New Near Open Set In Topological Space

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Abstract

The aim of this paper is to introduce new class of near open sets namely, b*-open set. And studies same of their properties, also we study the relation between this class among this classes. Also, we introduce some topological properties and we shall study same of their properties.

Keywords: b*-open set, b*-interior, b*-closure, b*-boundary, b*-neighbourhood.

Introduction


Definition 1.1: A subset A of topological space (X,τ) is called:

1. α-open if A ⊆ αint(cl(int(A))) [6]
2. preopen if A ⊆ αint(cl(A)) [8]
3. semi open if A ⊆ αcl(int(A)) [5]
4. Regular open if A = int(cl(A)) [4]
5. b-open (or semi pre open) if, A ∈ αcl(cl(int(A))) [9-15]
6. b-open. A ∈ cl(cl(A)) Ç int(cl(A)) [10]
7. A subset A of a space X is called Bc-open if for each x ∈ A ∈ bO(X), there exists a closed set F such that x ∈ F ⊆ A [11]

Remark 1.1: The complement of a α-open (resp. preopen, semi open, Regular open, β-open and b-open) set is called α-closed (resp. pre-closed, semi closed, Regular closed, β-closed and b-closed) set. The intersection of all α-closed (resp. pre-closed, semi closed, Regular closed, β-closed and b-closed) sets containing A is called the α-closure (resp. pre-closure, semi-closure, Regular closure, β-closure and b-closure) of A and is denoted by αcl(A) (resp. pcl(A), spcl(A), Rccl(A), bcl(A) or αcl(A) and bcl(A)).

The union of all α-open (resp. preopen, semi open, Regular open, β-open, and b-open) sets contained in A is called α-intrior (resp. pre-intrior, semi-intrior, Regular interior, β-intrior and b-intrior) of A and is denoted by αint(A) (resp. pint(A), sint(A), Rint(A), bint(A) or spint(A), and bint(A)). The family of all α-open (resp. α-closed, preopen, pre-closed, semi open, semi closed, Regular open, Regular closed, β-open, β-closed and b-open, b-closed) sets is denoted by αO(X) (resp cl(A), PO(X), PC(X), SO(X), SC(X), RO(A),C(A), βO(A), βC(A), bO(A) and bC(A)).

Proposition 1.1: For sub set A,B a space (X,r), the following statement hold:

1. pcl(A) = A ∪ cl(int(A)), pint(A) = A ∩ cl(int(A)) [10].
2. spcl(A) = A ∪ cl(int(A)), spint(A) = A ∩ cl(int(A)) [10].
3. pcl(A ∪ B) ⊆ pcl(A) ∪ pcl(B), spcl(A ∪ B) ⊆ spcl(A) ∪ spcl(B) [12,13].
4. pint(A ∪ B) ⊆ pint(A) ∩ pint(B), pint(A ∪ B) ⊇ pint(A) ∪ pint(B) [14].
5. X / (int(A)) = cl(X / (A)), int(X / A) = X / cl(A).

2 b*-Open sets

Definition 2.1: let (X,r) be topological space. Then a subset A of X is said to be

1. a b*-Open set if A ⊆ cl(int(cl(int(A)))) ∩ cl(int(A)).
2. a b*-closed set if A ⊇ int(cl(int(A))) ∪ cl(int(A)).

The family of all b*-Open set (resp. b*-closed set) subsets of a space (X,r) will be as always denoted by bO(X) (resp. bC(X)).

Example 2.1: Let X= [a,b,c,d] with topology τ={X,Φ, {a}, {a,b}, {a,c,d}}. Then the classes of b*-Open and b*-closed set

bO(X) = {X,Φ, {a}, {a,b}, {a,c}, {a,d}, {a,b,c}, {a,b,d}, {a,c,d},

and

bC(X) = {X,Φ, {b}, {c}, {d}, {c,d}, {b,d}, {b,c}, {b,c,d}}.

Proposition 2.1: Let A be a subset of a space (X,r). Then (1) Every pre open (resp. Bc-open) set is B*-open

Remark 2.1: The converse of the above proposition is not necessarily true as shown by the following example.
Example 2.2: Let $X=\{a,b,c\}$ with topology $\tau=\{X,\emptyset,\{a\},\{a,b\}\}$. Then

1. A subset $\{a,c\}$ of $X$ is $b^\star$-open but not preopen.
2. A subset $\{a\}$ of $X$ is $b^\star$-open but not $B_0$-open.

Remark 2.2: According to Definition (2.1) and Proposition (2.1), the following diagram holds for a subset $A$ of a space $X$:

Lemma 2.1: Let $(X, \tau)$ be topological space. Then the following statements are hold:

1. The union of $b^\star$-Open sets is $b^\star$-open.
2. The intersection of $b^\star$-closed sets is $b^\star$-closed.

Proof: (1) Let $\{A_i, i \in I\}$ be a family of $b^\star$-Open sets. Then $A_i \subseteq cl(int(cl(A_i))) \cap int(cl(A_i))$, hence $U_i \subseteq \overline{U_i} \subseteq \overline{\overline{U_i}} \subseteq \overline{\overline{\overline{\overline{U_i}}}}$, for all $i$. Thus $U_i$ is $b^\star$-Open.

(2) Let $\{\overline{A_i}, i \in I\}$ be a family of $b^\star$-closed sets. Then $A_i \supseteq int(cl(int(A_i))) \cap cl(int(A_i))$, hence $\bigcap_{i \in I} \overline{A_i} \supseteq \bigcap_{i \in I} cl(int(int(A_i))) \cap cl(int(A_i))$. Therefore, $A_i$ is $b^\star$-closed.

Remark 2.3: The intersection of any two $b^\star$-open sets is not $b^\star$-open. Let $X=\{a,b,c,d\}$, $\tau=\{X,\emptyset,\{a\},\{a,c\}\}$. Then $A=\{a,b\}$ and $B=\{a,c\}$ are $b^\star$-open sets, but $A \cap B=\{b\}$ is not $b^\star$-open.

Definition 2.2: Let $(X, \tau)$ be topological space. Then:

1. The union of all $b^\star$-open sets of $X$ contained in $A$ is called the $b^\star$-interior of $A$ and is denoted by $b^\star$-int($A$).
2. The intersection of all $b^\star$-closed sets of $X$ contained in $A$ is called the $b^\star$-closure of $A$ and is denoted by $b^\star$-Cl($A$).

Example 2.3: Let $X=\{a,b,c,d\}$ with topology $\tau=\{X,\emptyset,\{a\},\{a,c\}\}$ and $A=\{a,b\}$, $B=\{a,c\}$ are $b^\star$-open then $b^\star$-int$(A)=\{a,b\}$, $b^\star$-int$(B)=\{a\}$ and $b^\star$-Cl$(A)=\{a,b\}$, $b^\star$-Cl$(B)=X$.

Theorem 2.1: Let $(X, \tau)$ be topological space and $A \subset X$, then the following statement are equivalent:

1. $A$ is a $b^\star$-open set,
2. $A=b^\star$int$\cap$Cl$\cap$A.

Proof: (1)$\Rightarrow$(2). Let $A$ be a $b^\star$-open set. Then $A \subseteq cl(int(cl(A))) \cap int(cl(A))$, hence by proposition (1.1).

$\text{spin}(A) \cup \text{pint}(A) = (A \cap cl(int(cl(A)))) \cup (A \cap cl(int(A))) = A \cup cl(cl(A)) \cap cl(int(A)) = A \\
(2)$.$\Rightarrow$(1). Suppose that $A=\text{spin}(A)\cup \text{pint}(A)$. Then by proposition (1.1)

$A=(A \cap cl(int(cl(A)))) \cup (A \cap cl(int(A))) = cl(int(cl(A))) \cap cl(int(A))$. Therefore, $A$ is a $b^\star$-open.

Theorem 2.2: Let $(X, \tau)$ be topological space and $A \subset X$, then the following statement are equivalent:

1. $A$ is a $b^\star$-closed set,
2. $A=spcl(A)\cup pcl(A)$.

Proof: (1)$\Rightarrow$(2). Let $A$ be a $b^\star$-closed set. Then $A \supseteq int(cl(cl(A))) \cap cl(int(A))$, hence by proposition (1.1).

$\text{spcl}(A) \cap \text{pcl}(A) = (A \cap cl(int(cl(A)))) \cap (A \cap cl(int(A))) = A \cap cl(int(cl(A))) \cap cl(int(A)) = A \\
(2)$.$\Rightarrow$(1). Suppose that $A=\text{spcl}(A)\cap \text{pcl}(A)$. Then by proposition (1.1)

$A=(A \cap cl(int(cl(A)))) \cap (A \cap cl(int(A))) = int(cl(cl(A))) \cap cl(int(A))$. Therefore, $A$ is a $b^\star$-closed.

Theorem 2.3: Let $(X, \tau)$ be a subspace of a space $(X, \tau')$. Then

1. $b^\star$-Cl$(A) \subseteq \text{spcl}(A) \cup \text{pcl}(A)$, $b^\star$-Cl$(A) = \text{spcl}(A) \cup \text{pcl}(A)$.
2. $b^\star$-int$(A) \subseteq \text{spin}(A) \cup \text{pint}(A)$, $b^\star$-int$(A) = \text{spin}(A) \cup \text{pint}(A)$.

Proof: (1) It is easy to see that $b^\star$-Cl$(A) \subseteq \text{spcl}(A) \cup \text{pcl}(A)$. Also $\text{spcl}(A) \cap \text{pcl}(A) = (A \cap cl(int(cl(A)))) \cap (A \cap cl(int(A))) = A \cap cl(int(cl(A))) \cap cl(int(A))$. But, $b^\star$-Cl$(A)$ is $b^\star$-closed, hence $A \supseteq int(cl(cl(A))) \cap cl(int(A)) \supseteq A \cap b^\star$-Cl$(A) = b^\star$-Cl$(A)$. Therefore, $A \cap b^\star$-Cl$(A) \supseteq b^\star$-Cl$(A)$ there for, $\text{spcl}(A) \cap \text{pcl}(A) \subseteq b^\star$-Cl$(A) \cap b^\star$-Cl$(A)$. So $b^\star$-Cl$(A) \subseteq \text{spcl}(A) \cup \text{pcl}(A)$.

(2) It is easy to see that $b^\star$-int$(A) \subseteq \text{spin}(A) \cup \text{pint}(A)$. Also $\text{spin}(A) \cup \text{pint}(A) = (A \cap cl(int(cl(A)))) \cup (A \cap cl(int(A))) = A \cap cl(int(cl(A))) \cup int(cl(A))$. But, $b^\star$-int$(A)$ is $b^\star$-open, hence $b^\star$-int$(A) \subseteq \text{spint}(A) \cup \text{int}(A) \subseteq \text{spin}(A) \cup \text{int}(A) \subseteq \text{spcl}(A) \cup \text{int}(A)$. Therefore, $A \cap b^\star$-int$(A) \subseteq \text{spin}(A) \cup \text{int}(A) \subseteq b^\star$-Cl$(A)$ there for $\text{spin}(A) \cup \text{pint}(A) \subseteq b^\star$-int$(A)$. So $b^\star$-int$(A) \subseteq \text{spin}(A) \cup \text{pint}(A)$. 

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Theorem 2.4: Let A be a subset of a space (X, τ). Then
(1) A is a b'-open set if and only if A=b'−int(A)
(2) A is a b'-closed set if and only if A=b'−cl (A)

Proof: (1) Let A be a b'-open set. Then by theorem (2.1), A=spint(A)∪ pint (A) and by theorem (2.3), we have A=b'−int(A) Conversely, let A=b'−int(A) Then by theorem(2.3), A=spcl(A)∪pcl(A) and by theorem (2.1), A is b'-closed

(2) Let A be a b'-closed set. Then by theorem (2.1), A=spcl(A)∩ pcl(A) and by theorem (2.3), we have A=b'−cl (A) Conversely, let A=b'−cl (A) Then by theorem (2.3), A=spcl(A)∩ pcl(A) and by theorem (2.1), A is b'-closed

Theorem 2.5: Let A and B be a subsets of a space (X, τ). Then the following are hold
(1) \( b'−cl(X \setminus A) = X \setminus b'−int(A) \)
(2) \( b'−int(X \setminus A) = X \setminus b'−cl(A) \)
(3) If \( A \subseteq B \), then \( x \in b'−cl(A) \)
(4) \( x \in b'−cl(A) \) if and only if there exists a b'-open set U and \( x \in U \) such that \( U \cap A \neq \emptyset \).
(5) If \( x \in b'−int(A) \) if and only if there exists a b'-open set G and \( x \in G \) such that \( x \notin G \cap A \).
(6) \( b'−cl(b'−cl(A)) = b'−cl(A) \) and \( b'−int(b'−int(A)) = b'−int(A) \).
(7) \( b'−cl(A) \cup b'−cl(B) \subseteq b'−cl(A \cup B) \) and \( b'−int(A) \cup b'−int(B) \subseteq b'−int(A \cup B) \).
(8) \( b'−int(A \cap B) \subseteq b'−int(A) \cap b'−int(B) \) and \( b'−cl(A \cap B) \subseteq b'−cl(A) \cap b'−cl(B) \).

Proof: (1) Since \( (X \setminus A) \subseteq X \), by theorem (2.4), \( b'−cl(X \setminus A) = spcl(X \setminus A) \cap pcl(X \setminus A) \) and by proposition (1.1), \( b'−int(X \setminus A) = (X \setminus spint(A)) \cap (X \setminus pint(A)) \). Hence by theorem (2.4), \( b'−cl(X \setminus A) = X \setminus b'−int(A) \).
(2) Since \( (X \setminus A) \subseteq X \), by theorem (2.4), \( b'−int(X \setminus A) = spint(X \setminus A) \cup pint(X \setminus A) \) and by proposition (1.1), \( b'−int(X \setminus A) = (X \setminus spcl(A)) \cup (X \setminus pcl(A)) \). Hence by theorem (2.4), \( b'−int(X \setminus A) = X \setminus b'−cl(A) \).
(3) Since \( b'−cl(A) = spcl(A) \cap pcl(A) \) and \( A \subseteq B \), \( b'−cl(A) \subseteq spcl(B) \cap pcl(B) \).
(4) Let \( x \in b'−cl(A) \) then \( x \notin F \) where F is b'-closed with A ⊆ F, so \( x \notin X \setminus F \) and \( X \setminus F \) is a b'-open set containing x and hence \( (X \setminus F) \supseteq (X \setminus F) \). Conversely, suppose that exists a b'-open set containing x with \( A \cap U = \emptyset \).

Then \( A \subseteq X \setminus U \) and \( X \setminus U \) is a b'-closed. Hence \( x \notin b'−cl(A) \).

(5) Necessity. Let \( x \in b'−int(A) \). Then \( x \in \bigcup \{G : G \text{ is b'−open } G \subseteq A \} \) and hence there exists b'-open set G such that \( x \notin G \cap A \). Then \( A = U \setminus \{x : x \notin G \cap A \} \) which is the union of b'-open set. Then for, \( x \notin b'−cl(A) \).

Remark 2.4: The inclusion relation in part (6),(7) of the above theorem cannot be replaced by equality as shown by the following example.

Example 2.4: Let \( X = \{a, b, c, d\} \) with topology \( = \{X, \emptyset, \{a, b, c, d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}\} \).

Then \( (A \cup B) = \{a, b, d\} \)

(1) If \( A = \{a, b\} \) and \( B = \{d\} \), then \( b'−int(A) = A = b'−int(B) = b'−int(B) = b'−int(A) \cup b'−int(B) \) and \( b'−cl(A) \cap b'−cl(B) \).
(2) If \( C = \{b\} \) and \( B = \{c\} \), then \( b'−cl(C) = \{b, d\} \) and \( b'−cl(B) = \{b, c\} \), hence \( b'−cl(C) \cap b'−cl(B) = b'−cl(B) \).

Example 2.5: Let \( X = \{a, b, c, d\} \) with topology \( = \{X, \emptyset, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\} \) then

(1) If \( A = \{a, b\} \) and \( B = \{c, d\} \), then \( b'−int(A) = A = b'−int(B) = b'−int(B) = b'−int(A) \cup b'−int(B) \) and \( b'−cl(A) \cap b'−cl(B) \).
(2) If \( C = \{a\} \) and \( B = \{c\} \), then \( b'−cl(C) = \{a\} \) and \( b'−cl(B) = \{c, d\} \), hence \( b'−cl(C) \cap b'−cl(B) = b'−cl(B) \).

3 Some Topological Operations.
Definition 3.1: Let (X, τ) be a space and A ⊆ X. Then the b^-boundary of A (briefly, b^-b(A)) is given by b^-b(A)=b^-cl(A) ∩ b^-cl(X/A).

Example 3.1: From Example (2.1) we have A=[a, b] B=[a, b, c] C=[a, b, c, d] then b^-b(A)=[b, c, d], b^-b(B)=[c, d] and b^-b(C)=[c]

Remark 3.1: For any subset A of a space (X, τ) we have b^-b(A) ⊆ b(A) and b^-b(A) ⊆ p-b(A).

The inclusion of the above remark can be replaced as shows in the following example.

Example 3.2: From Example (2.3) and A=[a, b] then b^-b(A)=φ, p-b(A)=[b] we have p-b(A) ∋ b^-b(A).

Theorem 3.1: If A is a sub sets of a space (X, τ), then the following statements are hold:

(1) b^-b(b(A))=b^-b(X-A).
(2) b^-b(b(A))=b^-cl(A) \ b^-int(A).
(3) b^-b(b(A)) ∩ b^-int(A)=φ.
(4) b^-b(b(A)) ∩ b^-int(A)=b^-cl(A).

Proof: (1) Since b^-b(b(A))=b^-cl(A) \ b^-cl(X-A)=b^-b(X-A) and b^-b(b(A))=b^-cl(A) \ b^-cl(X-A) ∩ b^-cl(A)=b^-cl(A). Hence by (3.1), let A=b^-int(A) and hence A is b^-open.
(2) Since b^-b(b(A))=b^-cl(A) \ b^-cl(X-A)=b^-cl(A) \ b^-int(A) and b^-b(b(A))=b^-cl(A) \ b^-int(A) ∩ b^-int(A)=b^-cl(A) \ b^-int(A)∪ b^-int(A)=b^-int(A).
(3) By using (3) b^-b(b(A)) ∩ b^-int(A)=b^-b(A) \ b^-int(A) ∩ b^-int(A)=a b^-int(A).
(4) By using (3) b^-b(A) \ b^-int(A)=b^-b(A) \ b^-int(A) ∩ b^-int(A)=a b^-int(A).

Theorem 3.2: If A is a sub sets of a space (X, τ), then the following statement are holds:

(1) A is a b^-open set if and only if A=b^-int(A) hence by theorem (3.1)
(2) A is a b^-closed set if and only if b^-b(A) ⊆ A
(3) A is a b^-clopen set if and only if b^-b(A)=φ.

Proof: (1) Let A is a b^-open set. Then A=b^-int(A) hence by theorem (3.1). A=b^-int(A) ∩ b^-b(A) =φ
Conversely, let A=b^-int(A) then by theorem (3.1), A=b^-int(A) \ A \ b^-int(A) = A \ b^-int(A) = φ. So, A=b^-int(A) and hence A is b^-open.
(2) Let A is a b^-closed set. Then A=b^-cl(A), by theorem (3.1), but b^-b(A) \ b^-int(A) =b^-int(A) \ b^-int(A) =b^-int(A).
(3) Let A is a b^-clopen set. Then A=b^-int(A) and A=b^-cl(A), hence by theorem (3.1), b^-b(A) \ b^-int(A) =b^-int(A) \ b^-int(A) =b^-int(A).

Definition 3.2: Let (X, τ) be a space and A ⊆ X. Then the set X\(b^-xterior of A and is denoted by b^-xterior of A). Each point p ∈ X is called an b^-xterior point of A, if it is a b^-interior point of X\A.

Example 3.3: Let X=[a, b, c, d] with topology τ=[X, Φ [a], [b], [a, b], [b, c, d]]
If A=[a] B=[a, c] C=[b, c, d] then we have
b^-xterior (A)={b, c, d}, b^-xterior (B)={b, d} and b^-xterior (C)={a}

Remark 3.2: For any topology space (X, τ) and A ⊆ X, we have ext(A) ⊆ p-xterior (A) \ b^-xterior (A)

Proof: Since b^-cl(A) ⊆ cl(A), then X \ cl(A) ⊆ X \ b^-cl(A) and int(X \ A) ⊆ b^-b(A) \ int(X \ A). i.e. ext(A) ⊆ b^-xterior (A). Since ext(A) ⊆ p-xterior (A), then we have p-xterior (A) ⊆ b^-xterior (A). This implies that the relation hold.

Example 3.4: Let X=[a, b, c, d] with topology τ=[X, Φ [a], [b], [c], [b, c], [b, c], [b, c], [a, b, c, d]]. And A=[b, c], B=[c] we have
b^-xterior (A)={a, c}, p-xterior (A)={c}, p-xterior (B)={b, d}, ext (B)={d}

Theorem 3.3: If A and B are two sub sets of apace (X, τ), then the following statements are hold: ext(A) ∪ b^-xterior (A)

(1) b^-xterior (A) = b^-int(A).
(2) b^-xterior (A) is b^-open.
(3) b^-xterior (A) ∪ b^-int(A).
(4) b^-xterior (A) ∩ b^-b(A).
(5) $b^*\text{ext}(A)\cup b^*\text{d}(A)=b^*\text{cl}(X\setminus A)$.

(6) $|b^*\text{int}(A), b^*\text{cl}(A)\text{and } b^*\text{ext}(A)|$ from partition of X.

(7) If $A \subseteq B$, then $b^*\text{ext}(B) \subseteq b^*\text{ext}(A).

(8) $b^*\text{ext}(A \cup B) \subseteq b^*\text{ext}(A) \cup b^*\text{ext}(B).

(9) $b^*\text{ext}(A \cap B) \supseteq b^*\text{ext}(A) \cap b^*\text{ext}(B).

(10) $b^*\text{ext}(X)=\Phi$ and $b^*\text{ext}(\phi)=X$.

**Proof:** (1) by Definition (3.2) $b^*\text{ext}(A)=X \setminus b^*\text{cl}(A)-b^*\text{int}(X\setminus A)$.

(2) From (1) $b^*\text{ext}(A)=b^*\text{int}(X\setminus A)$. Since $b^*\text{int}(A)$ is the union of all $b^*\text{open}$ sets of X contained in A thus $b^*\text{ext}(A)$ is $b^*\text{open}$.

(3) Since $b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B)$

(4) By theorem (3.1), $b^*\text{ext}(A \cap B) \subseteq b^*\text{cl}(A \cap B)$.

(5) Also, by theorem (3.1) $b^*\text{ext}(A \cap B) \supseteq b^*\text{ext}(A) \cap b^*\text{ext}(B)$.

(6) From (3),(4) we have $b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B) \subseteq b^*\text{ext}(A) \cup b^*\text{ext}(B)$.

Now, we need to prove that $b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B)$.

(7) let $A \subseteq B$ then $(b^*\text{cl}(A) \subseteq b^*\text{cl}(B))$ and hence $X \subseteq X \setminus (b^*\text{cl}(A))$. So $b^*\text{ext}(B) \subseteq b^*\text{ext}(A)$.

(8) $b^*\text{ext}(A \cup B) \subseteq b^*\text{ext}(A \cup B) \subseteq b^*\text{ext}(A) \cup b^*\text{ext}(B) \subseteq b^*\text{ext}(A) \cup b^*\text{ext}(B)$.

(9) $b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B)$.

(10) $b^*\text{ext}(X \setminus b^*\text{cl}(X)) \subseteq X \setminus X = \phi$ and $b^*\text{ext}(\phi) = X \setminus \phi = X$.

**Remark 3.3:** The inclusion relation in part (5),(6) of the above theorem cannot be replaced by equality as is show by the following example.

**Example 3.5:** From Example (2.1) we have $A=\{b,c\}$ and $B=\{a,c\}$ then $b^*\text{ext}(A)=\{a,b\}, b^*\text{ext}(B)=\phi$. Therefore, $b^*\text{ext}(A \cup B) \supseteq b^*\text{ext}(A) \cup b^*\text{ext}(B)$. Also, $b^*\text{ext}(A \cap B) = \{a,b\}$, hence $b^*\text{ext}(A \cap B) \subseteq b^*\text{ext}(A) \cap b^*\text{ext}(B)$.

**Definition 3.3:** If $A$ is a subset of a space $(X,\tau)$, then a point $pX$ is called a $b^*\text{limit point}$ of a set $A \subseteq X$ if every $b^*\text{open}$ set $G$ containing $p$ contains a point of $A$ other than $p$. The set of all $b^*\text{limit point}$ of $A$ is called an $b^*\text{derived}$ set of $A$ and is denoted by $b^*\text{d}(A)$.

**Example 3.6:** let $X=\{a,b,c,d\}$ with topology $\tau=\{X,\emptyset,\{a\},\{a,c\},\{a,c,d\}\}$ and If $A=\{a,d\}$ B=\{a,c,d\} the $b^*\text{d}(A)=(\emptyset)$, and $b^*\text{d}(B)=\{b\}$.

**Theorem 3.4:** If $A$ and $B$ is two sub sets of aspace $(X,\tau)$, then the following statements are hold:

(1) If $A \subseteq B$, then $b^*\text{d}(A) \subseteq b^*\text{d}(B)$.

(2) A is a $b^*\text{closed}$ set if and only if it contains each of its $b^*\text{limit}$ points.

(3) $b^*\text{cl}(A)=A \cup b^*\text{d}(A)$.

(4) $b^*\text{d}(A \cup B) \subseteq b^*\text{d}(A) \cup b^*\text{d}(B)$.

(5) $b^*\text{d}(A \cap B) \subseteq b^*\text{d}(A) \cap b^*\text{d}(B)$.

**Proof:** (1) By definition (3.3), we have $p \in b^*\text{d}(A)$ if and only if $G \cap (A\setminus \{p\}) \neq \phi$ for every $b^*\text{open}$ set $G$ containing $p$. But $A \subseteq B$, then $G \cap (B\setminus \{p\}) \neq \phi$, for every $b^*\text{open}$ set $G$ containing $p$. Hence, so $p \in b^*\text{d}(B)$.

There for $b^*\text{d}(A) \subseteq b^*\text{d}(B)$.

(2) Let $A$ be $b^*\text{closed}$ set and $p \in \text{A}$ then $p \in \text{A}(X,A)$ which is $b^*\text{open}$, hence there exists $b^*\text{open}(X\setminus A)$ such that $(X\setminus A)\cap A = \phi$ so $p \notin b^*\text{d}(A)$, there for $b^*\text{d}(A) \subseteq A$. Conversely, suppose that $b^*\text{d}(A) \subseteq A$ and $p \notin \text{A}$, then $\notin b^*\text{d}(A)$, hence there exists $b^*\text{open}$ set $G$ containing $p$ such that $G \cap A = \phi$ and hence

$X \setminus A = \bigcup_{G \cap A = \phi} G \text{ is } b^*\text{ open there for } A \text{ is } b^*\text{closed}$

(3) Since, $b^*\text{d}(A) \subseteq b^*\text{cl}(A)$ and $A \subseteq b^*\text{cl}(A) b^*\text{d}(A) \subseteq A \subseteq b^*\text{cl}(A)$.
Conversely, suppose that \( p \notin b^* - d(A) \cup A \) Then \( p \notin b^* - d(A), p \notin A \) and hence there exists \( b^* - \text{open} \) set \( G \) containing \( p \) such that \( G \cap A = \emptyset. \) Thus \( p \notin b^* - c(A) \) which implies that \( b^* - c(A) \subset b^* - d(A) \cup A. \) Then for, \( b^* - c(A) = b^* - d(A) \cup A. \)

(5) Since \( A \notin b^* - d(A) \cup b^* - d(A) \cup b^* - d(A) \) and \( b^* - d(A) \notin b^* - d(A) \cup b^* - d(A) \). There for \( b^* - d(A) \notin b^* - d(A) \cup b^* - d(A). \)

**Definition 3.4:** Let \( (X, \tau) \) be a space and \( A \in X. \) Then the \( b^* \text{-border of } A \) (briefly, \( b^{*} - \text{Bd}(A) \)) is given by \( b^{*} - \text{Bd}(A) = A \setminus b^{*} - \text{int}(A). \)

**Example 3.7:** Let \( X = \{a, b, c, d\} \) with topologies \( \tau = \{\emptyset, \{a, b, c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}. \) If \( A = \{a, c\}, \) \( B = \{b, c\} \) and \( C = \{b, c\}, \) then \( b^{*} - \text{Bd}(A) = \{a, c\}, b^{*} - \text{Bd}(B) = \emptyset \) and \( b^{*} - \text{Bd}(C) = \{c\}. \)

**Theorem 3.5:** For a subset \( A \) of a space and \( X, \) the following statements are hold:

1. \( A = b^* - \text{int}(A) \cup b^* - \text{Bd}(A) \),
2. \( b^* - \text{int}(A) \cap b^* - \text{Bd}(A) = \emptyset \),
3. \( b^* - \text{Bd}(X) = b^* - \text{Bd} (\emptyset) = \emptyset \),
4. \( b^* - \text{Bd} (b^* - \text{int}(A)) = \emptyset \),
5. \( b^* - \text{int} (b^* - \text{Bd}(A)) = \emptyset \),
6. \( b^* - \text{Bd}(b^* - \text{Bd}(A)) = b^* - \text{Bd}(A) \),
7. \( b^* - \text{Bd}(A) = A \setminus b^* - c(X \setminus A) \),
8. \( b^* - \text{Bd}(A) = b^* - d(X \setminus A) \).

**Proof:**

1. \( b^* - \text{int}(A) \cup b^* - \text{Bd}(A) = b^* - \text{int}(A) \cup (A \setminus b^* - \text{int}(A)) = (b^* - \text{int}(A) \cup b^* - \text{int}(A) = A \setminus b^* - \text{int}(A) = A \).
2. \( b^* - \text{int}(A) \cap b^* - \text{Bd}(A) = b^* - \text{int}(A) \cap (A \setminus b^* - \text{int}(A)) = (b^* - \text{int}(A) \cap A \setminus b^* - \text{int}(A) = (b^* - \text{int}(A) \cap b^* - \text{int}(A)) = b^* - \text{int}(A) \cap b^* - \text{int}(A) = \emptyset \).
3. \( b^* - \text{Bd}(X) = X \setminus b^* - \text{int}(X) = X \setminus X = \emptyset \) and \( b^* - \text{Bd}(\emptyset) = \emptyset \). \( b^* - \text{int}(\emptyset) = \emptyset = \emptyset \).
4. \( b^* - \text{Bd} (b^* - \text{int}(A)) = b^* - \text{int}(A) \setminus b^* - \text{int}(A) = \emptyset \).
5. \( b^* - \text{int} (b^* - \text{Bd}(A)) = b^* - \text{int} (A \setminus b^* - \text{int}(A)) = b^* - \text{int} (A \setminus b^* - \text{int}(A)) = b^* - \text{int} (A \setminus b^* - \text{int}(A)) = \emptyset \).
6. \( b^* - \text{Bd} (b^* - \text{Bd}(A)) = b^* - \text{Bd} (A) \setminus b^* - \text{Bd}(A) = b^* - \text{Bd}(A) \setminus b^* - \text{Bd}(A) = b^* - \text{Bd}(A) \).
7. Also, from Theorem (2.5), \( b^* - \text{Bd}(A) = A \setminus b^* - \text{int}(A) \setminus (X \setminus b^* - c(X \setminus A)) = A \setminus b^* - c(X \setminus A) \).
8. Further, from Theorem 2.31 \( b^* - \text{Bd}(A) = A \setminus b^* - \text{int}(A) \setminus (A \setminus b^* - d(X \setminus A)) = b^* - d(X \setminus A) \).

**Theorem 3.6:** For a subset \( A \) of a space and \( X, \) the following statements are equivalence

1. \( A = b^* - \text{open} \),
2. \( A = b^* - \text{int}(A) \),
3. \( b^* - \text{Bd}(A) = \emptyset. \)

**Proof:**

1. \( \Rightarrow \) 2. By Theorem 3.4.
2. \( \Rightarrow \) 3. Suppose that \( A = b^* - \text{int}(A) \). Then by Definition 3.4,
3. \( b^* - \text{Bd}(A) = b^* - \text{int}(A) \setminus b^* - \text{int}(A) = \emptyset. \)
4. \( \Rightarrow \) 1. Let \( b^* - \text{Bd}(A) = \emptyset. \) Then by Definition 3.4, \( A = b^* - \text{int}(A) = \emptyset \).

**Definition 3.5:** A subset \( N \) of \( A \) is called \( b^* \text{-neighbourhood (briefly, } b^* - \text{nbd.}) \) of a point \( p \in X \) if there exists a \( b^* - \text{open} \) set \( \mathcal{W} \) such that \( p \in \mathcal{W} \cap N. \) The class of all \( b^* - \text{nbs} \) of \( p \) is called the \( b^* \text{-neighbourhood system of } p \) and denoted by \( b^* - \text{Np}. \)

**Example 3.8:** Let \( X = \{a, b, c\} \) with topology \( \tau = \{\emptyset, \{a, b, c\}, \{a, b\}, \{a, c\}\} \), then \( b^* - \text{Np} = \{a, c\}. \)

**Remark 3.4:** For any topology space \( (X, \tau) \) and for each \( x \in X \) we have \( N \subseteq p - N \subseteq b^* - N. \)

**Example 3.9:** From Example (2.2). We have \( \{a, c\} \in b^* - N \) but it is not in \( p - N \) and not in \( N. \)

**Theorem 3.7:** A subset \( G \) of a space \( X \) is \( b^* - \text{open} \) if and only if it is \( b^* \text{-nbd.} \) for every point \( p \in G. \)

**Proof:**

Sufficiency. Let \( G \) be an \( b^* - \text{open} \) set. Then \( G \) is \( b^* - \text{nbd.} \) for each \( p \in G. \) Then there exists a \( b^* - \text{open} \) set \( \mathcal{W} \) containing \( p \) such that \( p \in \mathcal{W} \subseteq G \), so \( G = \bigcup \{p \in \mathcal{W} \}. \) Therefore, \( G \) is \( b^* - \text{open}. \)
Theorem 3.8: For a space \((X, \tau)\). If \(b^*\text{-}N_p\) is the \(b^*\text{-}nbd.\) systems of a point \(p \in X\), then the following statements are hold:

1. \(b^*\text{-}N_p\) is not empty and \(p\) belongs to each member of \(b^*\text{-}N_p\).
2. Each superset of the members of \(b^*\text{-}N_p\) belongs to \(b^*\text{-}N_p\).
3. Each member \(N \in b^*\text{-}N_p\) is a superset of the member \(W \in b^*\text{-}N_p\), where \(W\) is \(b^*\text{-}nbd\) of each point \(p \in W\).

Proof: (1) Since \(X\) is a \(b^*\text{-}open\) set containing \(p\), then \(X \in b^*\text{-}N_p\). So, \(b^*\text{-}N_p \neq \phi\). Also, if \(N \in b^*\text{-}N_p\), then there exists a \(b^*\text{-}open\) set \(G\) such that \(p \in G \subseteq N\). Therefore, \(p\) belongs to each member of \(b^*\text{-}N_p\).

(2) Let \(M\) be a superset of \(N \in b^*\text{-}N_p\), then there exists a \(b^*\text{-}open\) set \(G\) such that \(p \in G \subseteq M\). Therefore, \(M \in b^*\text{-}N_p\).

(3) Let \(N\) be a \(b^*\text{-neighbourhood}\) of \(p \in X\), then there exists a \(b^*\text{-}open\) set \(W\) such that \(p \in W \subseteq N\). Then by Theorem 2.5.1, \(W\) is a \(b^*\text{-neighbourhood}\) of each of its points.

Definition 3.6: For a space \((X, \tau)\), a subset \(A\) of \(X\) is said to be \(b^*\text{-}dense\) in \(X\) if and only if \(b^*\text{-}cl\) \((A) = X\). The family of all \(b^*\text{-}dense\) sets in \((X, \tau)\) will be denoted by \(b^*\text{-}D(X, \tau)\).

Example 3.10: Let \(X=\{a, b, c\}\) with topology \(\tau=\{X, \Phi, \{a, b\}\}\). If, \(A=\{a, b\}\), then \(b^*\text{-}cl\) \((A) = X \text{ but } b^*\text{-}int\) \((A) = \emptyset\) (1).

Remark 3.5: Every \(b^*\text{-}dense\) set in \((X, \tau)\) is dense in \((X, \tau)\) by the fact that \(b^*\text{-}cl\) \((A) \subseteq cl\) \((A)\), while the converse may not be true.

Example 3.11: Let \(X=\{a, b, c, d\}\) with topology \(\tau=\{X, \Phi, \{a, c\}, \{b, d\}, \{a, c, d\}\}\). If \(A=\{b, c, d\}\), then \(cl\) \((A) = X \text{ but } b^*\text{-}cl\) \((A) = \{b, c, d\}\). Therefore, \(A\) is dense in \(X\) but not \(b^*\text{-}dense\) in \(X\).

Theorem 3.9: For a space \((X, \tau)\) and \(E \subseteq X\), the following statements are equivalent:

1. \(E\) is \(b^*\text{-}dense\) in \(X\).
2. If \(F\) is an \(b^*\text{-}closed\) set in \(X\) containing \(E\), then, \(F=X\).
3. \(M\text{-}int\) \((X/E) = \phi\).

Proof: (1)\&(2). Let \(E\) be an \(b^*\text{-}dense\) set of \(X\). Than \(b^*\text{-}Cl\) \((E) = X\). But \(F\) is an \(b^*\text{-}closed\) set contains \(E\), then \(b^*\text{-}Cl\) \((E) \subseteq F\) and therefore \(F=X\).

(2)\&(3). Since \(b^*\text{-}Cl\) \((E)\) is an \(b^*\text{-}closed\) set contains \(E\), By (2) we have \(b^*\text{-}Cl\) \((E) = X\). Hence \(\phi = X \setminus b^*\text{-}p - Cl\) \((E) = b^*\text{-}int\) \((X \setminus E)\).

(3)\&(1). Since \(b^*\text{-}int\) \((X/E)\) = \(E\). Then \(b^*\text{-}Cl\) \((E) = X\) Hence \(E\) is \(b^*\text{-}dense\) in \(X\).

Proposition 3.1: For a space \((X, \tau)\), \(E \subseteq X\), then the following statements are hold:

1. \(b^*\text{-}b(E) = b^*\text{-}cl(X \setminus E)\).
2. \(b^*\text{-}ext(E) = \phi\).

Proof: (1) From Definition (3.1), we have \(b^*\text{-}b(E) = b^*\text{-}cl(X \setminus E)\) and \(b^*\text{-}Cl\) \((E) = X\). (2) Also, by From Definition (3.2), \(b^*\text{-}ext\) \((E) = X \setminus b^*\text{-}cl\) \((E)\) but \(E \subseteq b^*\text{-}D(X, \tau)\), then \(b^*\text{-}ext\) \((E) = \phi\).

Definition 3.7: For a space \((X, \tau)\), \(AGX\) is called:

1. \(b^*\) nowhere dense if \(int\) \((A) \subseteq b^*\text{-}int\) \((b^*\text{-}cl\) \((A)) = \phi\).\n2. \(b^*\) residual if \(b^*\text{-}cl\) \((X/A) = X\) or \(b^*\text{-}int\) \((A) = \phi\).

\(b^*\) nowhere dense is \(b^*\text{-}residual\) from the fact that \(b^*\text{-}int\) \((A) \subseteq b^*\text{-}int\) \((b^*\text{-}cl\) \((A))\) for every \(AGX\).

Example 3.12: Let \(X=\{a, b, c\}\) with topology \(\tau=\{X, \Phi, \{a, b\}\}\) and \(A=\{b\}\) than \(b^*\text{-}int\) \((b^*\text{-}cl\) \((A)) = \phi\) and \(b^*\text{-}int\) \((A) = \phi\) so \(A\) is \(b^*\) nowhere dense and \(b^*\text{-}residual\).

Proposition 3.2: A subset \(A\) of a space \((X, \tau)\), \(AGX\) is \(b^*\text{-}nowhere dense\) of \(X\) if \(AG\) \(b^*\text{-}Cl\) \((X/b^*\text{-}cl\) \((A))\).

Proof: Let \(A\) is \(b^*\) nowhere dense then \(b^*\text{-}int\) \((b^*\text{-}cl\) \((A)) = \phi\).

Hence \(X \setminus b^*\text{-}cl\) \((A) = b^*\text{-}cl\) \((X \setminus A)\) = \(X \supseteq A\).

Theorem 3.10: The \(b^*\text{-}boundary\) of each \(b^*\text{-}open\) (resp. \(b^*\text{-}closed\)) set is \(b^*\text{-}nowhere dense\).

Proof: Let \(AE\) \(b^*\text{-}O(X)\) then:

\(b^*\text{-}int\) \((b^*\text{-}cl\) \((b^* - b(A))\) = \(b^*\text{-}int\) \((b^* - cl\) \((b^* - cl\) \((A) \setminus b^*\text{-}cl\) \((X \setminus A)\)) = \(b^*\text{-}int\) \((b^*\text{-}cl\) \((b^* - int\) \((A) \setminus b^*\text{-}cl\) \((X \setminus A)\)) = \(b^*\text{-}int\) \((b^*\text{-}cl\) \((b^*\text{-}int\) \((A) \setminus b^*\text{-}cl\) \((X \setminus A)\)) = \phi\).

Also if \(AE\) \(b^*\text{-}C(X)\) Then...
Proposition 3.3: For a space \((X, \tau)\), \(A \subseteq X\), then the sets \(A \cap b^* - \text{cl}(X \setminus A)\) and \(b^* - \text{cl}(A) \cap (X \setminus A)\) are \(b^*\)-residual.

Proof: Since

\[ b^* - \text{int}(A \cap b^* - \text{cl}(X \setminus A)) = b^* - \text{int}(b^* - \text{cl}(A) \cap (X \setminus A)) = b^* - \text{int}(b^* - \text{cl}(A)) \]

Then \(A \cap b^* - \text{cl}(X \setminus A)\) is residual. Similarly

\[ b^* - \text{int}(b^* - \text{cl}(A) \cap (X \setminus A)) = b^* - \text{int}(b^* - \text{cl}(A)) \cap (X \setminus A) = b^* - \text{cl}(A) \cap (X \setminus A) = \phi \]

and hence \(b^* - \text{cl}(A) \cap (X \setminus A)\) is \(b^*\)-residual.

Theorem 3.11: The \(b^*\)-boundary of any set contains the union of two \(b^*\)-residual sets.

Proof: Let \((X, \tau)\) be a space and \(A \subseteq X\). Then by Proposition (3.3), we have

\[
\begin{align*}
(A \cap b^* - \text{cl}(X \setminus A)) \cup (b^* - \text{cl}(A) \cap (X \setminus A)) &= ((A \cap b^* - \text{cl}(X \setminus A)) \cup b^* - \text{cl}(A)) \cap ((A \cup b^* - \text{cl}(X \setminus A)) \cup b^* - \text{cl}(A)) \\
&= ([A \cap b^* - \text{cl}(X \setminus A) \cup b^* - \text{cl}(A)] \cap (X \setminus A)) \cup (b^* - \text{cl}(A) \cap (X \setminus A)) \cup (b^* - \text{cl}(A) \cap (X \setminus A)) \\
&= [b^* - \text{cl}(A) \cap (X \setminus A)] \cap b^* - \text{cl}(X \setminus A) = b^* - \text{cl}(A) \cap (X \setminus A) = b^* - b(A).
\end{align*}
\]

References