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New Approximate Bayesian Confidence Intervals for the Shape and the Scale Parameters of the Two Parameter Gamma Distribution

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Abstract

The aim of the present study is to construct confidence intervals for the shape and the scale parameters of the two-parameter Gamma Distribution. Using the square error loss function, closed form approximate Bayesian confidence intervals are derived.

Numerical results show that the obtained Approximate Bayesian models rely only on the observations under study and have great coverage accuracy.

Keywords: Estimation; Loss function; Confidence interval; Central limit theorem; Statistical analysis

Introduction

Bayesian analysis implies the exploitation of suitable prior information and the choice of a loss function in association with Bayes' Theorem. It rests on the notion that a parameter within a model is not merely an unknown quantity but rather behaves as a random variable which follows some distribution. In the area of life testing, it is indeed realistic to assume that a life parameter is stochastically dynamic. This assertion is supported by the fact that the complexity of electronic and structural systems is likely to cause undetected component interactions resulting in an unpredictable fluctuation of life parameters. Drake gave an excellent account for the use of Bayesian statistics in reliability problems. As he points out "He (Bayesian) realizes that his selection of a prior (distribution) to express his present state of knowledge will necessarily be somewhat arbitrary. But he greatly appreciates this opportunity to make his entire assumptive structure clear to the world…" "Why should an engineer not use his engineering judgment and prior knowledge about the parameters in the statistical distribution he has picked? For example, if it is the mean time between failures (MTBF) of an exponential distribution that must be evaluated from some tests, he undoubtedly has some idea of what the value will turn to be.

Life testing in reliability has received a substantial amount of interest from theorists as well as reliability engineers. Their concern was a product of the increased complexity and sophistication in electronic and structural systems, which came into existence very rapidly during this time. In the early 1950's, Epstein and Sobel began to explore the field of parametric life testing. Under the assumption of an exponential time-to-failure distribution, they produced a series of papers (1953, 1954, and 1955) which were to influence future work in reliability and life parameter testing.

Shortly thereafter other failure distributions more complex than the exponential were used as failure models. For example, Kao brought attention to the Weibull probability distribution, while Birnhaum and Saunders suggested the gamma distribution. In this study, we will use the gamma probability as our underlying model. The gamma distribution is defined as follows: that is defined as follows:

$$f(x) = \frac{1}{\Gamma(\alpha)\,\beta^{\alpha}} x^{\alpha-1} e^{-\frac{x}{\beta}} \qquad (1)$$

$$0 \prec x \prec +Infinity, \qquad \alpha \succ 0, \qquad \beta \succ 0.$$

where α is the shape parameter and β is the scale parameter.

The Gamma distribution is widely used to model positive and continuous variables having skewed distributions.

The following estimator is frequently used to obtain point estimates of the shape parameter corresponding to the two-parameter Gamma distribution:

$$(\bar{x}/s)^2 \qquad (2)$$

The estimator listed below is also frequently used to obtaining point estimates of the shape parameters corresponding to the two-parameter Gamma distribution:

$$\frac{s^2}{\bar{x}} \qquad (3)$$

The Square Error loss function will be used to derive our Approximate Bayesian models that are presented in this research paper...

The "popular" Square Error loss function places a small weight on estimates that are near the true value and proportionately more weight on extreme deviation from the true value of the parameter. Its popularity is due to its analytical tractability in Bayesian modeling. The Square Error loss is defined as follows:

the Square Loss function that is defined as follows:

$$L_{SE}(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2 \qquad (4)$$

Methodology

Considering the Square Error Loss function, the following Approximate Bayesian confidence bounds for the variance [1,2], the mean [3-7] and the coefficient of variation [8-10], of a Normal distribution will be used along with the Central Limit Theorem, to

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construct Approximate Bayesian confidence bounds for the Shape and the Scale of a Gamma Distribution:

$$L_{\sigma^2(SE)} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-2-2\ln(\alpha/2)} \tag{5}$$

$$U_{\sigma^2(SE)} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-2-2\ln(1-\alpha/2)} \tag{6}$$

$$U_{U_\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}+\bar{x}^2-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)}\right)^{0.5} \tag{7}$$

$$L_{\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}+\bar{x}^2-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)}\right)^{0.5} \tag{8}$$

$$U_{Cv}(SE) == \left(\frac{n-1}{\left(1+\left(\frac{\bar{x}}{s}\right)^2\right)\left(n-2-2Ln(1-\alpha/2)\right)-n+1}\right)^{0.5} \tag{9}$$

$$L_{Cv}(SE) = \left(\frac{n-1}{\left(1+\left(\frac{\bar{x}}{s}\right)^2\right)\left(n-2-2Ln(\alpha/2)\right)-n+1}\right)^{0.5} \tag{10}$$

Per the Central Limit Theorem, for large samples, the distribution corresponding to the sample mean $\bar{x}$ is approximately Normal, irrespective of the distribution of X.

Therefore, with large samples, to derive our new Approximate Bayesian confidence bounds for the shape parameter of a Gamma distribution, we will use the Central Limit Theorem along with the above Approximate Bayesian confidence bounds for the variance and the mean of a Gaussian distribution.

Using equations (4) and (5), we have the followings:

$$\frac{\sum_{i=1}^{n}(\bar{x}-\mu)^2}{n-2-2\ln(\alpha/2)} \prec \frac{\sigma^2}{n} \prec \frac{\sum_{i=1}^{n}(\bar{x}-\mu)^2}{n-2-2\ln(1-\alpha/2)}$$

$$\frac{n\sigma^2}{n-2-2\ln(\alpha/2)} \prec \sigma^2 \prec \frac{n\sigma^2}{n-2-2\ln(1-\alpha/2)} \tag{11}$$

Thus, for large samples, we have the following Approximate Bayesian bounds for the population variance and the population mean of a Gamma distribution

Approximate Bayesian interval for the variance: $\alpha\beta^2$ when the Gamma population mean is known.

$$L_{(SE)} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-2-2\ln(\alpha/2)}$$

$$U_{(SE)} = \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{n-2-2\ln(1-\alpha/2)}$$

Approximate Bayesian interval for the variance $\alpha\beta^2$ of a Gamma distribution, when the Gamma population mean is not known.

$$L_{(SE)} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)}$$

$$U_{(SE)} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)}$$

Approximate Bayesian interval for the population mean $\alpha\beta$ of a Gamma distribution.

$$L_{\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}+\bar{x}^2-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)}\right)^{0.5}$$

$$UU_{\mu(SE)} = \left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-1}+\bar{x}^2-\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)}\right)^{0.5}$$

Using the above mentioned confidence intervals, for the mean and the variance of a Gamma distribution, we can easily derive the following confidence bounds for the inverse of the coefficient of variation of the Gamma distribution:

$$L = \sqrt{\frac{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(1-\alpha/2)\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}-1}$$

$$U = \sqrt{\frac{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(\alpha/2)\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}-1}$$

Thus, the confidence bounds for the shape parameter of a Gamma distribution are the following:

$$L_\alpha = \frac{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(1-\alpha/2)\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}-1 \tag{12}$$

$$U_\alpha = \frac{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(\alpha/2)\right)}{\sum_{i=1}^{n}(x_i-\bar{x})^2}-1 \tag{13}$$

Confidence bounds for the scale and the rate parameters

To derive the Approximate Bayesian confidence bounds for the scale parameter of the two-parameter Gamma distribution, we will use the Central limit along the obtained Approximate Bayesian confidence bounds for the variance and the shape parameter of the Gamma distribution (Equations 11 and12).

Using the confidence bounds corresponding to the variance of the Gamma distribution:

$$L_{\sigma^2(SE)} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)}$$

$$U_{\sigma^2(SE)} = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)}$$

Along with the confidence bounds corresponding to the inverse of the shape parameter

$$L = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(\alpha/2)\right)-\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

$$U = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(1-\alpha/2)\right)-\sum_{i=1}^{n}(x_i-\bar{x})^2}$$

We can easily obtain the following confidence bounds for the square of the scale parameter:

$$L = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(\alpha/2)}\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(\alpha/2)\right)-\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

$$U = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{n-2-2\ln(1-\alpha/2)}\left(\frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(1-\alpha/2)\right)-\sum_{i=1}^{n}(x_i-\bar{x})^2}\right)$$

Hence, the Approximate Bayesian confidence bounds for the scale parameter of the Gamma distribution are:

$$L = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sqrt{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(\alpha/2)\right)^2-\left(n-2-2\ln(\alpha/2)\right)\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

$$U = \frac{\sum_{i=1}^{n}(x_i-\bar{x})^2}{\sqrt{\left(s^2+\bar{x}^2\right)\left(n-2-2\ln(1-\alpha/2)\right)^2-\left(n-2-2\ln(1-\alpha/2)\right)\sum_{i=1}^{n}(x_i-\bar{x})^2}}$$

Numerical Results

For the numerical results, we will use large Gamma data from the collection of SAS data sets.

Example 1 SAS data

0.746, 0.357, 0.376, 0.327, 0.485, 1.741, 0.241, 0.777 0.768, 0.409, 0.252, 0.512, 0.534, 1.656, 0.742, 0.378, 0.714, 1.121, 0.597, 0.231, 0.541, 0.805, 0.682, 0.418, 0.506, 0.501, 0.247, 0.922, 0.880, 0.344, 0.519, 1.302, 0.275, 0.601, 0.388, 0.450, 0.845, 0.319, 0.486, 0.529, 1.547, 0.690, 0.676, 0.314, 0.736, 0.643, 0.483, 0.352, 0.636, 1.080

$\bar{x}$ =0.63362

S=0.098336807

SAS yielded the following point estimate of the shape and scale parameter (Table 1):

$(\bar{x}/s)^2$ =4.082646

SAS yields the following estimate of the shape and scale parameters:

Estimate of the scale parameter: $\frac{s^2}{\bar{x}}$ =0.155198

Estimate of the rate parameter: 6.443382 (Table 2).

Example 2 SAS data

620, 470, 260, 89, 388, 242, 103, 100, 39, 460, 284, 1285, 218, 393, 106, 158, 152, 477, 403, 103, 69, 158, 818, 947, 399, 1274, 32, 12, 134, 660, 548, 381, 203, 871, 193, 531, 317, 85, 1410, 250, 41, 1101, 32, 421, 32, 343, 376, 1512, 1792, 47, 95, 76, 515, 72,1585, 253, 6, 860, 89, 1055, 537, 101, 385, 176, 11, 565, 164, 16, 1267, 352, 160, 195, 1279, 356, 751, 500, 803, 560, 151, 24, 689, 1119, 1733, 2194, 763, 555, 14, 45, 776, 1.

$\bar{x}$ =468.74444444

S=475.927505248

Point estimate of shape parameter: 0.959264019 (Tables 3 and 4).

Example 3 SAS data

1747, 945, 12, 1453, 14, 150, 20, 41, 35, 69, 195, 89, 1090, 1868, 294, 96, 618, 44, 142, 892, 1307, 310, 230, 30, 403, 860, 23, 406, 1054, 1935, 561, 348, 130, 13, 230, 250, 317, 304, 79, 1793, 536, 12, 9, 256, 201, 733, 510, 660, 122, 27, 273, 1231, 182, 289, 667, 761, 1096, 43, 44, 87,

405, 998, 1409, 61, 278, 407, 113, 25, 940, 28, 848, 41,

646, 575, 219, 303, 304, 38, 195, 1061, 174, 377, 388, 10,

246, 323, 198, 234, 39, 308, 55, 729, 813, 1216, 1618, 539,

6, 1566, 459, 946, 764, 794, 35, 181, 147, 116, 141, 19,

380, 609, 546.

$\bar{x}$ =459.513513514

S=477.755849874

Point estimate of the shape parameter: 0.925091198 (Tables 5 and 6):

Exercise 4, Monte Carlo simulation has been used to generate the following 40 observations from a Gamma distribution.

4.5046, 8.9119, 66.7603, 0.2643, 6.1241, 30.5425, 29.7423, 60.2067, 16.3891, 8.2941, 52.0380, 0.1402, 12.6309, 19.4385, 22.8395, 52.3378, 3.4389, 19.3268, 8.2350, 3.4737, 56.0736, 22.6451, 0.8359, 7.3484, 7.7675, 15.3635, 4.05222, 36.2578, 5.6189, 8.7365, 7.6990, 15.3844, 23.2242, 11.8542, 63.6975, 14.8772, 32.9585, 2.2127, 5,4132, 44.2462

$\bar{x}$ =20.297643

S=19.3784360608

Point estimate of shape parameter: 1.097119087 (Tables 7 and 8).

Conclusion

In the present study, new and closed form Approximate Bayesian confidence intervals for the Shape and the Scale parameters of a Gamma Distribution have been obtained. The models that have been derived rely only on observations under study and have great coverage accuracy.

With these models, one may easily construct confidence intervals for the shape and the scale parameter of a Gamma distribution, at any level of significance.

Bayesian analysis contributes to reinforcing well-known statistical theories such as the Estimation and the Decision- making Theories.

C.L.%	Confidence bounds for the shape parameter
80	4.000775-4.456600
90	3.989559 -4.600397
95	3.984170- 4.744194
99	3.979958-5.078080

Table 1: Approximate Bayesian Confidence bounds for the shape parameter of the Gamma distribution.

C.L.%	Confidence bounds for the scale parameter
80	0.143864-0.158056
90	0.139282-0.158456
95	0.135428-0.158649
99	0.127254- 0.158800

Table 2: Confidence bounds for the scale parameter of the Gamma distribution.

C.L.%	Confidence bounds for the shape parameter
80	0.941889-1.038629
90	0.939508-1.069147
95	0.938365-1.099665
99	0.937471-1.170526

Table 3: Approximate Bayesian Confidence bounds for the shape parameter of the Gamma distribution.

C.L.%	Confidence bounds for the scale parameter
80	460.377645- 495.337918
90	450.400759-496.269320
95	440.867642-496.718142
99	420.281633-497.06955
Estimate of scale parameter: 488.650085.	

Table 4: Confidence bounds for the scale parameter of the Gamma distribution.

C.L.%	Confidence bounds for the shape parameter
80	0.9112782-0.988185
90	0.909386-1.012446
95	0.908477-1.036707
99	0.907766-1.093040

Table 5: Approximate Bayesian Confidence bounds for the shape parameter of the Gamma distribution.

C.L.%	Confidence bounds for the scale parameter
80	472.916267-502.278094
90	464.390820-503.049553
95	456.183203-503.421082
99	438.252898-503.711915
Estimate of the scale parameter: 496.7223931.	

Table 6: Confidence bounds for the scale parameter of the Gamma distribution.

C.L.%	Confidence bounds for the shape parameter
80	1.054678-1.290977
90	1.048863-1.365522
95	1.046070-1.440066
99	1.043886-1.613152

Table 7: Approximate Bayesian Confidence bounds for the shape parameter of the Gamma distribution.

C.L.%	Confidence bounds for the scale parameter
80	16.317761-19.063310
90	15.614121-19.143194
95	14.970570-19.181819
99	13.668167-19.212127
Estimate of the scale parameter: 18.50085668.	

Table 8: Confidence bounds for the scale parameter of the Gamma distribution.

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