Modules Over Color Hom-Poisson Algebras

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Abstract

In this paper we introduce color Hom-Poisson algebras and show that every color Hom-associative algebra has a non-commutative Hom-Poisson algebra structure in which the Hom-Poisson bracket is the commutator bracket. Then we show that color Poisson algebras (respectively morphism of color Poisson algebras) turn to color Hom-Poisson algebras (respectively morphism of Color Hom-Poisson Algebras) by twisting the color Poisson structure. Next we prove that modules over color Hom-associative algebras A extend to modules over the color Hom-Lie algebras $L(A)$, where $L(A)$ is the color Hom-Lie algebra associated to the color Hom-associative algebra A . Moreover, by twisting a color Hom-Poisson module structure map by a color Hom-Poisson algebra endomorphism, we get another one.

Keywords: Color hom-associative algebras; Color hom-Lie algebras; Homomorphism; Formal deformation; Hom-modules; Modules over color Hom-Lie algebras; Modules over color Hom-Poisson algebras

Introduction

Color Hom-Poisson algebras are generalizations of Hom-Poisson algebras introduced in [1], where they emerged naturally in the study of 1-parameter formal deformations of commutative Hom-associative algebras. Color Hom-Poisson algebras generalize, on the one hand, color Hom-associative [2,3] and color Hom-Lie algebras [2,3] which have been recently investigated by various authors. On the other hand, they generalize Hom-Lie superalgebras [4]. These structures are well-known to physicists and to mathematicians studying differential geometry and homotopy theory. The cohomology theory of Lie superalgebras [5] has been generalized to the cohomology of Hom-Lie superalgebras in [6]. A cohomology of color Lie algebras was introduced and investigated in [7], and the representations of color Lie algebras were explicitly described in [8]. Modules over Poisson algebras receive various definitions [9,10] we will use the one introduced in [9]. The aim of this paper is to study color Hom-Poisson algebras and modules over color Hom-Poisson algebras. The paper is organized as follows. In section 4, we recall some basic notions related to color Hom-associative algebras and color Hom-Lie algebras. In section 5, we define color Hom-Poisson algebras and point out that to any color Hom-associative algebra one can associate a color Hom-Poisson algebra. Next, starting from a color Poisson algebra and Poisson algebra morphism we get another one by twisting the associative product and Lie bracket. In section 6, we introduce modules over color Hom-Poisson algebras and prove that starting from a color Hom-Poisson module we get another one by twisting the module structure map by a Hom-Poisson algebra endomorphism. All vector spaces considered are supposed to be over fields of characteristics different from 2.

Preliminaries

Let G be an abelian group. A vector space V is said to be a G -graded if, there exist a family $(V_a)_a \in G$ of vector subspaces of V such that

$$V = \bigoplus_{a \in G} V_a$$

An element $x \in V$ is said to be homogeneous of degree $a \in G$ if $x \in V_a$. We denote $H(V)$ the set of all homogeneous elements in V .

Let $V = \bigoplus_{a \in G} V_a$ and $V' = \bigoplus_{a \in G} V'_a$ be two G -graded vector spaces. A linear mapping $f: V \rightarrow V'$ is said to be homogeneous of degree b if

$$f(V_a) \subseteq V_{a+b}, \forall a \in G$$

If f is homogeneous of degree zero i.e. $f(V_a) \subseteq V_a$ holds for any $a \in G$ then f is said to be even.

An algebra (A, μ) is said to be G -graded if its underlying vector space is G -graded i.e. $A = \bigoplus_{a \in G} A_a$ and if furthermore $\mu(A_a, A_b) \subseteq A_{a+b}$ for all $a, b \in G$.

Let A' be another G -graded algebra. A morphism $f: A \rightarrow A'$ of G -graded algebras is by definition an algebra morphism from A to A' which is, in addition, an even mapping.

Definition

Let G be an abelian group. A map $\varepsilon: G \times G \rightarrow K^*$ is called a skew-symmetric bicharacter on G if the following identities hold,

1. $\varepsilon(a, b) \varepsilon(b, a) = 1$
2. $\varepsilon(a, b+c) = \varepsilon(a, b) \varepsilon(a, c)$
3. $\varepsilon(a+b, c) = \varepsilon(a, c) \varepsilon(b, c)$

$a, b, c \in G$

Remark that

$$\varepsilon(a, 0) = \varepsilon(0, a) = 1, \varepsilon(a, a) = \pm 1 \text{ For all } a \in G$$

Where, 0 is the identity of G . If x and y are two homogeneous elements of degree a and b respectively and ε is a skew-symmetric bicharacter, then we shorten the notation by writing $\varepsilon(x, y)$ instead of $\varepsilon(a, b)$.

Definition

A color Hom-associative algebra is a quadruple $(A, \mu, \varepsilon, \alpha)$ consisting of a G -graded vector space A , an even bilinear map $\mu: A \times A \rightarrow A$ and an even linear map such $\alpha: A \rightarrow A$ that

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Received August 12, 2013; Accepted September 30, 2014; Published October 06, 2014

Citation: Bakayoko I (2014) Modules Over Color Hom-Poisson Algebras. J Generalized Lie Theory Appl 8: 212. doi: [10.4172/1736-4337.1000212](https://doi.org/10.4172/1736-4337.1000212)

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$$\mu(\alpha(x), \alpha(y)) = \alpha(\mu(x, y)) \quad (1)$$

$$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z)) \quad (2)$$

If in addition $\mu(x, y) = \varepsilon(x, y) \mu(y, x)$ the color Hom-associative algebra $(A, \mu, \varepsilon, \alpha)$ is said to be a ε -commutative color Hom-associative algebra.

Remark

When $\alpha = \text{Id}$ we recover the classical associative color algebra.

Recall that the Hom-associator, as_A of a Hom-algebra A is defined as $as_A: A \otimes A \otimes A \rightarrow A$, 1

Observe that $as_A = 0$ when A is a color-Hom-associative algebra.

Definition

Let $(A, \mu, \varepsilon, \alpha)$ and $(A', \mu', \varepsilon', \alpha')$ be two color Hom-associative algebras. An even linear map $f: A \rightarrow A'$ is said to be a morphism of color Hom-associative algebras if $f \circ \alpha = \alpha' \circ f$ and

$$f(\mu(x, y)) = \mu'(f(x), f(y))$$

For all $x, y \in A$.

Lemma

([17]) Let (A, μ, ε) be a color associative algebra and α be an even algebra endomorphism. Then $(A, \mu\alpha, \varepsilon, \alpha)$ where $\mu_\alpha = \alpha \circ \mu$ is a color Hom-associative algebra. Moreover, suppose that (A', μ', ε) be another color associative algebra and $\alpha': A' \rightarrow A'$ be an even algebra endomorphism such that $f \circ \alpha = \alpha' \circ f$ then $f: (A, \mu', \varepsilon, \alpha) \rightarrow (A', \mu', \varepsilon, \alpha')$ is also a morphism of color Hom-associative algebras.

Definition

([17]) A color Hom-Lie algebra is a quadruple $(A, \{\cdot, \cdot\}, \varepsilon, \alpha)$ consisting of a G-graded vector space A , an even bilinear map

$\{\cdot, \cdot\}: AXA \rightarrow A$ (i.e. $\{A_a, A_b\} \subseteq A_{a+b}$ for all $a, b \in G$) a bicharacter, and an even linear map $\alpha: A \rightarrow A$ such that for any $x, y \in H(A)$ we have

$$\{x, y\} = -\varepsilon(x, y) \{y, x\} \quad (\varepsilon\text{-Skew-symmetry}) \quad (3)$$

$$\alpha \circ \{\cdot, \cdot\} = \{\cdot, \cdot\} \circ \alpha^{\otimes 2} \quad (\text{Multiplicativity}) \quad (4)$$

$$\oint \varepsilon(z, x) \{ \alpha(x), \{y, z\} \} = 0 \quad (\varepsilon\text{-Hom-Jacobi identity}) \quad (5)$$

Where $\oint$ means cyclic summation.

By the ε skew symmetry 3 of the color Hom-Lie bracket $\{\cdot, \cdot\}$, the color Hom-Jacobi identity 5 is equivalent to

$$\{ \{x, y\}, \alpha(z) \} = \{ \alpha(x), \{y, z\} \} - \varepsilon(x, y) \{ \alpha(y), \{x, z\} \} \quad (6)$$

Remark that a color Lie algebra $(A, \{\cdot, \cdot\}, \varepsilon)$ is a color Hom-Lie algebra with $\alpha = \text{Id}$

Morphism of color Hom-Lie algebras are defined similarly to the Definition 4.3, where the color Hom-associative product is replaced by the color Hom-Lie bracket. Examples of color Hom-Lie algebras are provided in [2,3].

The following lemma connects color Hom-associative algebras to color Hom-Lie algebras.

Lemma

([17]) Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra.

Then $(A, \{\cdot, \cdot\} = \mu - \varepsilon(\cdot, \cdot) \mu^{op}, \varepsilon, \alpha)$ is a color Hom-Lie algebra, denoted by $L(A)$.

Color Hom-Poisson algebras

Definition

A color Hom-Poisson algebra consists of a G-graded vector space A , a multiplication $\mu: AXA \rightarrow A$, an even bilinear bracket $\{\cdot, \cdot\}: AXA \rightarrow A$ and an even linear map $\alpha: A \rightarrow A$ such that

1. $(A, \mu, \varepsilon, \alpha)$ is a color Hom-associative algebra,
2. $(A, \{\cdot, \cdot\}, \varepsilon, \alpha)$ is a color Hom-Lie algebra,
3. the color Hom-Leibniz identity is satisfied i.e.

$$\{ \alpha(x), \mu(y, z) \} = \mu(\{x, y\}, \alpha(z)) + \varepsilon(x, y) \mu(\alpha(y), \{x, z\}), \quad (7)$$

For $x, y, z \in H(A)$ any

If in addition μ is ε commutative, the color Hom-Poisson algebra $(A, \{\cdot, \cdot\}, \varepsilon, \alpha)$ is said to be a ε commutative color Hom-Poisson algebra.

The condition 7 expresses the compatibility between the color Hom-associative product μ and the color Hom-Lie bracket $\{\cdot, \cdot\}$ it can be written equivalently

$$\{ \mu(x, y), \alpha(z) \} = \mu(\alpha(x), \{y, z\}) + \varepsilon(x, y) \mu(\alpha(y), \{x, z\}). \quad (8)$$

Remark

We recover Poisson algebras ([6, 5]) when $\alpha = \text{Id}$ and $\varepsilon = 1$

We need the following lemma in Proposition 6.1.

Lemma

If $(A, \mu, \{\cdot, \cdot\}, \varepsilon, \alpha)$ is a ε commutative color Hom-Poisson algebra, then $(A, -\mu - \{\cdot, \cdot\}, \varepsilon, \alpha)$ is also a ε commutative color Hom-Poisson algebra.

The following theorem is the color version of ([11], Proposition 4.6).

Theorem

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra.

Then $(A, \mu, \{\cdot, \cdot\} = \mu - \varepsilon(\cdot, \cdot) \mu^{op}, \varepsilon, \alpha) \sqrt{b^2 - 4ac}$ is a color Hom-Poisson algebra.

Proof:

According to Lemma 4.2, it remains to prove the color Hom-Leibniz identity 7. For any $x, y, z \in H(A)$

$$\begin{aligned} & \{ \alpha(x), \mu(y, z) \} - \mu(\{x, y\}, \alpha(z)) - \varepsilon(x, y) \mu(\alpha(y), \{x, z\}) = \\ & = \mu(\alpha(x), \mu(y, z)) - \varepsilon(x, y = z) \mu(\mu(x, y), \alpha(z)) \\ & = \varepsilon(x, y) \mu(\mu(y, x), \alpha(z)) - \varepsilon(x, y) \mu(\alpha(y), \mu(x, z)) + \varepsilon(x, y) \varepsilon(x, z) \mu(\alpha(y), \mu(z, x)) \\ & = \mu(\alpha(x), \mu(y, z)) - \mu(\mu(x, y), \alpha(z)) - \varepsilon(x, y + z) \mu(\mu(y, z), \alpha(x)) \\ & + \varepsilon(x, y) \mu(\mu(y, x), \alpha(z)) + \varepsilon(x, y) \mu(\mu(y, x), \alpha(z)) - \varepsilon(x, y) \mu(\alpha(y), \mu(z, x)) \\ & = -as_{\mu, \alpha}(x, y, z) - \varepsilon(x, y) \varepsilon(x, z) as_{\mu, \alpha}(y, z, x) + \varepsilon(x, y) as_{\mu, \alpha}(y, x, z) = 0 \end{aligned}$$

This finishes the proof.

Corollary

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon, \alpha)$ be a color associative algebra and α an even color algebra endomorphism. Then $(A, \mu_\alpha, \{\cdot, \cdot\}_\alpha, \varepsilon, \alpha)$ where

$\mu_a = \alpha \circ \mu, \{\cdot, \cdot\}_a = \mu_a - \varepsilon(\cdot, \cdot) \mu_a^{op}$ is a color Hom-Poisson algebra.

Proof

The proof follows from Lemma 4.1 and Theorem 3.1.

Lemma

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon)$ be a color Poisson algebra and α be an even color Poisson algebra endomorphism. Then $(A, \mu_a, \{\cdot, \cdot\}_a, \varepsilon, \alpha)$ is a color Hom Poisson algebra.

Proof

By Lemma 4.1 and ([3] Example1.2), we only need to prove the color Hom-Leibniz identity. For any $x, y, z \in H(A)$

$$\begin{aligned} & \{\alpha(x), \mu_a(y, z)\}_a - \mu_a(\{x, y\}_a, \alpha(z)) - \varepsilon(x, y) \mu_a(\alpha(y), \{x, z\}_a) = \\ & = \alpha^2(x), \alpha^2(\mu(y, z)) - \mu(\alpha^2(\{x, y\}), \alpha^2(z)) - \varepsilon(x, y) \mu(\alpha^2(y), \alpha^2(\{x, z\})) \\ & = \alpha^2(\{x, \mu(y, z)\}) - \mu(\{x, y\}, z) - \varepsilon(x, y) \mu(y, \{x, z\}) \\ & = 0 \end{aligned}$$

This completes the proof.

Theorem

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon, \alpha)$ be a color Hom-Poisson algebra and $\beta: A \rightarrow A$ be an even color Poisson algebra endomorphism. Then, $A_\beta = (A, \mu_\beta = \beta \circ \mu, \{\cdot, \cdot\}_\beta = \beta \circ \{\cdot, \cdot\}, \varepsilon, \beta \circ \alpha)$ is a color Hom-Poisson algebra.

Moreover, suppose that $(A', \mu', \{\cdot, \cdot\}', \varepsilon)$ is a color Poisson algebra and $(A, \{\cdot, \cdot\}_\beta, \varepsilon, \beta \circ \alpha)$ is an even color Poisson algebra endomorphism. If $f: A \rightarrow A'$ is a color Poisson algebra morphism that satisfies $f \circ \beta = \alpha' \circ f$, then

$$f: (A, \mu_\beta, \{\cdot, \cdot\}_\beta, \varepsilon, \beta \circ \alpha) \rightarrow (A', \mu'_\alpha, \{\cdot, \cdot\}'_\alpha, \varepsilon, \alpha)$$

is a color Hom-Poisson algebra homomorphism.

Proof

It is straightforward to show that $(A, \mu_\beta, \varepsilon, \beta \circ \alpha)$ is a color Hom associative algebra and $(A, \{\cdot, \cdot\}_\beta, \varepsilon, \beta \circ \alpha)$ is a color Hom-Lie algebra ([3] Theorem1.1). The proof of the color Hom-Leibniz identity is similar to that of Lemma 5.2. For the second assertion, we have

$$\begin{aligned} f(\mu_\beta(x, y)) &= f(\mu(\beta(x), \beta(y))) = \mu'(f(\beta(x)), f(\beta(y))) \\ &= \mu'(\alpha'(f(x)), \alpha'(f(y))) = \mu'_\alpha(f(x), f(y)). \end{aligned}$$

We have a similar proof for the color Hom-Poisson bracket.

Corollary 5.2

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon, \alpha)$ be a color Hom-Poisson algebra. Then

$$A^n = (A, \{\cdot, \cdot\}^{(n)} = \alpha^n \circ \{\cdot, \cdot\}, \mu^{(n)} = \alpha^n \circ \mu, \varepsilon, \alpha^{n+1})$$

is a color Hom-Poisson algebra for each integer $n \geq 0$. We finish this section by studying deformations by composition of color HomPoisson algebras.

Definition 5.2

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon, \alpha)$ be a color Hom-Poisson algebra. A one parameter formal deformation of A is given by $K[[t]]$ -bilinear maps $\mu_t(\cdot, \cdot): A[[t]] \times A[[t]] \rightarrow A[[t]]$ and $\{\cdot, \cdot\}_t: A[[t]] \times A[[t]] \rightarrow A[[t]]$ of the

form $\mu_t = \sum_{i \geq 0} t^i \mu_i$ and $\{\cdot, \cdot\}_t = \sum_{i \geq 0} t^i \{\cdot, \cdot\}_i$ where each μ_i and $\{\cdot, \cdot\}_i$ are K-bilinear maps and (extended to $K[[t]]$ -bilinear maps), and such that for all following conditions be satisfied.

$$\mu_t(\alpha_t(x), \alpha_t(y)) = \alpha_t(\mu_t(x, y)), \quad (9)$$

$$\mu_t(\alpha_t(x), \mu_t(y, z)) = \mu_t(\mu_t(x, y), \alpha_t(z)), \quad (10)$$

$$\alpha_t \circ \{\cdot, \cdot\}_t = \{\cdot, \cdot\}_t \circ \alpha_t^{\otimes 2}, \quad (11)$$

$$\{x, y\}_t = -\varepsilon(x, y) \{y, x\}_t, \quad (12)$$

$$\oint \varepsilon(z, x) \{\alpha_t(x), \{y, z\}_t\}_t = 0, \quad (13)$$

$$\{\alpha_t(x), \mu_t(y, z)\}_t = \mu_t(\{x, y\}_t, \alpha_t(z)) + \varepsilon(x, y) \mu_t(\alpha_t(y), \{x, z\}_t). \quad (14)$$

The deformation is said to be of order k if $\mu_t = \sum_{i=0}^k t^i \mu_i$ and $\{\cdot, \cdot\}_t = \sum_{i=0}^k t^i \{\cdot, \cdot\}_i$

Proposition 5.1

Let $(A, \mu, \{\cdot, \cdot\}, \varepsilon)$ be a color Poisson algebra and α_t an even color Poisson algebra endomorphism of the form $\alpha_t = \alpha_0 + \sum_{i=0}^k t^i \alpha_i$ where α_i are endomorphism of A (as color Poisson algebra), t is a parameter in K and k is an entiger. Let $\mu_t = \mu$ and $\{\cdot, \cdot\}_t = \alpha_t \circ \{\cdot, \cdot\}$ then $(A, \mu_t, \{\cdot, \cdot\}_t, \varepsilon, \alpha_t)$ is a color Hom-Poisson algebra which is a deformation of the color Poisson algebra $(A, \mu, \{\cdot, \cdot\}, \varepsilon)$ viewed as a color Hom-Poisson algebra $(A, \mu, \{\cdot, \cdot\}, \varepsilon, Id)$.

Proof

The proof follows from Theorem.2.

As in the case of Poisson algebras ([10,12,13]), the cohomology of color Hom-Poisson algebras is described by the cohomology of the underlying color Hom-Lie algebras ([3]).

Modules Over Color Hom-Poisson Algebras

Definition 6.1

Let G be an abelian group. A Hom-module is a pair (M, α_M) in which M is a G-graded vector space and $\alpha_M: M \rightarrow M$ is an even linear map.

Definition 6.2

Let $(A, M_A, \varepsilon, \alpha_A)$ be a color Hom-associative algebra. An A-module is a Hom-module (M, α_M) together with a bilinear map $\mu_M: A \otimes M \rightarrow M$ called structure map, such that

$$\mu_M(A_a \cdot M_b) \subseteq M_{a+b} \quad (15)$$

$$\alpha_M \circ \mu_M = \mu_M \circ (\alpha_A \otimes \alpha_M), \quad (16)$$

$$\mu_M \circ (\alpha_A \otimes \mu_M) = \mu_M \circ (\mu_A \otimes \alpha_M) \quad (17)$$

Twisting a module structure map by an algebra endomorphism, we get another one as stated in the following Lemma.

Lemma 6.1

Let $(A, M_A, \varepsilon, \alpha_A)$ be a color Hom-associative algebra and M an A-module with structure map $\mu_M: A \otimes M \rightarrow M$ Define the map

$$\tilde{\mu}_M = \mu_M \circ (\alpha_A^2 \otimes Id_M): A \otimes M \rightarrow M. \quad (18)$$

Then $\tilde{\mu}_M$ is the structure map of another A-module structure on M.

Proof

The proof is similar to that of ([14], Lemma 4.5).

Definition 6.3

([3]) Let $(L, \{.,.\}, \varepsilon, \alpha_L)$ be a color Hom-Lie algebra and (M, α_M) a Hom-module. An L-module on M consists of a K-bilinear map $\mu_M : A \otimes M \rightarrow M$ such that

$$\mu_M(A_a, M_b) \subseteq M_{a+b} \quad (19)$$

$$\alpha_M(\mu_M(x, m)) = \mu_M(\alpha_L(x), \alpha_M(m)),$$

$$\mu_M(\{x, y\}, \alpha_M(m)) = \mu_M(\alpha_L(x), \mu_M(y, m)) \quad (20)$$

$$- \varepsilon(x, y) \mu_M(\alpha_L(y), \mu_M(x, m)), \quad (21)$$

for any $m \in H(M), x, y \in H(L)$

Remark 6.1

When $\alpha_M = Id_M$ and $\alpha = Id_L$ we recover the definition of Lie modules ([15-17]).

The following statement is the Lie analogue of Lemma 6.1.

Lemma 6.2

Let $(L, \{.,.\}, \varepsilon, \alpha_L)$ be a color Hom-Lie algebra and M an L-module with structure map $\mu_M = L \otimes M \rightarrow M$. Define the map

$$\tilde{\mu}_M = \mu_M \circ (\alpha^2 \otimes Id_M) : L \otimes M \rightarrow M \quad (22)$$

Then $\tilde{\mu}_M$ is the structure map of another L-module structure on M.

Proof

Equalities 19 and 20 are proved as in Lemma 6.1. Now, we prove

21 for $\tilde{\mu}_M$. For any $x, y \in L, m \in M$

$$\begin{aligned} \tilde{\mu}_M(\{x, y\}, \alpha_M(m)) &= \mu_M(\alpha^2 \otimes Id)(\{x, y\}, \alpha_M(m)) \\ &= \mu_M(\{\alpha^2(x), \alpha^2(y)\}, \alpha_M(m)) \\ &= \mu_M(\alpha^3(x), \mu_M(\alpha^2(y), m)) - \varepsilon(x, y) \mu_M(\alpha^3(y), \mu_M(\alpha^2(x), m)) \\ &= \mu_M(\alpha^3(x), \mu_M(\alpha^2 \otimes Id)(y \otimes m)) \\ &\quad - \varepsilon(x, y) \mu_M(\alpha^3(y), \mu_M(\alpha^2 \otimes Id)(x \otimes m)) \\ &= \mu_M(\alpha^2(\alpha(x), \tilde{\mu}_M(y \otimes m)) - \varepsilon(x, y) \mu_M(\alpha^2(\alpha(y)), \tilde{\mu}_M(x \otimes m)) \\ &= \mu_M(\alpha^2 \otimes Id)(\alpha(x) \otimes \tilde{\mu}_M(y \otimes m)) \\ &\quad - \varepsilon(x, y) \mu_M(\alpha^2 \otimes Id)(\alpha(y) \otimes \tilde{\mu}_M(x \otimes m)) \\ &= \tilde{\mu}_M(\alpha(x), \tilde{\mu}_M(y, m)) - \varepsilon(x, y) \tilde{\mu}_M(\alpha(y), \tilde{\mu}_M(x, m)). \end{aligned}$$

Hence the conclusion holds.

The following result shows that A-modules extend to L(A)-modules with samemodule structure map.

Theorem 6.1

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra and (M, α_M) an A-module with structure map μ_M . Then, M is a L(A)-module with structure map μ_M .

Proof

In fact, it suffices to show the relation 21. For any $x, y \in H(A), m \in H(M)$, We have

$$\begin{aligned} &\mu_M(\alpha(x), \mu_M(y, m)) - \varepsilon(x, y) \mu_M(\alpha(y), \mu_M(x, m)) \\ &= \mu_M(\mu(x, y), \alpha_M(m)) - \varepsilon(x, y) \mu_M(\mu(y, x), \alpha_M(m)) \\ &= \mu_M(\mu(x, y) - \varepsilon(x, y) \mu(y, x), \alpha_M(m)) \\ &= \mu_M(\{x, y\}, \alpha_M(m)). \end{aligned}$$

This establishes the Theorem.

The corollaries below give a large class of examples of L(A)-modules.

Corollary 6.1

Let $A_\alpha = (A, \mu_\alpha, \varepsilon, \alpha)$ be a color Hom-associative algebra as in Lemma 4.1 and (M, α_M) an A_α module with structure map μ_M . Then, M is an L(A)-module with structure map μ_M .

Proof

Prove 21. Indeed, for $x, y \in H(A), m \in H(M)$ we have

$$\begin{aligned} &\mu_M(\alpha(x), \mu_M(y, m)) - \varepsilon(x, y) \mu_M(\alpha(y), \mu_M(x, m)) \\ &= \mu_M(\mu_\alpha(x, y), \alpha_M(m)) - \varepsilon(x, y) \mu_M(\mu_\alpha(y, x), \alpha_M(m)) \\ &= \mu_M(\mu_\alpha(x, y) - \varepsilon(x, y) \mu_\alpha(y, x), \alpha_M(m)) \\ &= \mu_M(\{x, y\}, \alpha_M(m)). \end{aligned}$$

We conclude that M is an L(A)-module with structure map μ_M .

Corollary 6.2

Let $(A, \mu, \varepsilon, \alpha)$ be a color Hom-associative algebra and (M, α_M) an A-module with structure map μ_M . Put

$$\tilde{\mu}_M = \mu_M \circ (\alpha^2 \otimes Id_M)$$

Then M is an L(A)-module with structure map $\tilde{\mu}_M$.

Proof

We know from Lemma (6.1) that $\tilde{\mu}_M$ is an A-module structure map. And, for $x, y \in H(A), m \in H(M)$, one has

$$\begin{aligned} &\tilde{\mu}_M(\alpha(x), \tilde{\mu}_M(y, m)) - \varepsilon(x, y) \tilde{\mu}_M(\alpha(y), \tilde{\mu}_M(x, m)) \\ &= \mu_M(\alpha^2(\alpha(x)), \mu_M(\alpha^2(y), m)) - \varepsilon(x, y) \mu_M(\alpha^2(\alpha(y)), \mu_M(\alpha^2(x), m)) \\ &= \mu_M(\alpha(\alpha^2(x)), \mu_M(\alpha^2(y), m)) - \varepsilon(x, y) \mu_M(\alpha(\alpha^2(y)), \mu_M(\alpha^2(x), m)) \\ &= \mu_M(\mu(\alpha^2(x), \alpha^2(y)), \alpha_M(m)) - \varepsilon(x, y) \mu_M(\mu(\alpha^2(y), \alpha^2(x)), \alpha_M(m)) \\ &= \mu_M(\alpha^2(\mu(x, y) - \varepsilon(x, y) \mu(y, x)), \alpha_M(m)) \\ &= \mu_M(\alpha^2(\{x, y\}), \alpha_M(m)) \\ &= \tilde{\mu}_M(\{x, y\}, \alpha_M(m)). \end{aligned}$$

This is similar to the relation 21 for $\tilde{\mu}_M$.

Now we define modules for color Hom-Poisson algebras.

Definition 6.4

Let $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ be a ε commutative color Hom-Poisson algebra and (M, α_M) a Hom-module.

A color Hom-Poisson module structure on M consists of two K -bilinear maps $\mu_M : A \otimes M \rightarrow M$ and $\lambda_M : A \otimes M \rightarrow M$ such that

- (i) M is an A -module and an L -module,
- (ii) And for any $x, y \in H(A), m \in H(M)$,

$$\lambda_M(\alpha(x), \mu_M(y, m)) = \mu_M(\{x, y\}, \alpha_M(m)) + \varepsilon(x, y) \mu_M(\alpha(y), \lambda_M(x, m)), \quad (23)$$

$$\lambda_M(\mu(x, y), \alpha_M(m)) = \mu_M(\alpha(x), \lambda_M(y, m)) + \varepsilon(x, y) \mu_M(\alpha(y), \lambda_M(x, m)). \quad (24)$$

When $\alpha = Id_A, \alpha_M = Id_M$ and $\varepsilon \equiv 1$

We recover the definition of modules over Poisson algebras ([9]).

Example 6.1

(i) Any module over a ε commutative color Hom-associative algebra (resp. color Hom-Lie algebra) can be seen as a module over a ε commutative color Hom-Poisson algebra with the trivial color Hom-Lie bracket (resp. trivial color Hom-associative product).

(ii) Any ε commutative color Hom-Poisson algebra is a module over itself.

Example 6.2

Let $(V, \mu_V, \lambda_V, \alpha_V)$ and $(W, \mu_W, \lambda_W, \alpha_W)$ be two modules over the ε Commutative color Hom-Poisson algebra $(A, \mu, \{.,.\}, \varepsilon, \alpha)$

Then the direct product $M = V \times W$ is a module over A with structure maps $\mu_M : A \otimes M \rightarrow M, \lambda_M : A \otimes M \rightarrow M$ and $\alpha_M : M \rightarrow M$

Defined by

$$\mu_M(x(v, w)) = (\mu_V(x, v), \mu_W(x, w)), \lambda_M(x, (v, w)) = (\lambda_V(x, v), \lambda_W(x, w)) \text{ and } \alpha_M(v, w) = (\alpha_V(v), \alpha_W(w))$$

for any $x \in H(A), v \in H(V)$ and $w \in H(W)$.

Proposition 6.1

If $(M, \mu_M, \lambda_M, \alpha_M)$ is a module over the ε commutative color Hom-Poisson algebra $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ then $(M, -\mu_M, -\lambda_M, \alpha_M)$ is also a module

Over the ε commutative color Hom-Poisson algebra $(A, -\mu, -\{.,.\}, \varepsilon, \alpha)$

Proof

The proof comes from Definition 6.4 and Lemma 5.1.

Theorem 6.2

Let $(A, \mu, \{.,.\}, \varepsilon, \alpha)$ be a ε commutative color Hom-Poisson algebra and $(M, \mu_M, \lambda_M, \alpha_M)$ color Hom-Poisson module. Then

$$\tilde{\mu}_M = \mu_M \circ (\alpha^2 \otimes Id_M) : A \otimes M \rightarrow M, \quad (25)$$

$$\tilde{\lambda}_M = \lambda_M \circ (\alpha^2 \otimes Id_M) : A \otimes M \rightarrow M, \quad (26)$$

Define another color Hom-Poisson module structure on M .

Proof

We know that $\tilde{\mu}_M$ is a structure of another A -module structure

map on M (Lemma 6.1) and $\tilde{\lambda}_M$ is a structure of another L -module structure map on M (Lemma 6.2). Show relations 23 and 24 for $\tilde{\mu}_M$ and $\tilde{\lambda}_M$. For all $x, y \in H(A)$ and $m \in H(M)$

$$\begin{aligned} \tilde{\lambda}_M(\alpha \otimes \tilde{\mu}_M)(x \otimes y \otimes m) &= \tilde{\lambda}_M(\alpha(x), \tilde{\mu}_M(y \otimes m)) \\ &= \tilde{\lambda}_M(\alpha(x), \mu_M(\alpha^2 \otimes Id)(y \otimes m)) \\ &= \lambda_M(\alpha^2 \otimes Id)(\alpha(x), \mu_M(\alpha^2(y) \otimes m)) \\ &= \lambda_M(\alpha^3, \mu_M(\alpha^2 \otimes m)) \\ &= \mu_M(\{\alpha_A^2(x), \alpha_A^2(y)\}, \alpha_M(m)) \\ &\quad + \varepsilon(x, y) \mu_M(\alpha_A^3(y), \lambda_M(\alpha_A^2(x), m)) \quad (\text{by (23)}) \\ &= \mu_M(\alpha^2(\{x, y\}), \alpha_M(m)) \\ &\quad + \varepsilon(x, y) \mu_M(\alpha^2 \otimes Id)(\alpha(y), \lambda_M(\alpha^2 \otimes Id)(x \otimes m)) \\ &= \mu_M(\alpha^2 \otimes Id)(\{x, y\} \otimes \alpha_M(m)) + \varepsilon(x, y) \tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x \otimes m)) \\ &= \tilde{\mu}_M(\{x, y\} \otimes \alpha_M(m)) + \varepsilon(x, y) \tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x \otimes m)). \end{aligned}$$

And, for $x, y \in H(A)$ and $m \in H(M)$

$$\begin{aligned} \tilde{\lambda}_M(\mu \otimes \alpha_M)(x \otimes y \otimes m) &= \lambda_M(\alpha^2 \otimes Id)(\mu(x \otimes y) \otimes \alpha_M(m)) \\ &= \lambda_M(\alpha^2(\mu(x \otimes y)) \otimes \alpha_M(m)) \\ &= \mu_M(\alpha^3(x), \lambda_M(\alpha^2(y), m)) \\ &\quad + \varepsilon(x, y) \mu_M(\alpha^3(y), \lambda_M(\alpha^2(x), m)) \quad (\text{by (24)}) \\ &= \mu_M(\alpha^2 \otimes Id)(\alpha(x), \lambda_M(\alpha^2 \otimes Id)(y \otimes m)) \\ &\quad + \varepsilon(x, y) \mu_M(\alpha^2 \otimes Id)(\alpha(y), \lambda_M(\alpha^2 \otimes Id)(x \otimes m)) \\ &= \tilde{\mu}_M(\alpha(x), \tilde{\lambda}_M(y \otimes m)) + \varepsilon(x, y) \tilde{\mu}_M(\alpha(y), \tilde{\lambda}_M(x \otimes m)). \end{aligned}$$

Hence equations 23 and 24 hold for $\tilde{\mu}_M$ and $\tilde{\lambda}_M$. This completes the proof.

Corollary 6.3

Let $(A, \mu, \{.,.\}, \varepsilon)$ be a ε commutative color Poisson algebra and $(M, \mu_M, \lambda_M, \alpha_M)$ a module over the color Hom-Poisson algebra $(A, \mu_\alpha = \alpha \circ \mu, \{.,.\}_\alpha = \alpha \circ \{.,.\}, \varepsilon, \alpha)$. Then $\tilde{\mu}_M, \tilde{\lambda}_M$ define another color Hom-Poisson module structure on M .

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