New Exact Non-relativistic Energy Eigen Values for Modified Inversely Quadratic Hellmann Plus Inversely Quadratic Potential

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Abstract
In this current research, the solutions of modified Schrödinger equation (MSE) are presented for two companion potentials namely: modified inversely quadratic Hellmann potential and modified inversely quadratic potential (MIHQP), using generalization of Bopp’s shift method (instead to solving MSE with star product) and standard perturbation theory in extended quantum mechanics (EQM), we obtained modified Hamiltonian operator and corresponding modified eigenvalues in both three dimensional noncommutative space and phase (NC-3D: RSP) symmetries.

Keywords: Inversely quadratic hellmann; Inversely quadratic potential; Non-commutative space and phase; Star product

Introduction
It is well known that, the ordinary Schrodinger equation is one of the fundamental wave equations in physics. Recently, considerable efforts have been made towards obtaining exact analytic solution of MSE for central potentials in two and three dimensional space in different fields of nuclear physics, spectroscopy, quantum chemistry and many fields of matter sciences to search an profound physical and chemical interpretations at Nano and Plank’s scales [1-9], and in particular our works in this context [10-29]. The algebraic physical structure of EQM based on the following fundamental four NC canonical commutations relations (NCCRs), in both Schrödinger and Heisenberg pictures (SP and HP), respectively, as \((c=\hbar=1)\) [11-12],

\[
\begin{align*}
\hat{p}_x &= i\hbar \frac{\partial}{\partial x} \quad \text{and} \quad \hat{p}_y = i\hbar \frac{\partial}{\partial y} \\
\hat{p}_z &= i\hbar \frac{\partial}{\partial z} \quad \text{and} \quad \hat{p}_t = i\hbar \frac{\partial}{\partial t}
\end{align*}
\]

(1)

The very small two parameters \(\theta^{+}\) and \(\theta^{(-)}\) (compared to the energy) from the following equations [6-15]

\[
\begin{align*}
\delta (f * g) (x,p) &= \frac{1}{2} \left( \theta^+ \delta_{xx} f \partial_x^2 \partial_x g + \theta^+ \delta_{yy} f \partial_y^2 \partial_y g + \theta^+ \delta_{zz} f \partial_z^2 \partial_z g \right) (x,p)
\end{align*}
\]

(2)

Where \(\delta (f * g) (x,p) = (f * g - f \cdot g) (x,p)\), the new canonical coordinates \((\hat{x}, \hat{t})\) and new momentum \(\hat{p}(t)\) are determined from two projection relations, respectively, as follows [16],

\[
\begin{align*}
\{\hat{x}(t), \hat{t}(t)\} &= \exp(\partial_{\hat{t}} (t-t_0)) \{\hat{x}(t), \hat{p}(t)\} \exp(-\partial_{\hat{t}} (t-t_0))
\end{align*}
\]

(3)

By differentiating equation (3), we find that general two operators \((\hat{x}(t), \hat{p}(t))\) obey the Heisenberg equations of motion [16]:

\[
\frac{d\hat{x}(t)}{dt} = \left[ H_{nc} , \hat{x}(t) \right] \quad \text{and} \quad \frac{d\hat{p}(t)}{dt} = \left[ H_{nc} , \hat{p}(t) \right]
\]

(4)

The formalism of star product, Bopp’s shift method and the Seiberg-Witten map were played crucial roles in EQM. The Bopp’s shift method will be applied in this paper instead of solving MSE in global group symmetry (GGS) (NC-3D: RSP), the MSE will be treated by using directly the two new commutators, in addition to usual ordinary commutators on quantum mechanics, in the both SP and HP representation, respectively [17-22],

\[
\begin{align*}
[\hat{p}_x, \hat{p}_y] &= i\hbar \delta_{xy} \quad \text{and} \quad [\hat{p}_x, \hat{p}_z] = i\hbar \delta_{xz} \quad \text{and} \quad [\hat{p}_y, \hat{p}_z] = i\hbar \delta_{yz}
\end{align*}
\]

(5)

the new two operators \(p_x\) and \(p_y\) in GGS (NC-3D: RSP) are dependent with ordinary operator \(\hat{x}\) and \(\hat{p}_i\) from the following projections relations:

\[
\begin{align*}
\delta \hat{x} &= \hat{x} - x = -\frac{\theta^+}{2} p_x - \frac{\theta^+}{2} p_z \\
\delta \hat{y} &= \hat{y} - y = -\frac{\theta^+}{2} p_x + \frac{\theta^+}{2} p_z \\
\delta \hat{z} &= \hat{z} - z = -\frac{\theta^+}{2} p_x - \frac{\theta^+}{2} p_y
\end{align*}
\]

(6)

and

\[
\begin{align*}
\delta \hat{p}_x &= \hat{p}_x - p_x = \frac{\theta^+}{2} y^{+} + \frac{\theta^+}{2} z^{+} \\
\delta \hat{p}_y &= \hat{p}_y - p_y = \frac{\theta^+}{2} x^{+} + \frac{\theta^+}{2} z^{+} \\
\delta \hat{p}_z &= \hat{p}_z - p_z = \frac{\theta^+}{2} x^{+} + \frac{\theta^+}{2} y^{+}
\end{align*}
\]

(7)

Thus, the non-vanish 9-commutators in SP including in GGS (NC-3D: RSP) can be determined as follows:

\[
\begin{align*}
[\hat{x}, \hat{p}_x] &= [\hat{x}, \hat{p}_y] = [\hat{x}, \hat{p}_z] = \frac{\theta^+}{2} \\
[\hat{y}, \hat{p}_x] &= [\hat{y}, \hat{p}_y] = [\hat{y}, \hat{p}_z] = \frac{\theta^+}{2} \\
[\hat{z}, \hat{p}_x] &= [\hat{z}, \hat{p}_y] = [\hat{z}, \hat{p}_z] = \frac{\theta^+}{2} \\
[\hat{p}_x, \hat{p}_y] &= [\hat{p}_x, \hat{p}_z] = [\hat{p}_y, \hat{p}_z] = \frac{\theta^+}{2}
\end{align*}
\]

(8)

And a similarly non-vanish 9-commutators in HP. The aim of this work is to study the MIHQP in NC 3-D space and phase to discover the new spectrum in this new symmetries from which other modified potentials are deduced as special cases, this potential plays an important role in many fields of physics such as molecular physics, solid state and chemical physics on based to the main reference [30,31] in QM and our previously works [10-29] in EQM. The rest of this search paper is

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organized as follows: in the next section we briefly present and review the basic of eigenvalues and eigenfunctions for IHQP in ordinary 3-D spaces. In section 3, we give a brief review of Bopp’s shift method, then we derive the spin-orbital NC Hamiltonians \( \hat{H}_{\mu,\nu} \) for MIHQP in GGS (NC-3D: RSP). We find the exact spectrum produced by \( \hat{H}_{\mu,\nu} \) by applying ordinary perturbation theory and then we deduce the exact spectrum produced by NC magnetic Hamiltonians \( \hat{H}_{\mu,\nu} \) for MIHQP in GGS (NC-3D: RSP). In section four we resume the global spectrum for MIHQP and the main results. Finally, section 5 is kept for conclusive remarks.

The IHQP in Ordinary 3-Dimensional Spaces

The purpose of this section is to give a briefly review of eigenvalues and eigenfunctions for ordinary IHQP on based to the main reference [30],

\[
V(r) = a \frac{1}{r} + \frac{1}{r}(b + g) \quad \text{……..…….} \tag{9}
\]

where \( r \) represents the internuclear distance, \( a \) and \( \delta \) are the strengths of the coulomb and Yukawa potentials, respectively, while \( \delta \) is the screening parameter. The ordinary SE with potential \( V(r) \) can be written in spherical coordinate \((r, \theta, \phi)\) as [30],

\[
\frac{d^2 R_{\mu}(r)}{dr^2} + 2\mu \left( E - V(r) - \frac{\lambda}{2\mu r^2} \right) R_{\mu}(r) = 0 \quad \text{……..…….} \tag{10}
\]

where \( \lambda = l(l+1) \), according NU method, the normalized energy eigenfunctions \( \Psi(r) \) and corresponding eigenvalues \( E \) for IHQP [30],

\[
\Psi(r) = N_e \left[ \begin{array}{c}
\frac{1}{r_1} \sin \theta_1 \\
\frac{1}{r_2} \sin \theta_2 \\
\frac{1}{r_3} \sin \theta_3
\end{array} \right] \left( 2\sqrt{r_1r_2r_3} \right) Y_{\mu,\nu}(\theta, \phi)
\]

\[
E = b \delta^2 + \frac{\mu(a + \delta^2)}{2} \left( a + \frac{1}{2} + \sqrt{2}(a + \delta^2) \right) \quad \text{……..…….} \tag{13}
\]

\[
R_{\mu}(r) = 2\mu \left( E - V(r) - \frac{\lambda}{2\mu r^2} \right) R_{\mu}(r) = 0 \quad \text{……..…….} \tag{12}
\]

\[
\Psi(r) = N_e \left[ \begin{array}{c}
\frac{1}{r_1} \sin \theta_1 \\
\frac{1}{r_2} \sin \theta_2 \\
\frac{1}{r_3} \sin \theta_3
\end{array} \right] \left( 2\sqrt{r_1r_2r_3} \right) Y_{\mu,\nu}(\theta, \phi)
\]

\[
E = b \delta^2 + \frac{\mu(a + \delta^2)}{2} \left( a + \frac{1}{2} + \sqrt{2}(a + \delta^2) \right) \quad \text{……..…….} \tag{13}
\]

NC 3-D Phase-Spaces GGS (NC-3D: RSP) Hamiltonians for MIHQP

Overview of the Formalism of Bopp’s shift method

We have been given a brief description of the MSE in GGS (NC-3D: RSP) on based to our previously works [22-27]. To achieve this goal, we apply the important 4-steps on the ordinary SE:

1-Ordinary 3-D Hamiltonian operator \( \hat{H}_{\mu,\nu}(p, x) \) will be replacing by new NC Hamiltonian operator \( \hat{H}_{\mu,\nu}(\hat{p}, \hat{x}) \).

2-Ordinary complex wave functions \( \psi(r) \) will be replacing by new complex wave function \( \psi(\hat{r}) \).

3-Ordinary energies \( E \) will be replaced by new values \( E_{\mu,\nu} \).

While, the last step corresponds to replace the ordinary old product by new star product \( (\hat{p}, \hat{x}) \), which allow us to constructing the MSE in GGS (NC-3D: RSP) as:

\[
H_{\mu,\nu}(\hat{p}, \hat{x}) \Psi(\hat{r}) = E_{\mu,\nu} \Psi(\hat{r}) \quad \text{………………..} \tag{14}
\]

The Bopp’s shift method allows finding the reduced above MSE without star product as:

\[
H_{\mu,\nu}(\hat{p}, \hat{x}) \Psi(\hat{r}) = E_{\mu,\nu} \Psi(\hat{r}) \quad \text{………………..} \tag{15}
\]

the modified Hamiltonian operator \( H_{\mu,\nu}(\hat{p}, \hat{x}) \) for MIHQP obtained by replace both \( \chi(x, y, z) \) and \( \rho_<(p, p_-, p_-) \) by new two operators \( i \) and \( \rho_\perp \), respectively, in usual quantum Hamiltonian operator \( H_{\mu,\nu}(p, p_-) \):

\[
\Psi(\hat{r}) = \hat{p}^2 + V(\hat{r}) \quad \text{………………..} \tag{17}
\]

On based to our references [26-29], we can write the two operators \( \hat{r}_3^2 \) and \( \hat{p}^2 \) in GGS (NC-3D: RSP) as follows:

\[
\hat{r}_3^2 = \hat{p}^2 + V(\hat{r}) \quad \text{………………..} \tag{18}
\]

We can observe that the above operator is proportional with \( \hat{r}_3 \) perturbation term.

The Spin-Orbital NC Hamiltonian Operator for MIHQP in GGS (NC-3D: RSP)

In order to discover the new contribution of perturbative term \( V_{pert,\perp}(r, \theta, \phi) \) for MIHQP, we turn to the case of spin \( \frac{1}{2} \) particles described by the MSE, we make the following two simultaneously transformations:
\[ \hat{\Theta} \rightarrow 2\hat{\Theta} \hat{S} \hat{L} \quad \text{and} \quad \hat{\Theta} \rightarrow 2\hat{\Theta} \hat{S} \hat{L} \] ……… (22)

Then the perturbed operator \( V_{\text{pert-ih}}(r, \Theta, \hat{\Theta}) \) becomes as:

\[ V_{\text{pert-ih}}(r, \Theta, \hat{\Theta}) = 2 \left( \frac{b + g}{r^4} - \frac{a + h\delta}{2r^2} + \frac{1}{2\mu} \right) \hat{L} \hat{S} \quad \text{……… (23)} \]

Here \( \hat{S} \) denote to the spin of a fermionic particle (like electron in Hydrogen atom). Now, it is possible to replace the spin-orbit interaction \( \hat{L} \hat{S} \) by \( G^2 = \frac{1}{2} \left( \hat{J}^2 - \hat{L}^2 - \hat{S}^2 \right) \) to obtain directly the corresponding eigenvalues, and then new physical \( V_{\text{pert-ih}}(r, \Theta, \hat{\Theta}) \) can be expressed as:

\[ V_{\text{pert-ih}}(r, \Theta, \hat{\Theta}) = \left( \frac{b + g}{r^4} - \frac{a + h\delta}{2r^2} + \frac{1}{2\mu} \right) \left( \hat{J}^2 - \hat{L}^2 - \hat{S}^2 \right) \] …….. (24)

As it well known, the 4-operators \( (\hat{J}^2, \hat{L}^2, \hat{S}^2) \) form a complete basis on QM, then the operator \( \hat{L} \hat{S} \) will be gives 2-eigenvalues \( k \pm \left( \frac{1}{2} \hat{J}^2 + \frac{1}{2} \hat{L}^2 \right)^{1/2} \), corresponding \( j = \pm \frac{1}{2} \) respectively [23-29]. Then, one can form a diagonal matrix \( \hat{H}_{\text{anh}} \) of order \((3 \times 3)\), with null non elements:

\[ (\hat{H}_{\text{anh}})_{ij} = \begin{pmatrix} \hat{H}_{\text{anh}}(1,1) & \hat{H}_{\text{anh}}(1,2) & \hat{H}_{\text{anh}}(1,3) \\ \hat{H}_{\text{anh}}(2,1) & \hat{H}_{\text{anh}}(2,2) & \hat{H}_{\text{anh}}(2,3) \\ \hat{H}_{\text{anh}}(3,1) & \hat{H}_{\text{anh}}(3,2) & \hat{H}_{\text{anh}}(3,3) \end{pmatrix} \]

for MIHQPs in GGS (NC-3D: RSP):

\[ (\hat{H}_{\text{anh}})_{n,n} = k_{\text{anh}} e^{-\frac{\delta}{2}} \hat{J} \quad \text{and} \quad (\hat{H}_{\text{anh}})_{n,n+1} = (\hat{H}_{\text{anh}})_{n,n-1} = 0 \] ……… (25)

After profound calculation, one can show that, the radial function \( R_n(r) \) for MIHQPs satisfying the following differential equation, in EQM structure of GGS (NC-3D: RSP):

\[ \frac{d^2 R_n(r)}{d r^2} + 2 \frac{d R_n(r)}{d r} + \left[ \left( \frac{a + h\delta}{\mu} \right) - \frac{1}{r^2} \right] R_n(r) = 0 \] …….. (26)

The Exact Spectrum Produced by NC Spin-Orbital Hamiltonian \( \hat{H}_{\text{anh}} \) for MIHQPs using Standard Perturbation Theory in GGS (NC-3D: RSP)

The aim of this subsection is to obtain the modifications to the energy levels for \( n^* \) excited states \( E_{n^*} \) and \( E_{n^*} \) corresponding a fermionic particle with two polarizations spin up and spin down, respectively, at first order of two infinitesimal parameters \( \Theta \) and \( \hat{\Theta} \). In order to achieve this goal, we apply the standard perturbation theory using eq. (14) for MIHQPs:

\[ E_n \pm \frac{1}{2} \left( T_{\text{anh}}(\hat{\Theta}) \right) \left( \frac{\Delta}{\mu} \right) = k_n \] …….. (27)

It is possible to write both \( E_{n^*} \) and \( E_{d^*} \) as functions of three factors \( T_{\text{anh}} \), \( T_{\text{anh}} \), and \( \hat{T}_{\text{anh}} \) as follow:

\[ E_{n^*} = \frac{1}{2} \left( T_{\text{anh}} + \hat{T}_{\text{anh}} \right) + \frac{\hat{\Theta}}{\mu} \] …….. (28)

The explicit mathematical forms of 3- factors \( T_{\text{anh}} \), \( T_{\text{anh}} \), and \( \hat{T}_{\text{anh}} \): are given by:

\[ T_{\text{anh}} = \int_0^\infty e^{i \theta \gamma} \left( \frac{\Delta}{\mu} \right) d\gamma \]

\[ T_{\text{anh}} = \int_0^\infty e^{i \theta \gamma} \left( \frac{\Delta}{\mu} \right) d\gamma \]

\[ T_{\text{anh}} = \int_0^\infty e^{i \theta \gamma} \left( \frac{\Delta}{\mu} \right) d\gamma \]

Applying the following special integration [32],

\[ \int_0^\infty \exp(-i \gamma) \left( i \hat{D} \right) d\gamma = -i \left( \hat{D} \right) \]

To obtain the modifications to the energy levels for \( n^* \) excited states, where \( \hat{D} \) denote to the hypergeometric function, obtained from \( \hat{D} = \left( \alpha, \beta - \alpha, \beta, \gamma, \gamma \right) \) for \( \gamma = 3 \) and \( q = 2 \). After straightforward calculations, we can obtain the explicitly results:

\[ T_{\text{anh}} = \left( \frac{\Delta}{\mu} \right) \left( \hat{D} \right) \]

\[ T_{\text{anh}} = \left( \frac{\Delta}{\mu} \right) \left( \hat{D} \right) \]

\[ T_{\text{anh}} = \left( \frac{\Delta}{\mu} \right) \left( \hat{D} \right) \]

Inserting the above obtained expressions into equations (28), gives the following results for exact modifications \( E_{n^*} \) and \( E_{d^*} \) produced by new spin-orbital operator effect for MIHQPs:

\[ E_{n^*} = \frac{1}{2} \left( T_{\text{anh}} + \hat{T}_{\text{anh}} \right) \] …….. (34)

Where the new factor \( T_{\text{anh}} \) is sum of two factors \( T_{\text{anh}} \) and \( \hat{T}_{\text{anh}} \).
proportional's constants and $\theta$ is a uniform external magnetic field, we orient it to $(\psi_h)$ axis and then we can make the following two translations for MIHPQ:
\[
\left( \frac{b+g}{r} - \frac{a+b\delta^3}{2r^2} \right) + \frac{\beta}{2\mu} \beta L + B \left( \frac{b+g}{r} - \frac{a+b\delta^3}{2r^2} \right) + \frac{\sigma}{2\mu} \sigma L, \quad (36)
\]

Which allow us to introduce the modified new magnetic Hamiltonians $\hat{H}_{m-h}$ in GGS (NC-3D: RSP) for MIHPQ, as:
\[
\hat{H}_{m-h} = \left( \frac{b+g}{r} - \frac{a+b\delta^3}{2r^2} \right) + \frac{\beta}{2\mu} \beta J + \hat{H}_r, \quad (37)
\]

Where $\hat{H}_r = -\frac{\hbar^2}{2m} \nabla^2$ denote to the ordinary operator of Hamiltonian for Zeeman Effect in QM. To obtain the exact NC magnetic modifications of energy $E_{\text{mag-h}}$ for MIHPQ, it is sufficient to replace the 3-parameters $k_2$, $\Theta$ and $\tilde{\Theta}$ in the eq. (34) by the following new 3-parameters $m$, $\chi$ and $\tilde{\sigma}$:
\[
E_{\text{mag-h}} = \frac{1}{2} \left| N \right|^2 B \left( \chi T_{ih} + \frac{\sigma}{2\mu} \sigma T_{ih} \right) m \quad (38)
\]

Where $m$ denote to the eigenvalues of the operator $L_2$, which can be taking the discrete atomic values $-l, -l+1, ..., 0, ..., l$.

Results

Let us now resume the global exact spectrum of $n^a$ excited states ($E_{\text{exc-h}}, E_{\text{exc-h}},$ and $E_{\text{exc-h}}$) for MIHPQ in GGS (NC-3D: RSP) produced by the diagonal elements $(\hat{H}_{m-h}), (\hat{H}_{m-h}),$ and $(\hat{H}_{m-h})$ of the operator $\hat{H}_{m-h}$:
\[
\left\{ \begin{array}{l}
\left( \hat{H}_{m-h} \right)_{12} = -\frac{\Delta}{2\mu} + \frac{2b+g}{r} + \frac{a+b\delta^3}{2r^2} + k \left( \frac{b+g}{r} - \frac{a+b\delta^3}{2r^2} \right) + \frac{\beta}{2\mu} \beta L, \\
\left( \hat{H}_{m-h} \right)_{22} = -\frac{\Delta}{2\mu} + \frac{b+g}{r} + \frac{a+b\delta^3}{2r^2} + k \left( \frac{b+g}{r} - \frac{a+b\delta^3}{2r^2} \right), \\
\left( \hat{H}_{m-h} \right)_{33} = -\frac{\Delta}{2\mu} + \frac{b+g}{r} + \frac{a+b\delta^3}{2r^2},
\end{array} \right.
\]

As follows:
\[
E_{\text{exc-h}} = \frac{1}{2} \left| N \right|^2 B \left( \chi T_{ih} + \frac{\sigma}{2\mu} \sigma T_{ih} \right) m
\]

It is well known that the atomic quantum number $m$ can be takes $(2l+1)$ values and we have also two possible values for eigenvalues $j = l \pm \frac{1}{2}$, thus every state in usually 3-dimensional space for MIHPQ will be replace, in GGS (NC-3D: RSP) by $2(2l+1)$ sub-states and then the degenerated state can be take $2(2l+1) = 2m$ Values. It is important to noticing that our recent study can be extended to apply to molecular (spin $\frac{1}{2} \hat{S}$), we replace the one of two factors $k_2 = \left( \frac{1}{2} \left[ (l+1)^2 + (l-1)^2 \right] \right)^{\frac{1}{2}}$ by new factor $k(j,l,s) = \frac{1}{2} \left[ (l+1)^2 + (l-1)^2 - 2s(l+1) \right]$ with $\left| -l \leq s \leq l \right|$, which allow us to obtaining the new modifications to the energy levels $E_{\text{exc-h}}$ for MIHPQ:
\[
E_{\text{exc-h}} = \frac{1}{2} \left| N \right|^2 B \left( \chi T_{ih} + \frac{\sigma}{2\mu} \sigma T_{ih} \right) m
\]

And the corresponding NC Hamiltonian operators $\hat{H}_{m-h}$ can be fixed by the following results:
\[
\begin{align*}
\hat{H}_{m-h} &= \left( \frac{1}{2} \left[ (l+1)^2 + (l-1)^2 \right] \right)^{\frac{1}{2}} \left( \frac{b}{r} + \frac{a}{r} \right), \\
\hat{H}_{m-h} &= \left( \frac{1}{2} \left[ (l+1)^2 + (l-1)^2 \right] \right)^{\frac{1}{2}} \frac{b}{r}, \quad (41)
\end{align*}

We now look at some special cases and relationships between our recently results and some other existing results in our previously works. Case 1: If we set the parameters, $(b = g = 0)$ and $a = \alpha \xi^2$ it is easy to show that equations (21) and (34) are reduce to the modified interaction $V_{\text{pert-h}}(r, \Theta, \tilde{\Theta})$ of a particle in the modified Coulomb potential and corresponding NC spectrum $E_{\text{exc-h}}$ respectively:
\[
V_{\text{pert-h}}(r, \Theta, \tilde{\Theta}) = -\frac{\alpha \xi^2}{r} - \frac{1}{2} \left( \Theta + \tilde{\Theta} \right), \quad (42)
\]

With $\gamma = (l+1)$ and the three factors $T_{\gamma}(b = g = 0)$, $T_{\gamma}(b = g = 0)$ and $T_{\gamma}(b = g = 0)$ are given explicit by:
\[
T_{\gamma}(b = g = 0) = \left( \frac{1}{2} \left[ (l+1)^2 + (l-1)^2 \right] \right)^{\frac{1}{2}} \left( \frac{b}{r} + \frac{a}{r} \right), \quad (43)
\]

Case 2: Similarly, if we set $g = 0$ and $(a, b) = (0, 0)$ equations (21) and (34) are reduce to the results of modified inversely quadratic Hellmann potential in $V_{\text{pert-h}}(r, \Theta, \tilde{\Theta})$ and corresponding modified bound state energy spectrum $E_{\text{exc-h}}$ of a vibrating rotating diatomic molecule, respectively.
With \( \gamma = 2 \mu b + l(l + 1) \) and the three factors \( T_j^l(g = 0) \) and \( \tau_j^l(g = 0) \) are given explicitly by:

\[
T_j^l(g = 0) = (\sqrt{\gamma + j + \frac{1}{2}} - \sqrt{\gamma - j - \frac{1}{2}}) \Omega_j^l \quad \text{and} \quad \tau_j^l(g = 0) = \tau_j.
\]

It is important to notice that, the appearance of the polarization states of a fermionic particle for MIHPQ in the non-relativistic MIVE indicates a validity of obtained results at high energy where the two relativistic equations Klein-Gordon and Dirac are applied; this gives a positive indication of the possibility to apply these results of various Nano-particles at nano scales. Finally, if we make the two simultaneously limits \((\vec{r}, \vec{p}) \to (0,0)\) we obtain all results of QM which indicates the validity of our research.

### Conclusion

In this article, the Bopp’s shift method has been studied and applied to MIHPQ and the corresponding new energy eigenvalues of MSE are successfully investigated by applying the standard perturbation theory in GGS (NC-3D: RSP), we showed the obtained degenerated spectrum depended by ordinary discrete atomic quantum numbers \((m, j = l \pm \frac{1}{2})\) and \(s_z = \pm \frac{1}{2}\)). Furthermore, the validity of obtained corrections can be prolonged to Nano-particles at Nano and Plank’s scales. In addition, we recover the ordinary commutative spectrums when, we make the two simultaneously limits \((\vec{r}, \vec{p}) \to (0,0)\) for MIHPQ in GGS (NC-3D: RSP). The results are in excellent agreement with our reference no. [16].

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