New Types of 2D-Integrodifferential Equations and Their Properties

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Abstract
In this paper, we present new type of 2D-volterra integrodifferential equations and study the existence, uniquenesse and some other useful properties of solution of these equations. The main tools are based on application of the Banach fixed point theorem.

Keywords: 2D-Volterra nonlinear integrodifferential; ε-Approximate solution

Introduction
Consider the nonlinear integrodifferential equation
\[ D_x D_y u(x,y) = f(x,y,u(x,y),Gu(x,y),Hu(x,y)) \] (1.1)
with given data
\[ u(x,0) = \sigma(x), u(0,y) = \tau(y) \] (1.2)
for \( x, y \in \mathbb{R}_+ \), where
\[ Gu(x,y) = \int_0^y g(x,y,\xi) u(x,\xi)d\xi \]
\[ Hu(x,y) = \int_0^y h(x,y,m,n,u(m,n))dn \] (1.3)
and
\[ f \in C(\mathbb{R}_+^2, \mathbb{R}), \quad g \in C(\mathbb{R}_+ \times \mathbb{R}, \mathbb{R}), \quad h \in C(\mathbb{R}^2 \times \mathbb{R}, \mathbb{R}), \quad \sigma \in C(\mathbb{R}), \quad \tau \in C(\mathbb{R}), \quad \text{ that } E = \mathbb{R}_+ \times \mathbb{R} \]

Properties of Solution
Let \( S \) be the space of functions \( z, D_x D_y z \in C(E, \mathbb{R}) \) which fulfill the condition [1]
\[ z(x,y) = \sigma(x) + (\tau(x) - \sigma(x)) \exp(\lambda(x+y)) \] (2.1)
For \((x,y) \in E\), where \( \lambda > 0 \) is a constant. This space with the norm
\[ \|z\| = \sup \{ |z(x,y)| : (x,y) \in E \} \]
\[ \leq \sup \{ |\exp(\lambda(x+y))| : (x,y) \in E \} \]
\[ \leq \sup \{ N \exp(\lambda(x+y)) : (x,y) \in E \} \]
(2.2)
is a Banach space.

We note that the condition (2.2) implies that there exists a constant \( N \geq 2 \) such that
\[ \|z\| \leq N \exp(\lambda(x+y)) \] (2.3)
then
\[ \|z\| \leq \sup \{ |z(x,y)| \exp(-\lambda(x+y)) : (x,y) \in E \} \]
\[ \leq \sup \{ N \exp(\lambda(x+y)) \exp(-\lambda(x+y)) : (x,y) \in E \} \]
(2.4)
So \( z \leq N \)

Theorem 2.1
Assume that
Functions \( f, g, h \) in equation (1.1) satisfy the conditions [2]
\[ f(x,y,u,v,w) - f(x,y,u,v,w) \leq k(x,y)|u - \bar{u}| + v - \bar{v} + |w - \bar{w}| \] (2.5)
\[ g(x,y,u,v) - g(x,y,u,v) \leq \xi(u,v)|u - \bar{u}| \] (2.6)
\[ h(x,y,m,n,u) - h(x,y,m,n,u) \leq \eta(m,n)|u - \bar{u}| \] (2.7)
where \( k \in C(E, \mathbb{R}), \quad a \in C(E \times E), \quad b \in C(E), \quad \xi \in C(E), \quad \eta \in C(E) \).

Proof:
Note that by take integration of (1.1), we have
\[ u(x,y) = \sigma(x) + (\tau(x) - \sigma(x)) \exp(\lambda(x+y)) \int_0^y f(s,t,u(s,t),gu(s,t),hu(s,t))ds \] (2.10)
Let \( u(x,y) \in S \) and define the operator \( T \) by
\[ (Tu)(x,y) = \sigma(x) + (\tau(x) - \sigma(x)) \exp(\lambda(x+y)) \int_0^y f(s,t,u(s,t),gu(s,t),hu(s,t))ds \] (2.11)
Now we shall show that \( T \) maps \( S \) into itself. Evidently, \( Tu \) is continuous on \( E \) because \( \sigma, \tau, u, f \) are continuous and \( Tu \in E \).

From (2.11), we observe that
\[ D_x D_y (Tu)(x,y) = f(x,y,u(x,y),Gu(x,y),Hu(x,y)) \]
So \( D_x D_y Tu \) is continuous and \( D_x D_y Tu \in E \) from (2.11) and using the hypotheses and (2.4) we have
\[ \|Tu(x,y)\| \leq \sup \{ f(s,t,u(s,t),Gu(s,t),Hu(s,t)) : (s,t) \in E, \tau(x), \lambda \} \]
(2.8)
\[ \leq \sup \{ k(x,y)|u - \bar{u}| + v - \bar{v} + |w - \bar{w}| : (x,y) \in E \} \]
(2.9)
and
\[ \leq \sup \{ g(x,y,u,v) : (x,y) \in E \} \]
(2.10)
and
\[ \leq \sup \{ h(x,y,m,n,u) : (x,y) \in E \} \]
(2.11)
Therefore, according to the Banach fixed point theorem, \( T \) has a unique fixed point \( u \) on \( E \).

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\[ \text{some equations and lemmas here} \]

**Some Results**

**Lemma:** Let \( u \in L^1(E, \mathbb{R}) \) and \( c \geq 0 \). If
\[
|u(x,t)| \leq c + \int_0^t \int_0^s r(s,t) u(s,t) \, ds \, dt \quad \text{for } (x,t) \in E,
\]
then
\[
|u(x,t)| \leq c \exp\left(\int_0^t \int_0^s r(s,t) \, ds \, dt\right)
\]

**Proof:** [1, p. 60]

**Theorem 3.1**

Suppose that the functions \( f, g, h \) in (1.1) satisfy the conditions
\[
|f(x,y,u,v) - f(x,y,v,w)| + |g(x,y,u,v) - g(x,y,u,w)| + |h(x,y,u,m,n) - h(x,y,u,m,n)|
\]

where \( k \in C(E, \mathbb{R}_+), a \in C(E, \mathbb{R}_+), b \in C(E^2, \mathbb{R}_+) \) and
\[
\alpha \leq \sup_{(x,y) \in E} \left| f(x,y,u,v) - u(0,0) - \int_0^t \int_0^s f(s,t,0,0) \, ds \, dt \right|
\]

If \( u(x,y) \) is any solution of equation (1.1) on \( E \), then
\[
|u(x,y)| \leq \min\left\{ z(x,y) \right\}
\]

where
\[
A(x,y) = \exp\left( \int_0^t \int_0^s a(s,t) \, ds \, dt \right)
\]

**Proof:**

We know
\[
|u(x,y)| \leq \min\left\{ z(x,y) \right\}
\]

Define a function \( z(x,y) \) by
\[
z(x,y) = \exp\left( \int_0^t \int_0^s a(s,t) \, ds \, dt \right)
\]

Then (3.1) can be restated as
\[
|u(x,y)| \leq z(x,y) \exp\left( \int_0^t \int_0^s a(s,t) \, ds \, dt \right)
\]

It is easy to observe that \( z(x,y) \) is positive, continuous and non decreasing function and using lemma, we get
\[
|u(x,y)| \leq z(x,y) \exp\left( \int_0^t \int_0^s a(s,t) \, ds \, dt \right)
\]

**We observe that**
\[
z(x,y) \leq c + \int_0^t \int_0^s b(s,t,m,n) u(m,n) \, dn \, dm \quad \text{for } (x,t) \in E
\]

Define a function \( v(x,y) \) by the right hand side of (3.3). Then \( v(x,y) \geq 0 \) and
\[
D_2v(x,y) \leq k(x,y) v(x,y) + f(x,y,v(x,y),w(x,y))-z(x,y)v(x,y)
\]

Because \( w(x,y) \geq v(x,y) \geq 0 \) then
\[
D_2w(x,y) \leq A(x,y) \leq A(x,y) + b(x,y,v(x,y),w(x,y))
\]

Lemma: Let \( u(x,t) \leq c + \int_0^t \int_0^s r(s,t) u(s,t) \, ds \, dt \quad \text{for } (x,t) \in E \),
then
\[
|u(x,t)| \leq c \exp\left(\int_0^t \int_0^s r(s,t) \, ds \, dt\right)
\]

**Proof:** [1, p. 60]
\[
\ln w(x, y) \leq \ln c + \int_0^x \int_0^y A(s, t) \left[ k(s, t) + b(x, y, s, t) \right] ds dt
\]
\[
w(x, y) \leq c \exp \left( \int_0^x \int_0^y A(s, t) \left[ k(s, t) + b(x, y, s, t) \right] ds dt \right)
\]
Using \( z(x, y) \leq v(x, y) \leq w(x, y) \) and (3.2), we get
\[
\left| u(x, y) \right| \leq c A(x, y) \exp \left( \int_0^x \int_0^y A(s, t) \left[ k(s, t) + b(x, y, s, t) \right] ds dt \right)
\]
The proof is complete.

We call the function \( u \in C(E, \mathbb{R}) \) an \( \varepsilon \)-approximate solution to equation (1.1), if there exists a constant \( \varepsilon \geq 0 \) such that \( D_2D_1 u(x, y) \) exists and satisfies the inequality
\[
\left| D_2D_1 u(x, y) - f(x, y, u(x, y), Gu(x, y), Hu(x, y)) \right| \leq \varepsilon
\] (4.4)
The following theorem deals with the estimate on the difference between two approximate solutions of equation (1.1).

**Theorem 3.3**

Assume that functions \( f, g, h \) in equation (1.1) satisfy the conditions
\[
f(x, y, u, v, w) - f(x, y, \xi, \eta, \zeta) \leq \varepsilon \]
(3.5)
\[
g(x, y, \xi, u) - g(x, y, \xi, \eta) \leq \varepsilon \]
(3.6)
\[
h(x, y, m, n, u) - h(x, y, m, n, \eta) \leq \varepsilon \]
(3.7)

where \( k, c, q, r, \varepsilon \in C(E, \mathbb{R}) \). Let \( u_1(x, y), u_2(x, y) \) be respectively \( \varepsilon_1 - \) and \( \varepsilon_2 - \) approximate solutions of equation (1.1) on \( E \) that
\[
u_1(x, 0) = \sigma_1(x), \quad u_1(0, y) = \tau_1(y) \quad i = 1, 2
\] (3.8)
\[
|\sigma_1(x) - \sigma_2(x)| + |\tau_1(y) - \tau_2(y)| + \varepsilon_1(0, 0) - \varepsilon_2(0, 0) \leq \delta
\] (3.9)
That \( \delta \geq 0 \) is constant and \( c = \sup_{(x,y) \in \tilde{E}} |v(x, y) + \delta| \)
(3.10)
\[
p(x, y) = \max \{ k(x, y), k(x, y)c(x, y), k(x, y)q(x, y) \}
\] (3.11)
\[
\varepsilon \in \varepsilon_1 + \varepsilon_2
\] (3.12)
Then
\[
\left| u_1(x, y) - u_2(x, y) \right| \leq c A(x, y) \exp \left( \int_0^x \int_0^y A(s, t) \left[ k(s, t) + b(x, y, s, t) \right] ds dt \right)
\]
where \( A(x, y) = \exp \left( \int_0^x \int_0^y a(\xi, \eta) d\xi d\eta \right) \) ds dt .

**Proof.**

Since \( u_i(x, y) \) is \( \varepsilon_i - \) approximate solution to equation (1.1), we have
\[
D_2D_1 u_i(x, y) - f(x, y, u_i(x, y), Gu_i(x, y), Hu_i(x, y)) \leq \varepsilon_i \quad i = 1, 2
\] (3.13)
with integration of (3.13) and using the elementary inequalities
\[| - \| \leq \| \] we get
\[
\varepsilon_i \int_0^x \int_0^y D_2D_1 u_i(s, t) - f(s, t, u_i(s, t), Gu_i(s, t), Hu_i(s, t)) \] ds dt
\[
\geq \left| \int_0^x \int_0^y \left[ u_i(s, t) - \sigma_i(s, t) - \tau_i(y) - \varepsilon_i(0, 0) - \int_0^x \int_0^y f(s, t, u_i(s, t), Gu_i(s, t), Hu_i(s, t)) \right] ds dt \right|
\]
i=1, 2. We observe that
\[
\left| u_1(x, y) - u_2(x, y) \right| \leq c A(x, y) \exp \left( \int_0^x \int_0^y A(s, t) \left[ k(s, t) + b(x, y, s, t) \right] ds dt \right)
\]
References


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