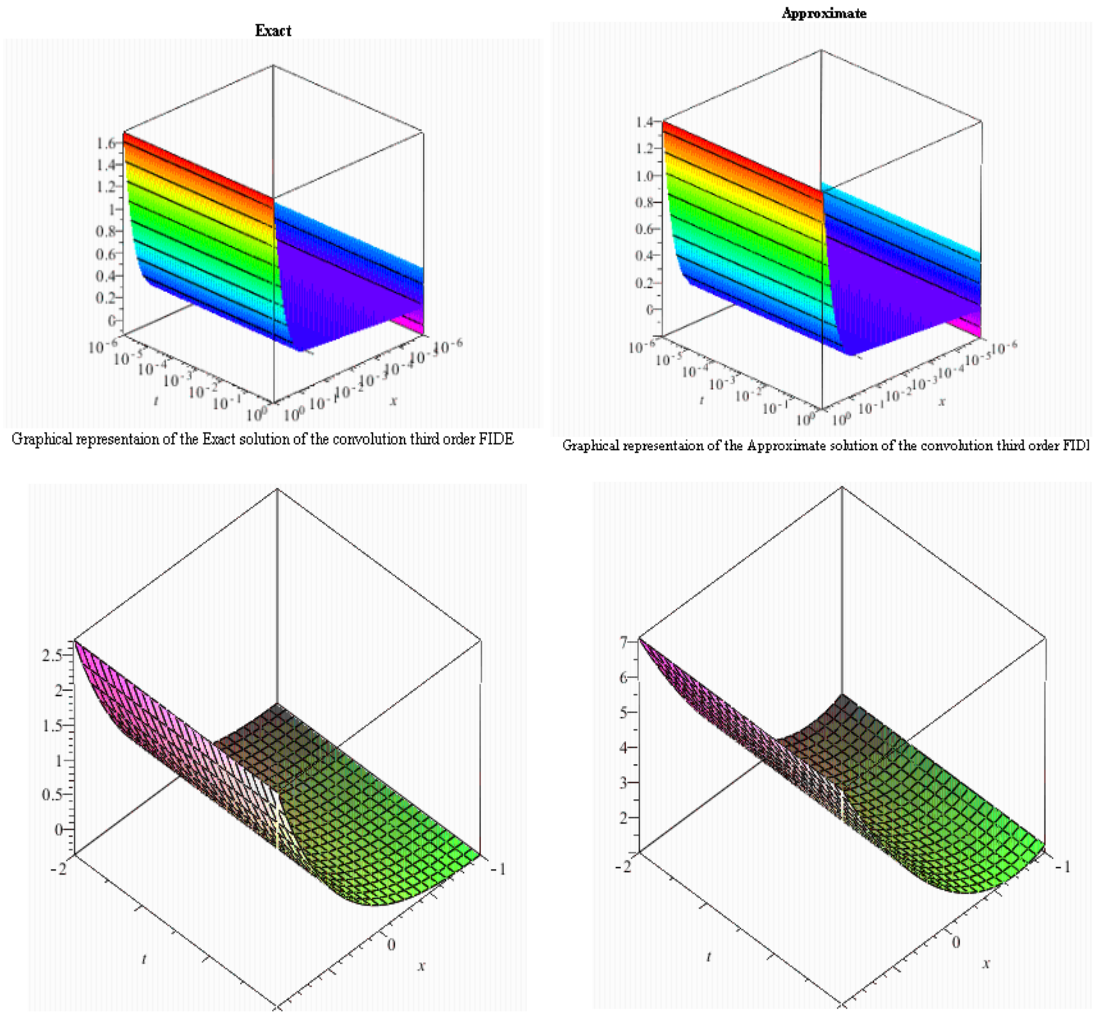

Figure 1: Graphical representation of the exact solution of the convolution third order FIDE.

Non-linear Fredholm-integral differential equation of the second kind are considered by using the modified vibrational iteration (MVIM) and this:

- Help to reduce some inherited problems and weakness associated with other method as outlined in literatures.
- Comparison of the approximate, exact solutions shows that MVIM is more an efficient tool and more practical for solving non-linear systems of integral-differential equations and plot confirm.
- It will now be possible to investigate the approximate solution of nonlinear applied problems, particularly of the nonlinear problems in dynamic model of a chemical reactor and the present method reduces the computational difficulty of other traditional methods and all the calculation are made simple.

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