Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras

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Abstract

The main purpose of this paper is to study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. We provide construction results dealing with tensor product and direct sums of two (non-commutative) ternary (Hom-) Nambu-Poisson algebras. Moreover, we explore twisting principle of (non-commutative) ternary Nambu-Poisson algebras along with algebra morphism that lead to construct (non-commutative) ternary Hom-Nambu-Poisson algebras. Furthermore, we provide examples and a 3-dimensional classification of non-commutative ternary Nambu-Poisson algebras.

Keywords: Hom-nambu poisson algebra; Ternary nambu poisson; Non-commutative ternary; n-ary

Introduction

In the 70’s, Nambu proposed a generalized Hamiltonian system based on a ternary product, the Nambu-Poise bracket, which allows to use more than one hamiltonian [1]. More recent motivation for ternary brackets appeared in string theory and M-branes, ternary Lie type structure was closely linked to the super-symmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes and was applied to the study of Bagger-Lambert theory. Moreover ternary operations appeared in the study of some quarks models. In 1996, quantizations of Nambu-Poise brackets were investigated [2], it was presented in a novel approach of Zariski, and this quantization is based on the factorization on \( \mathbb{R} \) of polynomials of several variables.

The algebraic formulation of Nambu mechanics was discussed [3] and Nambu algebras was studied [4] as a natural generalization of a Lie algebra for higher-order algebraic operations. By definition, Nambu algebra of order \( n \) over a field \( \mathbb{K} \) of characteristic zero consists of a vector space \( V \) over \( \mathbb{K} \) together with a \( \mathbb{K} \)-multilinear skew-symmetric operation \( \{ \ldots, \; ; \ldots \} : V^n \rightarrow V \), called the Nambu bracket that satisfies the following generalization of the Jacobi identity. Namely, for any \( x_1, \ldots, x_n \in V \), define an adjoint action \( \text{ad}(x_1, \ldots, x_{n-1}) : V \rightarrow V \) by \( \text{ad}(x_1, \ldots, x_{n-1})x_n = [x_1, \ldots, x_n, x_n] \) \( x_n \in V \). Then the fundamental identity is a condition saying that the ad joint action is a derivation with respect the Nambu bracket, i.e. for all \( x_1, \ldots, x_n, y_1, \ldots, y_s \in V \)

\[
\text{ad}(x_1, \ldots, x_{n-1})([y_1, \ldots, y_s]) = \sum_{k=1}^{n} \left\{ y_1, \ldots, \text{ad}(x_1, \ldots, x_{n-1})y_1, \ldots, y_s \right\}. \tag{0.1}
\]

When \( n=2 \), the fundamental identity becomes the Jacobi identity and we get a definition of a Lie algebra.

Different aspects of Nambu mechanics, including quantization, deformation and various algebraic constructions for Nambu algebras have recently been studied. Moreover a twisted generalization, called Hom-Nambu algebras, was introduced [5]. This kind of algebras called Hom-algebras appeared as deformation of algebras of vector fields using \( \sigma \)-derivations. The first examples concerned \( q \)-deformations of Witt and Virasoro algebras. Then Hartwig, Larson and Silvestrov introduced a general framework and studied Hom-Lie algebras [6], in which Jacobi identity is twisted by a homomorphism. The corresponding associative algebras, called Hom-associative algebras, were introduced [7]. Non-commutative Hom-Poisson algebras were discussed [8]. Likewise, \( n \)-ary algebras of Hom-type were introduced [5,9-13].

We aim in this paper to explore and study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. The paper includes five Sections. In the first one, we summarize basic definitions of (non-commutative) ternary Nambu-Poisson algebras and discuss examples. In the second Section, we recall some basics about Hom-algebras structures and introduce the notion of (non-commutative) ternary Hom-Nambu-Poisson algebra. Section 3 is dedicated to construction of (non-commutative) ternary Hom-Nambu-Poisson algebras using direct sums and tensor products. In Section 4, we extend twisting principle to ternary Hom-Nambu-Poisson algebras. It is used to build new structures with a given ternary (Hom-) Nambu-Poisson algebra and algebra morphism. This process is used to construct ternary Hom-Nambu-Poisson algebras corresponding to the ternary algebra of polynomials where the bracket is defined by the Jacobian. We provide in the last section a classification of 3-dimensional ternary Nambu-Poisson algebras and then compute corresponding Hom-Nambu-Poisson algebras using twisting principle. Notice that a complete classification of 3-dimensional Hom-Nambu-Poisson algebras is difficult to obtain since so far the classification of 3-dimensional Hom-Nambu-Lie algebras is not known.

Ternary (Non-Commutative) Nambu-Poisson Algebra

In the section we review some basic definitions and fix notations. In the sequel, \( A \) denotes a vector space over \( \mathbb{K} \), where \( \mathbb{K} \) is an algebraically closed field of characteristic zero. Let \( \mu : A \times A \rightarrow A \) be a bilinear map, we denote by \( \mu^\tau : A^2 \rightarrow A \) the opposite map, i.e., \( \mu^\tau = \mu \circ \tau \) where \( \tau : A^2 \rightarrow A^2 \) interchanges the two variables.

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A ternary algebra is given by a pair \((A, m)\), where \(m\) is a ternary operation on \(A\), that is a trilinear map \(m : A \times A \times A \to A\), which is denoted sometimes by brackets.

**Definition 1.1.** A ternary Nambu algebra is a ternary algebra \((A, \{., ., .\})\) satisfying the fundamental identity defined as
\[
\{x_1, x_2, x_3, x_4, x_5\} = \\
\{x_1, x_2, x_3\} + \{x_4, x_5, x_1\} + \{x_5, x_1, x_2\} + \{x_3, x_4, x_5\} + \{x_1, x_2, x_3\}
\]
(1.1)
for all \(x_1, x_2, x_3, x_4, x_5 \in A\).

This identity is sometimes called Filippov identity or Nambu identity, and it is equivalent to the identity (0.1) with \(n=3\).

A ternary Nambu-Lie algebra or 3-Lie algebra is a ternary Nambu algebra for which the bracket is skew-symmetric, that is for all \(\sigma \in S_3\), where \(S_3\) is the permutation group,
\[
\{x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\} = Sgn(\sigma)\{x_1, x_2, x_3\}
\]
Let \(A\) and \(A'\) be two ternary Nambu algebras (resp. Nambu-Lie algebras). A linear map \(f : A \to A'\) is a morphism of a ternary Nambu algebras (resp. ternary Nambu-Lie algebras) if it satisfies \(f(\{x_1, x_2, x_3\}) = \{f(x_1), f(x_2), f(x_3)\}\).

**Example 1.2.** The polynomials of variables \(x, y, z\) with the ternary operation defined by the Jacobian function:
\[
\begin{array}{c|cc}
 & x & y & z \\
\hline
x & 1 & 0 & 0 \\
y & 0 & 1 & 0 \\
z & 0 & 0 & 1 \\
\end{array}
\]
(1.2)
is a ternary Nambu-Lie algebra.

**Example 1.3.** Let \(V = \mathbb{R}^4\) be the 4-dimensional oriented Euclidean space over \(\mathbb{R}\). The bracket of 3 vectors \(\tilde{x}, \tilde{y}, \tilde{z}\) is given by
\[
\{x, y, z\} = \tilde{x} \times \tilde{y} \times \tilde{z} = \\
\begin{array}{c|c|c|c|c}
 & x & y & z & e_1 \\
\hline
x & 1 & 0 & 0 & e_1 \\
y & 0 & 1 & 0 & e_2 \\
z & 0 & 0 & 1 & e_3 \\
\tilde{e}_4 & 0 & 0 & 0 & e_4 \\
\end{array}
\]
where \(\{e_1, e_2, e_3, e_4\}\) is a basis of \(\mathbb{R}^4\) and
\[
\tilde{x} = \sum_{i=1}^3 x_i e_{i+1}, \quad \tilde{y} = \sum_{i=1}^3 y_i e_{i+1}, \quad \tilde{z} = \sum_{i=1}^3 z_i e_{i+1}
\]
Then \((V, \{., ., .\})\) is a ternary Nambu-Lie algebra.

Now, we introduce the notion of (non-commutative) ternary Nambu-Poisson algebra.

**Definition 1.4.** A non-commutative ternary Nambu-Poisson algebra is a triple \((A, \mu, \{., ., .\})\) consisting of a \(\mathbb{K}\)-vector space \(A\), a bilinear map \(\mu : A \times A \to A\) and a ternary map \(\{., ., .\} : A \otimes A \otimes A \to A\) such that

(1) \((A, \mu)\) is a binary associative algebra,
(2) \((A, \{., ., .\})\) is a ternary Nambu-Lie algebra,
(3) the following Leibniz rule
\[
\{x_1, x_2, \mu(x_3, x_4)\} = \mu\{x_1, x_2, x_3\} + \mu\{x_1, x_2, x_4\} + \mu\{x_1, x_2, x_5\}
\]
holds for all \(x_1, x_2, x_3, x_4, x_5 \in A\).

A ternary Nambu-Poisson algebra is a non-commutative ternary Nambu-Poisson algebra \((A, \mu, \{., ., .\})\) for which \(\mu\) is commutative, then \(\mu\) is commutative unless otherwise stated.

In a (non-commutative) ternary Nambu-Poisson algebra, the ternary bracket \(\{., ., .\}\) is called Nambu-Poisson bracket.

Similarly, a non-commutative n-ary Nambu-Poisson algebra is a triple \((A, \mu, \{., ., ., \ldots\})\)
where \((A, \{., ., ., \ldots\})\) defines an n-Lie algebra satisfying similar Leibniz rule with respect to \(\mu\).

A morphism of (non-commutative) ternary Nambu-Poisson algebras is a linear map that is a morphism of the underlying ternary Nambu-Lie algebras and associative algebras.

**Example 1.5.** Let \(C^\infty(\mathbb{R}^3)\) be the algebra of \(C^\infty\) functions on \(\mathbb{R}^3\) and \(x, y, z\) the coordinates on \(\mathbb{R}^3\). We define the ternary brackets as in (1.2), then \((C^\infty(\mathbb{R}^3), \{., ., .\})\) is a ternary Nambu-Lie algebra. In addition the bracket satisfies the Leibniz rule:
\[
\{fg, x\} = f \{g, x\} + g \{f, x\}
\]
where \(f, g, x \in C^\infty(\mathbb{R}^3)\) and the multiplication being the point wise multiplication that is \(fg(x) = f(x)g(x)\). Therefore, the algebra is a ternary Nambu-Poisson algebra.

This algebra was considered already in 1973 by Nambu [9] as a possibility of extending the Poisson bracket of standard hamiltonian mechanics to bracket of three functions defined by the Jacobian. Clearly, the Nambu bracket may be generalized further to a Nambu-Poisson allowing for an arbitrary number of entries.

In particular, the algebra of polynomials of variables \(x, y, z\) with the ternary operation defined by the Jacobian function in (1.2), is a ternary Nambu-Poisson algebra.

**Remark 1.6.** The \(n\)-dimensional ternary Nambu-Lie algebra of Example 1.3 does not carry a non-commutative Nambu-Poisson algebra structure except that one given by a trivial multiplication.

**Hom-type (Non-Commutative) Ternary Nambu-Poisson Algebras**

In this section, we present various Hom-algebra structures. The main feature of Hom-algebra structures is that usual identities are deformed by an endomorphism and when the structure map is the identity, we recover the usual algebra structure.

A Hom-algebra (resp. ternary Hom-algebra) is a triple \((A, \nu, \alpha)\) consisting of a \(\mathbb{K}\)-vector space \(A\), a linear map \(\nu : A \times A \to A\) (resp. a trilinear map \(\nu : A \times A \times A \to A\)) and a linear map \(\alpha : A \to A\). A Hom-algebra \((A, \nu, \alpha)\) is said to be multiplicative if \(\alpha \circ \nu = \nu \circ \alpha\) and it is called commutative if \(\mu = \nu \circ \alpha\). A ternary Hom-algebra \((A, \nu, \alpha)\) is said to be multiplicative if \(\alpha \circ m = m \circ \alpha\). Classical algebras (resp. ternary algebras) are regarded as Hom-algebras (resp. ternary Hom-algebras) with identity twisting map. We will often use the abbreviation \(\nu(x, y)\) when there is no ambiguity. For a linear map \(\alpha : A \to A\), denote by \(a^\alpha\) the \(n\)-fold composition of \(n\)-copies of \(a\), with \(a^\alpha \equiv \nu \nu \cdots \nu I d\).

**Definition 2.1.** A Hom-algebra \((A, \nu, \alpha)\) is a Hom-associative algebra if it satisfies the Hom-associativity condition, that is
\[
\nu(\alpha(x), \nu(y, z)) = \nu(\nu(x, y), \alpha(z)) = \nu(y, \nu(x, z))
\]
for all \(x, y, z \in A\).

**Remark 2.2.** When \(\alpha\) is the identity map, we recover the classical associativity condition, then usual associative algebras.

**Definition 2.3.** A ternary Hom-Nambu algebra is a triple \((A, \{., ., .\}, \tilde{\alpha})\) consisting of a \(\mathbb{K}\)-vector space \(A\), a ternary map \(\{., ., .\}\)


A ternary (non-commutative) Hom-Nambu-Poisson algebra. A linear map \( f : A \rightarrow A \) is a morphism of (non-commutative) ternary Hom-Nambu-Poisson algebras if it satisfies for all \( x, y, z \in A \):

\[
 f(x, y, z) = (f(x), f(y), f(z))',
\]

(2.2)

\[
 f \circ \mu = \mu' \circ f^\otimes 2.
\]

(2.3)

\[
 f \circ \alpha = \alpha' \circ f.
\]

(2.4)

It said to be a weak morphism if hold only the two first conditions.

**Tensor Product and Direct Sums**

In this section we discuss direct sums and define tensor product of ternary (non-commutative) Hom-Nambu-Poisson algebra and a totally Hom-associative symmetric ternary algebra. In the following, we define a direct sum of two ternary (non-commutative) Hom-Nambu-Poisson algebras.

**Theorem 3.1.** Let \((A, \mu, \alpha, \beta)\) and \((A, \mu', \alpha', \beta')\) be two ternary (non-commutative) Hom-Nambu-Poisson algebras. Let \(\mu_{A\oplus A'}\) be a bilinear map on \(A \oplus A'\) defined for \(x, y, z \in A\) and \(x', y', z' \in A'\) by \(\mu_{A\oplus A'}(x + x', y + y', z + z') = \mu(x, y, z) + \mu(x', y', z')\). \(A \oplus A'\) is a ternary algebra defined by \(\{x + x', y + y', z + z'\} = \{x, y, z\} + \{x', y', z'\}\) and \(\mu_{A\oplus A'}(x + x', y + y', z + z')\) is a linear map defined by \(\alpha_{A\oplus A'}(x + x') = \alpha_A(x') + \alpha_{A'}(x)\).

Then \((A \oplus A', \mu_{A\oplus A'}, \alpha_{A\oplus A'}, \beta_{A\oplus A'})\) is a ternary (non-commutative) Hom-Nambu-Poisson algebra.

**Proof.** The commutativity of \(\mu_{A\oplus A'}\) is obvious since \(\mu\) and \(\mu'\) are commutative. The skew-symmetry of the bracket follows from the skew-symmetry of \([..., ...]\) and \([..., ...]\). So it remains to check the Hom-associativity, the Hom-Nambu and the Hom-Leibniz identities. For Hom-associativity identity, we have

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_A(\alpha_A(x), \alpha_A(y), \alpha_A(z)),
\]

(2.5)

\[
\mu_{A\oplus A'}(\mu_A(x, y, z) + \mu_A(x', y', z')) = \mu_A(\mu_A(x, y, z), \alpha_A(x'), \alpha_A(y'), \alpha_A(z')).
\]

(2.6)

\[
\mu_{A\oplus A'}(\alpha_A(x), \mu_A(y, z) + \mu_A(y', z')) = \mu_A(\alpha_A(x), \mu_A(y, z), \alpha_A(y'), \alpha_A(z')).
\]

(2.7)

\[
\mu_{A\oplus A'}(\mu_A(x, y, z) + \mu_A(x', y', z'), \alpha_A(x'), \alpha_A(y'), \alpha_A(z')) = \mu_A(\mu_A(x, y, z) + \mu_A(x', y', z'), \alpha_A(x'), \alpha_A(y'), \alpha_A(z')).
\]

(2.8)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.9)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.10)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.11)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.12)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.13)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.14)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.15)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.16)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.17)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.18)

\[
\mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')) = \mu_{A\oplus A'}(\alpha_{A\oplus A'}(x + x', y + y', z + z')).
\]

(2.19)

\[
\mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')) = \mu_{A\oplus A'}(\mu_{A\oplus A'}(x + x', y + y', z + z'), \alpha_{A\oplus A'}(x + x'), \alpha_{A\oplus A'}(y + y'), \alpha_{A\oplus A'}(z + z')).
\]

(2.20)
Finally, for Hom-Leibniz identity we have
\[
\{\mu_{\otimes} \circ (x, x', x'' + x), \alpha_{\otimes} \circ (x, x'), \alpha_{\otimes} \circ (x, x')\}_{A \otimes A} = \{\mu_{\otimes} (x, x', x''), \alpha_{\otimes} (x, x'), \alpha_{\otimes} (x, x')\}_{A \otimes A} + \{\mu_{\otimes} (x, x', x''), \alpha_{\otimes} (x, x'), \alpha_{\otimes} (x, x')\}_{A \otimes A}.
\]

Let \( A_1, (A, m, a) \), where \( \alpha = \langle \alpha \rangle_{A \otimes A} \) and \( A_2 = (A', m', a') \) where \( \alpha = \langle \alpha \rangle_{A \otimes A} \) be two ternary (non-commutative) Hom-algebras of a given he tensor product \( A \otimes A_1 \) is a ternary Hom-algebra defined by the triple \((A \otimes A_1, m, a')\) where \( A \otimes a = (\alpha \otimes \alpha \otimes \alpha)_{A \otimes A} \) with
\[
m \circ (m(x, x', x''), (A \otimes A_1, m, a') = m(m(x, x', x''), x, x''), m(x, x', x''), x, x'').
\]

This ends the proof.

**Proposition 3.2.** Let \( (A_1, \mu_1, \ldots, \mu_1, \alpha_1) \) and \((A_2, \mu_2, \ldots, \alpha_2)\) be two ternary (non-commutative) Hom-Nambu-Poisson algebras. A linear map \( \phi: A \to A_1 \) is a morphism of ternary (non-commutative) Hom-Nambu-Poisson algebras if and only if
\[
\phi \circ (\Gamma_1) \subseteq \phi \circ (\Gamma_2).
\]

**Proof.** Let \( \phi: \langle A, \mu_1, \ldots, \alpha_1 \rangle \to \langle A, \mu_2, \ldots, \alpha_2 \rangle \) be a morphism of ternary Hom-Nambu-Poisson algebras. We have
\[
\{\phi(x), \phi(x'), \phi(x'')\}_{A \otimes A} = \{\phi(x, x', x''), \phi(x, x', x'')\}_{A \otimes A} = \{\phi(x), \phi(x'), \phi(x'')\}_{A \otimes A},
\]
for all \( x, x', x'' \in A \). Therefore \( \phi \) is a morphism of ternary Hom-Nambu-Poisson algebras.

Now, we define the tensor product of two ternary Hom-algebras. Moreover, we consider a tensor product of a ternary Hom-Nambu-Poisson algebra and a totally Hom-associative symmetric ternary algebra.
Using ternary Nambu identity of \(\{.,.,.\}\) we have \(a=e+e+g\), and \(b=d+f= fh\) using the symmetry of \(\tau\) and Hom-associativity of \(\mu\), then the left hand side is equal to the right hand side from where the ternary Hom- Nambu identity of bracket \(\{.,.,.\}\) is verified.

For the Hom-Leibniz identity, we have

\[
\begin{align*}
\text{LHS} &= (\mu \otimes (a \otimes b), a, a \otimes (a \otimes b))_\beta \\
&= (\mu(a, b)) \otimes (a \otimes (a \otimes b))_\beta \\
&= (\mu(a, b), a, a \otimes (a \otimes b))_\beta
\end{align*}
\]

Then it remains to show Hom-Leibniz identity

\[
\mu(a, b)(\mu(x, y), \beta(x, y)) = \beta^2(\mu(x, \mu(x, y)), \mu(y, z))
\]

Therefore \(A'=\langle\{.,.,.\}, \mu, \beta, \beta\rangle\) is a ternary (non-commutative) Hom-Nambu-Poisson algebra. For the multiplicity assertion, suppose that \(A\) is multiplicative and \(\beta\) is an algebra morphism. We have

\[
(\beta a \circ o)(\mu) = \beta a \circ o \mu = \mu \circ o \beta^a \circ o^\beta = \mu \circ o(\beta a \circ o^\beta)
\]

And

\[
\beta a \circ o(\{.,.,.\}) = \beta a \circ o(\{.,.,.\}) = \{.,.,.\} \circ o(\beta a \circ o^\beta)
\]

Then \(A_\beta\) is multiplicative.

**Corollary 4.2.** Let \((A, \mu, \{.,.,.\}, a)\) be a multiplicative ternary (non-commutative) Hom-Nambu-Poisson algebra. Then

\[
A_\beta = (\mu, \{.,.,.\}, \beta, \beta)\text{ is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra for each integer } n \geq 0.
\]

**Proof.** The multiplicity of \(A\) implies that \(\alpha^* : A \rightarrow A\) is a Nambu-Poisson algebra morphism. By Theorem 4.1. \(A_\beta\) is a multiplicative ternary (non-commutative) Hom-Nambu-Poisson algebra.

**Corollary 4.3.** Let \(A, \mu, \{.,.,.\}, a\) be a ternary (non-commutative) Nambu-Poisson algebra and \(\beta : A \rightarrow A\) be a Nambu-Poisson algebra morphism. Then

\[
A_\beta = (\mu, \{.,.,.\}, \beta, \beta)\text{ is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra.}
\]

**Remark 4.4.** Let \((A, \mu, \{.,.,.\}, a)\) and \((A', \mu', \{.,.,.\}', a')\) be two (non-commutative) ternary Nambu-Poisson algebra and \(\beta : A \rightarrow A\) be ternary Nambu-Poisson endomorphisms. If \(\phi : A \rightarrow A\) is a ternary Nambu-Poisson algebra morphism that satisfies \(\phi \circ o = \phi \circ o^\beta\) then

\[
\phi : (A, \mu, \{.,.,.\}, \beta, \beta) \rightarrow (A', \mu', \{.,.,.\}', \beta, \beta)
\]

is a (non-commutative) ternary hom-nambu poisson algebra morphism.

Indeed, we have

\[
\phi (\{.,.,.\}) = \phi (\{.,.,.\}) = \{.,.,.\} \circ o^\phi = \{.,.,.\} \circ o^\beta = \{.,.,.\} \circ o^\beta
\]

Then it remains to show Hom-Leibniz identity

\[
\mu(a, b)(\mu(x, y), \beta(x, y)) = \beta^2(\mu(x, \mu(x, y)), \mu(y, z))
\]

Theorem 4.1. Let \((A, \mu, \{.,.,.\}, a)\) be a (non-commutative) ternary Hom-Nambu-Poisson algebra and \(\beta : A \rightarrow A\) be a weak Hom-Nambu-Poisson morphism, then \(A_\beta = \{A, \mu, \{.,.,.\}, \beta, \beta\}\) is also a ternary (non-commutative) Hom-Nambu-Poisson algebra. Moreover, if \(A\) is multiplicative and \(\beta\) is an algebra morphism, then \(A_\beta\) is a multiplicative (non-commutative) Hom-Nambu-Poisson algebra.

**Construction of Ternary Hom-Nambu-Poisson Algebras.**

In this section, we provide constructions of ternary Hom-Nambu-Poisson algebras using twisting principle.

Theorem 4.1. Let \((A, \mu, \{.,.,.\}, a)\) be a (non-commutative) ternary Hom-Nambu-Poisson algebra and \(\beta : A \rightarrow A\) be a weak Hom-Nambu-Poisson morphism, then \(A_\beta = \{A, \mu, \{.,.,.\}, \beta, \beta\}\) is also a ternary (non-commutative) Hom-Nambu-Poisson algebra. Moreover, if \(A\) is multiplicative and \(\beta\) is an algebra morphism, then \(A_\beta\) is a multiplicative (non-commutative) Hom-Nambu-Poisson algebra.

Proof. If \(\mu\) is commutative, then clearly so is \(\mu_\beta\). The rest of the proof applies whether \(\mu\) is commutative or not. The skew-symmetry follows from the skew- symmetry of the bracket \(\{.,.,.\}\). It remains to prove Hom-associativity condition, Hom-Nambu-identity and Hom-Liebnniz identity.
Theorem 4.5. A morphism \( \alpha : \mathbb{R} [x, y, z] \rightarrow \mathbb{R} [x, y, z] \) which gives a structure of ternary Hom-Nambu-Poisson algebra \((\mathbb{R} [x, y, z], \alpha, \cdots, \alpha)\) satisfies the following equation:

\[
1 - \begin{vmatrix}
\frac{\partial \alpha (x)}{\partial x} & \frac{\partial \alpha (x)}{\partial y} & \frac{\partial \alpha (x)}{\partial z} \\
\frac{\partial \alpha (y)}{\partial x} & \frac{\partial \alpha (y)}{\partial y} & \frac{\partial \alpha (y)}{\partial z} \\
\frac{\partial \alpha (z)}{\partial x} & \frac{\partial \alpha (z)}{\partial y} & \frac{\partial \alpha (z)}{\partial z}
\end{vmatrix} = 0
\]  

(4.1)

Proof. Let \( \alpha \) be a Nambu-Poisson algebra morphism, then it satisfies for all \( f, g, h \in \mathbb{R} [x, y, z] \)

\[
\alpha (f \cdot g) = \alpha (f) \cdot \alpha (g),
\]

\[
\alpha (f \cdot g \cdot h) = [\alpha (f), \alpha (g), \alpha (h)].
\]

The first equality shows that it is sufficient to just set \( \alpha \) on \( x, y \) and \( z \). For the second equality, we suppose by linearity that

\[
f(x, y, z) = x^i y^j z^k,
\]

\[
g(x, y, z) = x^i y^j z^k, \quad f(x, y, z) = x^i y^j z^k.
\]

Then we can write the second equation as follows

\[
1 - \begin{vmatrix}
\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\
\frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\
\frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}
\end{vmatrix} = 0
\]  

which can be simplified to

\[
1 - \begin{vmatrix}
\frac{\partial \alpha (x)}{\partial x} & \frac{\partial \alpha (x)}{\partial y} & \frac{\partial \alpha (x)}{\partial z} \\
\frac{\partial \alpha (y)}{\partial x} & \frac{\partial \alpha (y)}{\partial y} & \frac{\partial \alpha (y)}{\partial z} \\
\frac{\partial \alpha (z)}{\partial x} & \frac{\partial \alpha (z)}{\partial y} & \frac{\partial \alpha (z)}{\partial z}
\end{vmatrix} = 0
\]  

(4.2)

Example 4.6. We set polynomials:

\[
\alpha (x) = P_1 (x, y, z) = \sum_{a_1, a_2, a_3} a_{a_1 a_2 a_3} x^{a_1} y^{a_2} z^{a_3},
\]

\[
\alpha (y) = P_2 (x, y, z) = \sum_{b_1, b_2, b_3} b_{b_1 b_2 b_3} x^{b_1} y^{b_2} z^{b_3},
\]

\[
\alpha (z) = P_3 (x, y, z) = \sum_{c_1, c_2, c_3} c_{c_1 c_2 c_3} x^{c_1} y^{c_2} z^{c_3},
\]

Where \( P_1, P_2, P_3 \in \mathbb{R} [x, y, z] \), and \( d \) the largest degree for each variable. We assume that \( a_{a} = b_{b} = c_{c} = 0 \)

Case of polynomials of degree one. We take

\[
P_1 (x, y, z) = a_{111} x + a_{112} y + a_{113} z,
\]

\[
P_2 (x, y, z) = b_{111} x + b_{112} y + b_{113} z,
\]

\[
P_3 (x, y, z) = c_{111} x + c_{112} y + c_{113} z
\]

Equation (2.5) becomes

\[
\begin{vmatrix}
\frac{\partial P_1 (x, y, z)}{\partial x} & \frac{\partial P_1 (x, y, z)}{\partial y} & \frac{\partial P_1 (x, y, z)}{\partial z} \\
\frac{\partial P_2 (x, y, z)}{\partial x} & \frac{\partial P_2 (x, y, z)}{\partial y} & \frac{\partial P_2 (x, y, z)}{\partial z} \\
\frac{\partial P_3 (x, y, z)}{\partial x} & \frac{\partial P_3 (x, y, z)}{\partial y} & \frac{\partial P_3 (x, y, z)}{\partial z}
\end{vmatrix} = 0,
\]

(4.3)

Whence

\[
1 - \frac{a_1}{b_1} - \frac{a_2}{c_2} - \frac{a_3}{c_3} = 0.
\]

(4.4)

The polynomials \( P_1, P_2, P_3 \) and \( P_4 \) are of one of this form

\[
P_1 (x, y, z) = a_{11} x + a_{12} y + a_{13} z,
\]

\[
P_2 (x, y, z) = b_{11} x + b_{12} y + b_{13} z,
\]

\[
P_3 (x, y, z) = c_{11} x + c_{12} y + c_{13} z,
\]

\[
P_4 (x, y, z) = d_{11} x + d_{12} y + d_{13} z.
\]

(4.5)

(4.6)

(4.7)

(4.8)

(4.9)

(4.10)

(4.11)

(4.12)

(4.13)

(4.14)

Particular case of polynomials of degree two. We take one of the polynomials of degree two

\[
P_1 (x, y, z) = a_{11} x + a_{12} y + a_{13} z,
\]

\[
P_2 (x, y, z) = b_{11} x + b_{12} y + b_{13} z,
\]

\[
P_3 (x, y, z) = c_{11} x + c_{12} y + c_{13} z.
\]

The polynomials \( P_1, P_2, P_3 \) and \( P_4 \) are of one of this form

\[
P_1 (x, y, z) = a_{11} x + a_{12} y + a_{13} z,
\]

\[
P_2 (x, y, z) = b_{11} x + b_{12} y + b_{13} z,
\]

\[
P_3 (x, y, z) = c_{11} x + c_{12} y + c_{13} z.
\]

(1)
Classification

In this section, we provide the classification of 3-dimensional ternary non-commutative Nambu-Poisson algebras. By straightforward calculations and using a computer algebra system we obtain the following result.

**Theorem 5.1.** Every 3-dimensional ternary Nambu-Lie algebra is isomorphic to the ternary algebra defined with respect to basis \( \{ e_1, e_2, e_3 \} \), by the skew-symmetric bracket defined as

\[
[e_1, e_2, e_3] = e_1
\]

Moreover it defines a 3-dimensional ternary non-commutative Nambu-Poisson algebra \((A, \{ , , \}, \mu)\) if and only if \( \mu \) is one of the following non-commutative associative algebra defined as

\[
\begin{align*}
\mu(e_1, e_2) &= acc_1, \\
\mu(e_2, e_3) &= bce_1, \\
\mu(e_3, e_1) &= a(he_1 + ge_2 + e_1).
\end{align*}
\]

where \( a, b \) are parameters.

The opposite algebra of (2)

The multiplication which are not mentioned are equal to zero.

**Remark 5.2.** The 3-dimensional ternary Nambu-Lie algebra is endowed with a commutative Nambu-Poisson algebra structure only when the multiplication is trivial.

Using the twisting principle described in Theorem 4.1, we obtain the following 3-dimensional non-commutative ternary Hom-Nambu-Poisson algebras.

**Proposition 5.3.** Any 3-dimensional ternary non-commutative Hom-Nambu-Poisson algebra \((A, \{ , , \}, \mu, \alpha)\) obtained by a twisting defined with respect to the basis \( \{ e_1, e_2, e_3 \} = e_1 \), where \( \alpha \) is a parameter, is one of the following binary Hom-associative algebras defined by \( \mu_\alpha \) and a corresponding structure map

\[
\begin{align*}
\mu_\alpha(e_1, e_2) &= acc_1, \\
\mu_\alpha(e_2, e_3) &= bce_1, \\
\mu_\alpha(e_3, e_1) &= a(he_1 + ge_2 + e_1).
\end{align*}
\]

With

\[
\alpha_\alpha_1(e_1) = cce_1, \alpha_\alpha_2(e_1) = dce_1 + e_2, \alpha_\alpha_3(e_1) = he_1 + ge_2 + e_1.
\]

\[
\mu_\alpha(e_1, e_2) = acc_1, \quad \mu_\alpha(e_2, e_3) = bce_1, \quad \mu_\alpha(e_3, e_1) = a(he_1 + ge_2 + e_1),
\]

With

\[
\alpha_\alpha_1(e_1) = cce_1, \alpha_\alpha_2(e_1) = dce_1 + e_2, \alpha_\alpha_3(e_1) = he_1 + ge_2 + e_1.
\]

\[
\mu_\alpha(e_1, e_2) = acc_1, \quad \mu_\alpha(e_2, e_3) = bce_1, \quad \mu_\alpha(e_3, e_1) = a(he_1 + ge_2 + e_1),
\]

With

\[
\alpha_\alpha_1(e_1) = cce_1, \alpha_\alpha_2(e_1) = dce_1 + e_2, \alpha_\alpha_3(e_1) = he_1 + ge_2 + e_1.
\]
References