Journal of Generalized Lie Theory and Applications

Makhlouf and Amri, J Generalized Lie Theory Appl 2015, 9:1
DOI: 10.4172/1736-4337.1000221

Research Article Open Access

Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras

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Abstract

The main purpose of this paper is to study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. We provide construction results dealing with tensor product and direct sums of two (non- commutative) ternary (Hom-) Nambu-Poisson algebras. Moreover, we explore twisting principle of (non-commutative) ternary Nambu-Poisson algebras along with algebra morphism that lead to construct (non-commutative) ternary Hom-Nambu-Poisson algebras. Furthermore, we provide examples and a 3-dimensional classification of non-commutative ternary Nambu-Poisson algebras.

Keywords: Hom-nambu poisson algebra; Ternary nambu poisson; Non-commutative ternary; *n*-ary

Introduction

In the 70's, Nambu proposed a generalized Hamiltonian system based on a ternary product, the Nambu-Poisson bracket, which allows to use more than one hamiltonian [1]. More recent motivation for ternary brackets appeared in string theory and M-branes, ternary Lie type structure was closely linked to the super-symmetry and gauge symmetry transformations of the world-volume theory of multiple coincident M2-branes and was applied to the study of Bagger-Lambert theory. Moreover ternary operations appeared in the study of some quarks models. In 1996, quantizations of Nambu-Poisson brackets were investigated [2], it was presented in a novel approach of Zariski, and this quantization is based on the factorization on $\mathbb{R}$ of polynomials of several variables.

The algebraic formulation of Nambu mechanics was discussed [3] and Nambu algebras was studied [4] as a natural generalization of a Lie algebra for higher- order algebraic operations. By definition, Nambu algebra of order n over a field $\mathbb{K}$ of characteristic zero consists of a vector space V over $\mathbb{K}$ together with a $\mathbb{K}$-multilinear skew-symmetric operation $[.,\cdots,.]:\Lambda^n V\rightarrow V$, called the Nambu bracket that satisfies the following generalization of the Jacobi identity. Namely, for any $x_1,\dots,x_{n-1}\in V$ define an adjoint action $ad(x_1, \dots, x_{n-1}): V\rightarrow V$ by $ad(x_1, \dots, x_{n-1})x_n=[x_1, \dots, x_{n-1}, x_n]$, $x_n\in V$. Then the fundamental identity is a condition saying that the ad joint action is a derivation with respect the Nambu bracket, i.e. for all $x_1, \dots, x_{n-1}, y_1, \dots, y_n\in V$

$$ad(x_1,\dots,x_{n-1})[y_1,\dots,y_n]=\sum_{k=1}^{n}[y_1,\dots,ad(x_1,\dots,x_{n-1})y_k,\dots,y_n]. \quad (0.1)$$

When n=2, the fundamental identity becomes the Jacobi identity and we get a definition of a Lie algebra.

Different aspects of Nambu mechanics, including quantization, deformation and various algebraic constructions for Nambu algebras have recently been studied. Moreover a twisted generalization, called Hom-Nambu algebras, was introduced [5]. This kind of algebras called Hom-algebras appeared as deformation of algebras of vector fields using σ-derivations. The first examples concerned q-deformations of Witt and Virasoro algebras. Then Hartwig, Larsson and Silvestrov introduced a general framework and studied Hom-Lie algebras [6], in which Jacobi identity is twisted by a homomorphism. The corresponding associative algebras, called Hom-associative algebras were introduced [7]. Non-commutative Hom-Poisson algebras were discussed [8]. Likewise, n-ary algebras of Hom-type were introduced [5,9-13].

We aim in this paper to explore and study non-commutative ternary Nambu-Poisson algebras and their Hom-type version. The paper includes five Sections. In the first one, we summarize basic definitions of (non-commutative) ternary Nambu- Poisson algebras and discuss examples. In the second Section, we recall some basics about Hom-algebra structures and introduce the notion of (non-commutative) ternary Hom-Nambu-Poisson algebra. Section 3 is dedicated to construction of (non-commutative) ternary Hom-Nambu-Poisson algebras using direct sums and tensor products. In Section 4, we extend twisting principle to ternary Hom-Nambu- Poisson algebras. It is used to build new structures with a given ternary (Hom-) Nambu-Poisson algebra and algebra morphism. This process is used to construct ternary Hom-Nambu-Poisson algebras corresponding to the ternary algebra of polynomials where the bracket is defined by the Jacobian. We provide in the last section a classification of 3-dimensional ternary Nambu-Poisson algebras and then compute corresponding Hom-Nambu-Poisson algebras using twisting principle. Notice that a complete classification of 3-dimensional Hom-Nambu-Poisson algebras is difficult to obtain since so far the classification of 3-dimensional Hom-Nambu-Lie algebras is not known.

Ternary (Non-Commutative) Nambu-Poisson Algebra

In the section we review some basic definitions and fix notations. In the sequel, A denotes a vector space over $\mathbb{K}$, where $\mathbb{K}$ is an algebraically closed field of characteristic zero. Let $\mu : A\times A\rightarrow A$ be a bilinear map, we denote by $\mu^{op}: A^{\times 2}\rightarrow A$ the opposite map, i.e., $\mu^{op}=\mu\circ\tau$ where $\tau: A^{\times 2}\rightarrow A^{\times 2}$ interchanges the two variables.

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Received October 17, 2014; **Accepted** June 30, 2015; **Published** July 07, 2015

Citation: Makhlouf A, Amri A (2015) Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras. J Generalized Lie Theory Appl 9: 221. doi:10.4172/1736-4337.1000221

A ternary algebra is given by a pair (A, m), where m is a ternary operation on A, that is a trilinear map $m : A \times A \times A \to A$, which is denoted sometimes by brackets.

Definition 1.1. *A ternary Nambu algebra is a ternary algebra* $(A, \{\ ,\ ,\ \})$ *satisfying the fundamental identity defined as*

$$\{x_1, x_2, \{x_3, x_4, x_5\}\}=$$

$$\{\{x_1, x_2, x_3\}, x_4, x_5\} + \{x_3, \{x_1, x_2, x_4\}, x_5\} + \{x_3, x_4, \{x_1, x_2, x_5\}\} \quad (1.1)$$

for all $x_1, x_2, x_3, x_4, x_5 \in A$.

This identity is sometimes called Filippov identity or Nambu identity, and it is equivalent to the identity (0.1) with n=3.

A *ternary Nambu-Lie algebra* or 3-Lie algebra is a ternary Nambu algebra for which the bracket is skew-symmetric, that is for all $\sigma \in S_3$, where S_3 is the permutation group,

$$\left[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}\right] = Sgn(\sigma)\left[x_1, x_2, x_3\right].$$

Let A and A' be two ternary Nambu algebras (resp. Nambu-Lie algebras). A linear map $f : A \to A'$ is a *morphism* of a ternary Nambu algebras (resp. ternary Nambu-Lie algebras) if it satisfies $f(\{x, y, z\}_A) = \{f(x), f(y), f(z)\}_{A'}$.

Example 1.2. *The polynomials of variables* x_1, x_2, x_3 *with the ternary operation defined by the Jacobian function:*

$$\{f_1, f_2, f_3\} = \begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \frac{\partial f_3}{\partial x_3} \end{vmatrix}, \quad (1.2)$$

is *a ternary Nambu-Lie algebra.*

Example 1.3. *Let* $V = \mathbb{R}^4$ *be the 4-dimensional oriented Euclidian space over* $\mathbb{R}$. *The bracket of 3 vectors* $\vec{x}, \vec{y}, \vec{z}$ *is given by*

$$[x, y, z] = \vec{x} \times \vec{y} \times \vec{z} = \begin{vmatrix} x_1 & y_1 & z_1 & e_1 \\ x_2 & y_2 & z_2 & e_2 \\ x_3 & y_3 & z_3 & e_3 \\ x_4 & y_4 & z_4 & e_4 \end{vmatrix},$$

Where $\{e_1, e_2, e_3, e_4\}$ *is a basis of* $\mathbb{R}^4$ *and* $\vec{x} = \sum_{i=1}^{3} x_i \vec{e_i}$, $\vec{y} = \sum_{i=1}^{3} y_i \vec{e_i}$ *and* $\vec{z} = \sum_{i=1}^{3} z_i \vec{e_i}$. *Then* $(V, [.,.,.])$ *is a ternary Nambu – lie algebra.*

Now, we introduce the notion of (non-commutative) ternary Nambu-Poisson algebra.

Definition 1.4. A *non-commutative ternary Nambu-Poisson algebra* is a triple $(A, \mu, \{., ., .\})$ consisting of a $\mathbb{K}$-vector space A, a bilinear map $\mu : A \times A \to A$ and a trilinear map $\{., ., .\} : A \otimes A \otimes A \to A$ such that

(1) (A, μ) is a binary associative algebra,

(2) $(A, \{., ., .\})$ is a ternary Nambu-Lie algebra,

(3) the following Leibniz rule

$\{x_1, x_2, \mu(x_3, x_4)\} = \mu(x_3, \{x_1, x_2, x_4\}) + \mu(\{x_1, x_2, x_3\}, x_4)$

holds for all $x_1, x_2, x_3 \in A$.

A ternary Nambu-Poisson algebra is a non-commutative ternary Nambu-Poisson algebra $(A, \mu, \{., ., .\})$ for which μ is commutative, then μ is commutative unless otherwise stated.

In a (non-commutative) ternary Nambu-Poisson algebra, the ternary bracket $\{., ., .\}$ is called Nambu-Poisson bracket.

Similarly, a non-commutative n-ary Nambu-Poisson algebra is a triple $(A, \mu, \{., \bullet\bullet\bullet, .\})$

where $(A, \{., \bullet\bullet\bullet, .\})$ defines an n-Lie algebra satisfying similar Leibniz rule with respect to μ.

A morphism of (non-commutative) ternary Nambu-Poisson algebras is a linear map that is a morphism of the underlying ternary Nambu-Lie algebras and associative algebras.

Example 1.5. *Let* $C^\infty(\mathbb{R}^3)$ *be the algebra of* C^∞ *functions on* $\mathbb{R}^3$ *and* x_1, x_2, x_3 *the coordinates on* $\mathbb{R}^3$. *We define the ternary brackets as in (1.2), then* $(C^\infty(\mathbb{R}^3), \{., ., .\})$ *is a ternary Nambu-Lie algebra. In addition the bracket satisfies the Leibniz rule:* fg, $\{fg, f_2, f_3\} = f\{g, f_2, f_3\} + \{f, f_2, f_3\}g$ *where* $f, g, f_2, f_3 \in C^\infty(\mathbb{R}^3)$ *and the multiplication being the point wise multiplication that is* $fg(x) = f(x)g(x)$. *Therefore, the algebra is a ternary Nambu-Poisson algebra.*

This algebra was considered already in 1973 by Nambu [9] *as a possibility of extending the Poisson bracket of standard hamiltonian mechanics to bracket of three functions defined by the Jacobian. Clearly, the Nambu bracket may be generalized further to a Nambu-Poisson allowing for an arbitrary number of entries.*

In particular, the algebra of polynomials of variables x_1, x_2, x_3 *with the ternary operation defined by the Jacobian function in (1.2), is a ternary Nambu-Poisson algebra.*

Remark 1.6. The n-dimensional ternary Nambu-Lie algebra of Example 1.3 does not carry a non-commutative Nambu-Poison algebra structure except that one given by a trivial multiplication.

Hom-type (Non-Commutative) Ternary Nambu-Poisson Algebras

In this section, we present various Hom-algebra structures. The main feature of Hom-algebra structures is that usual identities are deformed by an endomorphism and when the structure map is the identity, we recover the usual algebra structure.

A Hom-algebra (resp. ternary Hom-algebra) is a triple (A, ν, α) consisting of a $\mathbb{K}$-vector space A, a bilinear map $\nu : A \times A \to A$ (resp. a trilinear map $\nu : A \times A \times A \to A$) and a linear map $\alpha : A \to A$. A Hom-algebra (A, μ, α) is said to be multiplicative if $\alpha \circ \mu = \mu \circ \alpha^{\otimes 2}$ and it is called commutative if $\mu = \mu^{op}$. A ternary Hom-algebra (A, m, α) is said to be multiplicative if $\alpha \circ m = m \circ \alpha^{\otimes 3}$. Classical algebras (resp. ternary algebras) are regarded as Hom-algebras (resp. ternary Hom-algebras) with identity twisting map. We will often use the abbreviation xy for $\mu(x, y)$ when there is no ambiguity. For a linear map $\alpha : A \to A$, denote by α^n the n-fold composition of n-copies of α, with $\alpha^0 \equiv Id$.

Definition 2.1. A Hom-algebra (A, μ, α) is a *Hom-associative algebra* if it satisfies the Hom-associativity condition, that is

$\mu(\alpha(x), \mu(y, z)) = \mu(\mu(x, y), \alpha(z))$ for all $x, y, z \in A$.

Remark 2.2. When α is the identity map, we recover the classical associativity condition, then usual associative algebras.

Definition 2.3. A ternary Hom-Nambu algebra is a triple $(A, \{., ., .\}, \tilde{\alpha})$ consisting of a $\mathbb{K}$-vector space A, a ternary map $\{., .,$

.} : $A \times A \times A \to A$ and a pair of $\tilde{\alpha} = (\alpha_1, \alpha_2)$ where $\alpha_1, \alpha_2 : A \to A$, satisfying [5]

$$\{\alpha_1(x_1),\alpha_2(x_2),\{x_3,x_4,x_5\}\} = \{\{x_1,x_2,x_3\},\alpha_1(x_4),\alpha_2(x_5)\} + \{\alpha_1(x_3),\{x_1,x_2,x_4\},\alpha_2(x_5)\} + \{\alpha_1(x_3),\alpha_2(x_4),\{x_1,x_2,x_5\}\}. \quad (2.1)$$

We call the above condition the ternary Hom-Nambu identity.

Generally, the n-ary Hom-Nambu algebras are defined by an n-ary bracket and maps $\alpha_1, \bullet\bullet\bullet, \alpha_{n-1}$, satisfying the following Hom-Nambu identity

$$\{\alpha_1(x_1), \dots, \alpha_{n-1}(x_{n-1}), \{x_n, \dots, x_{2n-1}\}\}$$

$$= \sum_{i=n}^{2n-1} \{\alpha_1(x_n),\dots,\alpha_{i-n}(x_{i-1}),\{x_1,\dots,x_{n-1},x_i\},\alpha_{i-n+1}(x_{i+1})\dots,\alpha_{n-1}(x_{2n-1})\}$$

for all $(x_1, \dots, x_{2n-1}) \in A^{2n-1}$.

Remark 2.4. A Hom-Nambu algebra is a *Hom-Nambu-Lie* algebra if the bracket is skew-symmetric.

We introduce now the definition of non-commutative ternary Hom-Nambu-Poisson algebra in its general form, involving three linear maps. Next, we will discuss the class in which these three maps are equal. This particular case suits to provide a twisting construction.

Definition 2.5. A *non-commutative ternary Hom-Nambu-Poisson algebra* is a tuple $(A, \mu, \beta, \{.,.,.\}, \tilde{\alpha})$ consisting of a vector space A, a ternary operation $\{.,.,.\} : A\times A\times A \to A$, a binary operation $\mu : A\times A \to A$, a pair of linear maps $\tilde{\alpha} = (\alpha_1, \alpha_2)$

where $\alpha_1, \alpha_2 : A \to A$, and a linear map $\beta : A \to A$ such that:

(A, μ, β) is a binary Hom-associative algebra, (1)

$(A,\{.,.,.\}, \tilde{\alpha})$ is a ternary Hom-Nambu-Lie algebra,

$\{\mu(x_1, x_2), \alpha_1(x_3), \alpha_2(x_4)\}=\mu(\beta(x_1), \{x_2, x_3, x_4\}) + \mu(\{x_1, x_3, x_4\}, \beta(x_2))$.

The third condition is called Hom-Leibniz identity.

When, all the linear maps are equal $\alpha=\alpha_1=\alpha_2=\beta$, we refer to the ternary

Hom-Nambu-Poisson algebra by a quadruple $(A, \mu, \{.,.,.\}, \alpha)$.

Remark 2.6. Notice that μ is not assumed to be commutative. When μ is a commutative multiplication, then $(A, \mu, \beta, \{.,.,.\}, \tilde{\alpha})$ is said to be a ternary Hom-Nambu-Poisson algebra.

We recover the classical (non-commutative) ternary Nambu-Poisson algebra when $\alpha_1=\alpha_2=\beta=Id$.

Similarly, a non-commutative n-ary Hom-Nambu-Poisson algebra is a tuple $(A, \mu, \beta,\{., \dots ,.\}, \tilde{\alpha})$ where $(A,\{., \dots ,.\}, \tilde{\alpha})$ with linear maps $\tilde{\alpha} = (\alpha_1, \dots, \alpha_{n-1})$ that defines an n-ary Hom-Nambu-Lie algebra satisfying similar Leibniz rule with respect to (A, μ, β).

In the sequel we will mainly interested in the class of non-commutative ternary Nambu-Poisson algebras where $\alpha=\alpha_1=\alpha_2=\beta$.

Definition 2.7. Let $(A, \mu, \{.,.,.\}, \alpha)$ be a (non-commutative) ternary Hom-Nambu-Poisson algebra. It is said to be *multiplicative* if

$\alpha(\{x1, x2, x3\})=\{\alpha(x_1), \alpha(x_2), \alpha(x_3)\}$,

$\alpha \circ \mu=\mu \circ \alpha\otimes 2$.

If in addition α is bijective then it is called *regular*.

Definition 2.8. Let $(A, \mu, \{.,.,.\}, \alpha)$ and $(A',\mu',\{.,.,.\}',\alpha')$ be two (non-commutative) ternary Hom-Nambu-Poisson algebras. A linear map $f : A \to A'$ is a morphism of (non-commutative) ternary Hom-Nambu-Poisson algebras if it satisfies for all $x_1, x_2, x_3 \in A$:

$$f(\{x_1,x_2,x_3\}) = \{f(x_1), f(x_2), f(x_3)\}', \quad (2.2)$$

$$f \circ \mu = \mu' \circ f^{\otimes 2} \quad (2.3)$$

$$f \circ \alpha = \alpha' \circ f. \quad (2.4)$$

It said to be a weak morphism if hold only the two first conditions.

Tensor Product and Direct Sums

In this section we discuss direct sums and define tensor product of ternary (non-commutative) Hom-Nambu-Poisson algebra and a totally Hom-associative sym-metric ternary algebra. In the following, we define a direct sum of two ternary (non-commutative) Hom-Nambu-Poisson algebras.

Theorem 3.1. *Let $(A_1, \mu_1, \{.,.,.\}_1, \alpha_1)$ and $(A_2, \mu_2, \{.,.,.\}_2, \alpha_2)$ be two ternary (non-commutative) Hom-Nambu-Poisson algebras. Let $\mu_{A1\oplus A2}$ be a bilinear map on $A_1 \oplus A_2$ defined for $x_1, y_1, z_1 \in A_1$ and $x_2, y_2, z_2 \in A_1$ by $\mu(x_1+x_2, y_1+y_2)=\mu_1(x_1, y_1) + \mu_2(x_2, y_2)$, $\{.,.,.\}A_1\oplus A_2$ a trilinear map defined by $\{x_1+x_2, y_1+y_2, z_1+z_2\}_{A_1\oplus A_2} = \{x_1,y_1,z_1\}_1 + \{x_2,y_2,z_2\}_2$ and $\alpha_{A_1\oplus A_2}$ a linear map defined by $\alpha_{A_1\oplus A_2}(x_1+x_2) = \alpha_1(x_1)+\alpha_2(x_2)$.*

Then $(A_1 \oplus A_2, \mu_{A_1\oplus A_2}, \{.,.,.\}_{A_1\oplus A_2}, \alpha_{A_1\oplus A_2})$ is a ternary (non-commutative) Hom-Nambu-Poisson algebra.

Proof. The commutativity of $\mu_{A_1\oplus A_2}$ is obvious since μ_1 and μ_2 are commutative. The skew-symmetry of the bracket follows from the skew-symmetry of $\{.,.,.\}_1$ and $\{.,.,.\}_2$. So it remains to check the Hom-associativity, the Hom-Nambu and the Hom-Leibniz identities. For Hom-associativity identity, we have

$$\begin{aligned}
&\mu_{A_1\oplus A_2}(\mu_{A_1\oplus A_2}(x_1+x_1', x_2+x_2'), \alpha_{A_1\oplus A_2}(x_3+x_3'))\\
&= \mu_{A_1\oplus A_2}(\mu_1(x_1,x_2)+\mu_2(x_1',x_2'), \alpha_1(x_3)+\alpha_2(x_3'))\\
&= \mu_1(\mu_1(x_1,x_2),\alpha_1(x_3)) + \mu_2(\mu_2(x_1',x_2'),\alpha_2(x_3'))\\
&= \mu_1(\alpha_1(x_1),\mu_1(x_2,x_3)) + \mu_2(\alpha_2(x_1'),\mu_2(x_2',x_3'))\\
&= \mu_{A_1\oplus A_2}(\alpha_1(x_1)+\alpha_2(x_1'), \mu_1(x_2,x_3)+\mu_2(x_2',x_3'))\\
&= \mu_{A_1\oplus A_2}(\alpha_{A_1\oplus A_2}(x_1,x_1'), \mu_{A_1\oplus A_2}(x_2+x_2', x_3+x_3'))
\end{aligned}$$

Now we prove the Hom-Nambu identity

$$\begin{aligned}
&\{\alpha_{A_1\oplus A_2}(x_1+x_1'), \alpha_{A_1\oplus A_2}(x_2+x_2'), \{x_3+x_3', x_4+x_4', x_5+x_5'\}_{A_1\oplus A_2}\}_{A_1\oplus A_2}\\
&= \{\alpha_1(x_1)+\alpha_2(x_1'), \alpha_1(x_2)+\alpha_2(x_2'), \{x_3,x_4,x_5\}_1 + \{x_3',x_4',x_5'\}_2\}_{A_1\oplus A_2}\\
&= \{\alpha_1(x_1),\alpha_1(x_2),\{x_3,x_4,x_5\}_1\}_1 + \{\alpha_2(x_1'),\alpha_2(x_2'),\{x_3',x_4',x_5'\}_2\}_2\\
&= \{\{x_1,x_2,x_3\}_1,\alpha_1(x_4),\alpha_1(x_5)\}_1 + \{\alpha_1(x_3),\{x_1,x_2,x_4\}_1,\alpha_1(x_5)\}_1\\
&+\{\alpha_1(x_3),\alpha_1(x_4),\{x_1,x_2,x_5\}_1\}_1 + \{\{x_1',x_2',x_3'\}_2,\alpha_2(x_4'),\alpha_2(x_5')\}_2\\
&+\{\alpha_2(x_3'),\{x_1',x_2',x_4'\}_2,\alpha_2(x_5')\}_2 + \{\alpha_2(x_3'),\alpha_2(x_4'),\{x_1',x_2',x_5'\}_2\}_2\\
&= \{\{x_1,x_2,x_3\}_1 + \{x_1',x_2',x_3'\}_2, \alpha_1(x_4)+\alpha_2(x_4'), \alpha_1(x_5)+\alpha_2(x_5')\}_{A_1\oplus A_2}\\
&+\{\alpha_1(x_3)+\alpha_2(x_3'), \{x_1,x_2,x_4\}_1 + \{x_1',x_2',x_4'\}_2, \alpha_1(x_5)+\alpha_2(x_5')\}_{A_1\oplus A_2}\\
&+\{\alpha_1(x_3)+\alpha_2(x_3'), \alpha_1(x_3)+\alpha_2(x_3'), \{x_1,x_2,x_5\}_1 + \{x_1',x_2',x_5'\}_2\}_{A_1\oplus A_2}\\
&= \{\{x_1+x_1', x_2+x_2', x_3+x_3'\}_{A_1\oplus A_2}, \alpha_{A_1\oplus A_2}(x_4+x_4'), \alpha_{A_1\oplus A_2}(x_5+x_5')\}_{A1\oplus A_2}\\
&+\{\alpha_{A_1\oplus A_2}(x_3+x_3'), \{x_1+x_1', x_2+x_2', x_4+x_4'\}_{A_1\oplus A_2}, \alpha_{A_1\oplus A_2}(x_5+x_5')\}_{A1\oplus A_2}\\
&+\{\alpha_{A_1\oplus A_2}(x_3+x_3'), \alpha_{A_1\oplus A_2}(x_4+x_4'), \{x_1+x_1', x_2+x_2', x_5+x_5'\}_{A_1\oplus A_2}\}_{A_1\oplus A_2}
\end{aligned}$$

Finally, for Hom-Leibniz identity we have

$$\{\mu_{A_1\oplus A_2}(x_1+x_1',x_2+x_2'),\alpha_{A_1\oplus A_2}(x_3,x_3'),\alpha_{A_1+A_2}(x_4,x_4')\}_{A_1\oplus A_2}$$
$$=\{\mu_1(x_1,x_2)+\mu_2(x_1',x_2'),\alpha_1(x_3)+\alpha_2(x_3'),\alpha_1(x_4)+\alpha_2(x_4')\}_{A_1\oplus A_2}$$
$$=\{\mu_1(x_1,x_2),\alpha_1(x_3),\alpha_1(x_4)\}_1+\{\mu_2(x_1',x_2'),\alpha_2(x_3'),\alpha_2(x_4')\}_2$$
$$=\mu_1(\alpha_1(x_1),\{x_2,x_3,x_4\}_1)+\mu_1(\{x_1,x_3,x_4\}_1,\alpha_1(x_2))$$
$$+\mu_2(\alpha_2(x_1'),\{x_2',x_3',x_4'\}_2)+\mu_2(\{x_1',x_3',x_4'\}_2,\alpha_2(x_2'))$$
$$=\mu_{A_1\oplus A_2}(\alpha_{A_1\oplus A_2}(x_1,x_1'),\{x_2+x_2',x_3+x_3',x_4+x_4'\}_{A_1\oplus A_2})$$
$$+\mu_{A_1\oplus A_2}(\{x_1+x_1',x_3+x_3',x_4+x_4'\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2}(x_2,x_2'))$$

This ends the proof.

Proposition 3.2. *Let* $(A_1,\mu_1,\{.,.,.\}_1,\alpha_1)$ *and* $(A_2,\mu_2,\{.,.,.\}_2,\alpha_2)$ *be two ternary (non-commutative) Hom-Nambu-Poisson algebras. A linear map* $\phi: A_1\to A_2$ *is a morphism of ternary (non-commutative) Hom-Nambu-Poisson algebras if and only if* $\Gamma_\phi\subseteq A_1\oplus A_2$ *is a Hom-Nambu-Poisson subalgebra of*

$$(A_1\oplus A_2,\mu_{A_1\oplus A_2},\{.,.,.\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2})$$

Where $\Gamma_\phi=\{(x,\phi(x)):x\in A_1\}\subset A_1\oplus A_2$.

Proof. Let $\phi:(A_1,\mu_1,\{.,.,.\}_1,\alpha_1)\to(A_2,\mu_2,\{.,.,.\}_2,\alpha_2)$ be a morphism of ternary

Hom-Nambu-Poisson algebras.

We have

$$\{x_1+\phi(x_1),x_2+\phi(x_2),x_3+\phi(x_3)\}_{A_1\oplus A_2}=\{x_1,x_2,x_3\}_1+\{\phi(x_1),\phi(x_2),\phi(x_3)\}_2$$
$$=\{x_1,x_2,x_3\}_1+\phi\{x_1,x_2,x_3\}_1$$

Then Γ_ϕ is closed under the bracket $\{.,.,.\}_{A_1\oplus A_2}$.

We have also $(\alpha_1+\alpha_2)(x_1+\phi(x_1))=\alpha_1(x_1)+\alpha_2{}^\circ\phi(x_1)=\alpha_1(x_1)+\phi^\circ\alpha_1(x_1)$, which implies that $(\alpha_1+\alpha_2)\Gamma_\phi\subseteq\Gamma_\phi$.

Moreover Γ_ϕ is closed under the multiplication indeed

$$\mu_{A_1\oplus A_2}(x_1+\phi(x_1),x_2+\phi(x_2))=\mu_1(x_1,x_2)+\mu_2(\phi(x_1),\phi(x_2))$$
$$=\mu_1(x_1,x_2)+\phi^\circ\mu_1(x_1,x_2)\subseteq\Gamma_\phi$$

Conversely, if the graph $\Gamma_\phi\subseteq A_1\oplus A_2$ is a Hom-subalgebra of

$(A_1\oplus A_2,\mu_{A_1\oplus A_2},\{.,.,.\}_{A_1\oplus A_2},\alpha_{A_1\oplus A_2})$,

Then we have

$$\{\phi(x_1),\phi(x_2),\phi(x_3)\}_2=\phi\{x_1,x_2,x_3\}_1,$$

and

$$\alpha_1+\alpha_2)(x+\phi(x))=\alpha_1(x)+\alpha_2\,o\,\phi(x)\in\Gamma_\phi$$
$$=\alpha_1(x)+\phi\,o\,\alpha_1(x)\,.$$

Finally

$$\mu_{A_1\oplus A_2}(x_1+\phi(x_1),x_2+\phi(x_2))=\mu_1(x_1,x_2)+\mu_2(\phi(x_1),\phi(x_2))$$
$$=\mu_1(x_1,x_2)+\phi\,o\,\mu_1(x_1,x_2)\subseteq\Gamma_\phi.$$

Therefore ϕ is a morphism of ternary (non-commutative) Hom-Nambu-Poisson algebras.

Now, we define the tensor product of two ternary Hom-algebras. Moreover, we consider a tensor product of a ternary Hom-Nambu-Poisson algebra and a totally Hom-associative symmetric ternary algebra.

Let $A_1=(A, m, \alpha)$, where $\alpha=(\alpha_i)_{i=1,2}$ and $A_2=(A',m',\alpha')$ where $\alpha'=(\alpha_i')_{i=1,2}$ be two ternary (non- commutative) Hom-algebras of a given he tensor product $A_1\otimes A_2$ is a ternary Hom-algebra defined by the triple $(A\otimes A',m\otimes m',\alpha\otimes\alpha')$ where $\alpha\otimes\alpha'=(\alpha_i\otimes\alpha_i')_{i=1,2}$ with

$$m\otimes m'(x_1\otimes x_1',x_2\otimes x_2',x_3\otimes x_3')=m(x_1,x_2,x_3)\otimes m'(x_1',x_2',x_3'),\tag{3.1}$$
$$\alpha_i\otimes\alpha_i'(x_1\otimes x_{1'})=\alpha_i(x_1)\otimes\alpha_i'(x_1'),\tag{3.2}$$

Where $x_1, x_2, x_3\in A$ and $x_1',x_2',x_3'\in A_2$

Recall that (A, m, α) is a totally Hom- associative ternary algebra if

$$m(\alpha_1(x_1),\alpha_2(x_2),m(x_3,x_4,x_5))=m(\alpha_1(x_1),m(x_2,x_3,x_4),\alpha_2(x_5))$$
$$=m(m(x_1,x_2,x_3),\alpha_1(x_4),\alpha_2(x_5)).$$

for all $x_1\cdots,x_5\in A$, and the ternary multiplication m is symmetric if

$$m(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})=m(x_1,x_2,x_3)\,.\tag{3.3}$$

for all $\sigma\in S_3,x_1,x_2,x_3\in A$

Lemma 3.3. *Let* $A_1=(A, m, \alpha)$ *and* $A_2=(A',m',\alpha')$ *be two ternary Hom- algebras of given type (Hom-Nambu, totally Hom-associative). If* m *is symmetric and* m' *is skew–symmetric then* $m\otimes m'$ *is skew symmetric.*

Proof. Straight forward

Theorem 3.4. *Let* $(A, \mu, \beta, \{.,.,.\}, (\alpha1, \alpha2))$ *be a ternary (non-commutative) Hom- Nambu-Poisson algebra,* $(B,\tau,(\alpha_1',\alpha_2'))$ *be a totally Hom-associative symmetric ternary algebra, and* (B,μ',β') *be a Hom-associative algebra, then*

$$(A\otimes B,\mu\otimes\mu',\beta\otimes\beta',\{.,.,.\}_{A\otimes B},(\alpha_1\otimes\alpha_1',\alpha_2\otimes\alpha_2'))$$

a (non-commutative) ternary Hom-Nambu-Poisson algebra if and only if

$$\tau(\mu'(b_1,b_2),b_3,b_4)=\mu'(b_1,\tau(b_2,b_3,b_4))=\mu'(\tau(b_1,b_3,b_4),b_2)\,.\tag{3.4}$$

Proof. Since μ and μ' are both Hom-associative multiplication whence a tensor product $\mu\otimes\mu'$ is Hom-associative. Also the commutativity of $\mu\otimes\mu'$, the skew- symmetry of $\{.,.,.\}$ and the symmetry of τ simply the skew-symmetry of $\{.,.,.\}_{A\otimes B}$.

Therefore, it remains to check Hom-Nambu identity and Hom-Leibniz identity

We have

$$LHS=\{\alpha_1\otimes\alpha_1'(a_1\otimes b_1),\alpha_2\otimes\alpha_2'(a_2\otimes b_2),\{a_3\otimes b_3,a_4\otimes b_4,a_5\otimes b_5\}_{A\otimes B}\}_{A\otimes B}$$
$$=\{\alpha_1(a_1)\otimes\alpha_1'(b_1),\alpha_2(a_2)\otimes\alpha_2'(b_2),\{a_3,a_4,a_5\}_A\otimes\tau(b_3,b_4,b_5)\}_{A\otimes B}$$
$$=\underbrace{\{\alpha_1(a_1),\alpha_2(a_2),\{a_3,a_4,a_5\}\}}_{a}\otimes\underbrace{\tau(\alpha_1'(b_1),\alpha_2'(b_2),\tau(b_3,b_4,b_5))}_{b},$$

and

$$RHS=\{\{a_1\otimes b_1,a_2\otimes b_2,a_3\otimes b_3\}_{A\otimes B},\alpha_1\otimes\alpha_1'(a_4\otimes b_4),\alpha_2\otimes\alpha_2'(a_5\otimes b_5)\}_{A\otimes B}$$
$$+\{\alpha_1\otimes\alpha_1'(a_3\otimes b_3),\{a_1\otimes b_1,a_2\otimes b_2,a_4\otimes b_4\}_{A\otimes B},\alpha_2\otimes\alpha_2'(a_5\otimes b_5)\}_{A\otimes B}$$
$$+\{\alpha_1\otimes\alpha_1'(a_3\otimes b_3),\alpha_2\otimes\alpha_2'(a_4\otimes b_4),\{a_1\otimes b_1,a_2\otimes b_2,a_5\otimes b_5\}_{A\otimes B}\}_{A\otimes B}$$
$$=\{\{a_1,a_2,a_3\}_A\otimes\tau(b_1,b_2,b_3),\alpha_1(a_4)\otimes\alpha_1'(b_4),\alpha_2(a_5)\otimes\alpha_2'(b_5)\}_{A\otimes B}$$
$$+\{\alpha_1(a_3)\otimes\alpha_1'(b_3),\{a_1,a_2,a_4\}_A\otimes\tau(b_1,b_2,b_4),\alpha_2(a_5)\otimes\alpha_2'(b_5)\}_{A\otimes B}$$
$$+\{\alpha_1(a_3)\otimes\alpha_1'(b_3),\alpha_2(a_4)\otimes\alpha_2'(b_4),\{a_1,a_2,a_5\}_A\otimes\tau(b_1,b_2,b_5)\}_{A\otimes B}$$
$$=\underbrace{\{\{a_1,a_2,a_3\},\alpha_1(a_4),\alpha_2(a_5)\}}_{c}\otimes\underbrace{\tau(\tau(b_1,b_2,b_3),\alpha_1'(b_4),\alpha_2'(b_5))}_{d}$$
$$+\underbrace{\{\alpha_1(a_3),\{a_1,a_2,a_4\},\alpha_2(a_5)\}}_{e}\otimes\underbrace{\tau(\alpha_1'(b_3),\tau(b_1,b_2,b_4),\alpha_2'(b_5))}_{f}$$
$$+\underbrace{\{\alpha_1(a_3),\alpha_2(a_4),\{a_1,a_2,a_5\}\}}_{g}\otimes\underbrace{\tau(\alpha_1'(b_3),\alpha_2'(b_4),\tau(b_1,b_2,b_5))}_{h}$$

Citation: Makhlouf A, Amri A (2015) Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras. J Generalized Lie Theory Appl 9: 221. doi:10.4172/1736-4337.1000221

Using ternary Nambu identity of {., ., .} we have $a=c+e+g$, and $b=d=f=h$ using the symmetry of τ and Hom-associativity of μ', then the left hand side is equal to the right hand side from where the ternary Hom- Nambu identity of bracket $\{.,.,.\}_{A\otimes B}$ is verified

For the Hom-Leibniz identity, we have

$$LHS=\{\mu\otimes\mu'(a_1\otimes b_1,a_2\otimes b_2),\alpha_1\otimes\alpha_1'(a_3\otimes b_3),\alpha_2\otimes\alpha_2'(a_4\otimes b_4)\}_{A\otimes B}$$

$$=\{\mu(a_1,b_1)\otimes\mu'(a_2,b_2),\alpha_1(a_3)\otimes\alpha_1'(b_3),\alpha_2(a_4)\otimes\alpha_2'(b_4)\}_{A\otimes B}$$

$$=\underbrace{\{\mu(a_1,b_1),\alpha_1(a_3),\alpha_2(a_4)\}_A}_{a'}\otimes\underbrace{\tau(\mu'(a_2,b_2),\alpha_1'(b_3),\alpha_2'(b_4))}_{b'}$$

And

$$RHS=\mu\otimes\mu'(\beta\otimes\beta'(a_1\otimes b_1),\{a_2\otimes b_2,a_3\otimes b_3,a_4\otimes b_4\}_{A\otimes B})$$

$$+\mu\otimes\mu'(\{a_1\otimes b_1,a_3\otimes b_3,a_4\otimes b_4\}_{A\otimes B},\beta\otimes\beta'(a_2\otimes b_2))$$

$$=\mu\otimes\mu'(\beta(a_1)\otimes\beta'(b_1),\{a_2,a_3,a_4\}\otimes\tau(b_2,b_3,b_4))$$

$$+\mu\otimes\mu'(\{a_1,a_3,a_4\}\otimes\tau(b_1,b_3,b_4),\beta(a_2)\otimes\beta'(b_2))$$

$$=\underbrace{\mu(\beta(a_1),\{a_2,a_3,a_4\})}_{c'}\otimes\underbrace{\mu'(\beta'(b_1),\tau(b_2,b_3,b_4))}_{d'}$$

$$+\underbrace{\mu(\{a_1,a_3,a_4\},\beta(a_2))}_{e'}\otimes\underbrace{\mu'(\tau(b_1,b_3,b_4),\beta'(b_2)}_{f'}$$

With Hom-Leibniz identity we have $a'=c'+e'$, and using condition (3.4) we have $b'=d'=f'$, therefore the left hand side is equal to the right hand side and the Hom-Leibniz identity is proved. Then

$$(A\otimes B,\mu\otimes\mu',\beta\otimes\beta',\{.,.,.\}_{A\otimes B},(\alpha_1\otimes\alpha_1',\alpha_2\otimes\alpha_2'))$$

is a (non-commutative) ternary Hom-Nambu-Poisson algebra.

Construction of Ternary Hom-Nambu-Poisson Algebras

In this section, we provide constructions of ternary Hom-Nambu-Poisson algebras using twisting principle.

Theorem 4.1. *Let (A, μ, {., ., .}, α) be a (non-commutative) ternary Hom-Nambu- Poisson algebra and β : A → A be a weak Hom-Nambu-Poisson morphism, then* $A_\beta=(A,\{.,.,.\}_\beta=\beta\circ\{.,.,.\},\mu\beta=\beta\circ\mu,\beta\alpha)$ *is also a ternary (non-commutative)Hom-Nambu-Poisson algebra. Moreover, if A is multiplicative and β is an algebra morphism, then* A_β *is a multiplicative (non-commutative) Hom-Nambu-Poisson al- gebra.*

Proof. If μ is commutative, then clearly so is μ_β. The rest of the proof applies whether μ is commutative or not. The skew-symmetry follows from the skew- symmetry of the bracket {., ., .}. It remains to prove Hom-associativity condition, Hom-Nambu-identity and Hom-Leibniz identity. Indeed

$$\mu_\beta(\mu_\beta(x,y),\beta\alpha(z))=\mu_\beta(\beta(\mu(x,y),\beta\alpha(z)))=\beta^2(\mu(\mu(x,y),\alpha(z)))$$

$$=\beta^2(\mu(\alpha(x),\mu(y,z)))=\mu_\beta(\beta\alpha(x),\mu_\beta(y,z)).$$

We check the Hom-Nambu identity

$$\{\beta\alpha(x_1),\beta\alpha(x_2),\{x_3,x_4,x_5\}_\beta\}_\beta=\beta^2\{\alpha(x_1),\alpha(x_2),\{x_3,x_4,x_5\}\}$$

$$=\beta^2(\{\{x_1,x_2,x_3\},\alpha(x_4),\alpha(x_5)\}+\{\alpha(x_3),\{x_1,x_2,x_4\},\alpha(x_5)\}$$

$$+\{\alpha(x_3),\alpha(x_4),\{x_1,x_2,x_5\}\})$$

$$=\{\{x_1,x_2,x_3\}_\beta,\beta\alpha(x_4),\beta\alpha(x_5)\}_\beta+\{\beta\alpha(x_3),\{x_1,x_2,x_4\}\beta,\beta\alpha(x_5)\}_\beta$$

$$+\{\beta\alpha(x3),\beta\alpha(x_4),\{x_1,x_2,x_5\}_\beta\}_\beta.$$

Then it remains to show Hom-Leibniz identity

$$\{\mu_\beta(x_1,x_2),\beta\alpha(x_3),\beta\alpha(x_4)\}_\beta=\beta^2(\{\mu(x_1,x_2),\alpha(x_3),\alpha(x_4)\})$$

$$=\beta^2(\mu(\alpha(x_1),\{x_2,x_3,x_4\})+\mu(\{x_1,x_3,x_4\},\alpha(x_2)))$$

$$=\mu_\beta(\beta\alpha(x_1),\{x_2,x_3,x_4\}_\beta)+\mu_\beta(\{x_1,x_3,x_4\}_\beta,\beta\alpha(x_2))$$

Therefore $A_\beta=(A,\{.,.,.\}_\beta,\mu_\beta,\beta\alpha)$ is a ternary (non-commutative) Hom-Nambu- Poisson algebra. For the multiplicativity assertion, suppose that A is multiplicative and β is an algebra morphism. We have

$$(\beta\alpha)\circ(\mu_\beta)=\beta\alpha\circ\beta\circ\mu=\mu_\beta\circ\alpha^{\otimes 2}\beta^{\otimes 2}=\mu_\beta\circ(\beta\alpha)^{\otimes 2}$$

And

$$\beta\alpha\circ\{.,.,.\}_\beta=\beta\alpha\circ\beta\circ\{.,.,.\}=\{.,.,.\}_\beta\circ(\beta\alpha)^{\otimes 3}$$

Then A_β is multiplicative.

Corollary 4.2. *Let (A, μ, {., ., .}, α) be a multiplicative ternary (non-commutative) Hom-Nambu-Poisson algebra. Then*

$$A^n=(A,\mu^{(n)}=\alpha^n\circ\mu,\{.,.,.\}^{(n)}=\alpha^{(n)}\circ\{.,.,.\},\alpha^{n+1})$$

is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra for each integer n ≥ 0.

Proof. The multiplicativity of A implies that $\alpha^n:A\to A$ is a Nambu-Poisson algebra morphism. By Theorem 4.1 $A_{\alpha^n}=A^n$ is a multiplicative ternary (non- commutative) Hom-Nambu-Poisson algebra.

Corollary 4.3. *Let (A, μ, {., ., .}) be a ternary (non-commutative) Nambu-Poisson algebra and β : A → A be a Nambu-Poisson algebra morphism. Then*

$$A_\beta=(A,\mu_\beta=\beta\circ\mu,\{.,.,.\}_\beta=\beta\circ\{.,.,.\},\beta)$$

is a multiplicative (non-commutative) ternary Hom-Nambu-Poisson algebra.

Remark 4.4. Let (A, μ, {., ., .}, α) and $(A',\mu',\{.,.,.\}',\alpha')$ be two (non-commutative) ternary Nambu-Poisson algebras and $\beta:A\to A,\beta':A'\to A'$ be ternary Nambu- Poisson algebra endomorphisms. If $\varphi:A\to A'$ is a ternary Nambu-Poisson algebra morphism that satisfies $\varphi\circ\beta=\beta'\circ\varphi$ then

$$\varphi:(A,\mu_\beta,\{.,.,.\}_\beta,\beta\alpha)\to(A',\mu'_{\beta'},\{.,.,.\}'_{\beta'},\beta'\alpha')$$

Is a (non-commutative) ternary hom-nambu poisson algebra morphism.

Indeed, we have

$$\varphi\circ\{.,.,.\}_\beta=\varphi\circ\beta\circ\{.,.,.\}=\beta'\circ\varphi\circ\{.,.,.\}=\beta'\circ\{.,.,.\}'\circ\varphi^{\times 3}=\{.,.,.\}'_{\beta'}\circ\varphi^{\times 3}$$

$$\varphi\circ\mu_\beta=\varphi\circ\beta\circ\mu=\beta'\circ\varphi\circ\mu=\beta'\circ\mu'\circ\varphi^{\times 2}=\mu'_{\beta'}\circ\varphi^{\times 2}$$

In the sequel, we aim to construct Hom-type version of the ternary Nambu- Poisson algebra of polynomials of three variables $(\mathbb{R}[x,y,z],\cdot,\{.,.,.\})$, defined in Example 1.5. The Poisson bracket of three polynomials is defined in (1.2).

The twisted version is given by a structure of ternary Hom-Nambu-Poisson algebra where $(\mathbb{R}[x,y,z],\cdot_\alpha=\alpha\circ\cdot,\{.,.,.\}_\alpha=\alpha\circ\{.,.,.\},\alpha)$ where $\alpha:\mathbb{R}[x,y,z]\to\mathbb{R}[x,y,z]$ is an algebra morphism satisfying for all $f,g\in\mathbb{R}[x,y,z]$

$$\alpha(f\cdot g)=\alpha(f)\cdot\alpha(g)$$

$$\alpha\{f,g,h\}=\{\alpha(f),\alpha(g),\alpha(h)\}.$$

Citation: Makhlouf A, Amri A (2015) Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras. J Generalized Lie Theory Appl 9: 221. doi:10.4172/1736-4337.1000221

Theorem 4.5. *A morphism* $\alpha:\mathbb{R}[x,y,z]\to\mathbb{R}[x,y,z]$ *which gives a structure of ternary Hom-Nambu-Poisson algebra* $(\mathbb{R}[x,y,z],\cdot_\alpha=\alpha\circ\cdot,\{.,.,.\}_\alpha=\alpha\circ\{.,.,.\},\alpha)$ *satisfies the following equation:*

$$1-\begin{vmatrix}\frac{\partial\alpha(x)}{\partial x} & \frac{\partial\alpha(x)}{\partial y} & \frac{\partial\alpha(x)}{\partial z}\\ \frac{\partial\alpha(y)}{\partial x} & \frac{\partial\alpha(y)}{\partial y} & \frac{\partial\alpha(y)}{\partial z}\\ \frac{\partial\alpha(z)}{\partial x} & \frac{\partial\alpha(z)}{\partial y} & \frac{\partial\alpha(z)}{\partial z}\end{vmatrix}=0 \tag{4.1}$$

Proof. Let α be a Nambu-Poisson algebra morphism, then it satisfies for all $f,g,h\in\mathbb{R}[x,y,z]$

$$\alpha(f\cdot g)=\alpha(f)\cdot\alpha(g),$$
$$\alpha\{f,g,h\}=\{\alpha(f),\alpha(g),\alpha(h)\}.$$

The first equality shows that it is sufficient to just set α on x, y and z. For the second equality, we suppose by linearity that

$$f(x,y,z)=x^iy^jz^k,$$
$$g(x,y,z)=x^ly^mz^p,$$
$$f(x,y,z)=x^qy^rz^s.$$

Then we can write the second equation as follows

$$\alpha\begin{vmatrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z}\\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z}\\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} & \frac{\partial h}{\partial z}\end{vmatrix}=\begin{vmatrix}\frac{\partial\alpha(f)}{\partial x} & \frac{\partial\alpha(f)}{\partial y} & \frac{\partial\alpha(f)}{\partial z}\\ \frac{\partial\alpha(g)}{\partial x} & \frac{\partial\alpha(g)}{\partial y} & \frac{\partial\alpha(g)}{\partial z}\\ \frac{\partial\alpha(h)}{\partial x} & \frac{\partial\alpha(h)}{\partial y} & \frac{\partial\alpha(h)}{\partial z}\end{vmatrix},$$

which can be simplified to

$$1=\begin{vmatrix}\frac{\partial\alpha(x)}{\partial x} & \frac{\partial\alpha(x)}{\partial y} & \frac{\partial\alpha(x)}{\partial z}\\ \frac{\partial\alpha(y)}{\partial x} & \frac{\partial\alpha(y)}{\partial y} & \frac{\partial\alpha(y)}{\partial z}\\ \frac{\partial\alpha(z)}{\partial x} & \frac{\partial\alpha(z)}{\partial y} & \frac{\partial\alpha(z)}{\partial z}\end{vmatrix}. \tag{4.2}$$

Example 4.6. *We set polynomials*:

$$\alpha(x)=P_1(x,y,z)=\sum_{0\le i,j,k\le d}a_{ijk}x^iy^jz^k,$$
$$\alpha(y)=P_2(x,y,z)=\sum_{0\le i,j,k\le d}b_{ijk}x^iy^jz^k,$$
$$\alpha(z)=P_3(x,y,z)=\sum_{0\le i,j,k\le d}c_{ijk}x^iy^jz^k,$$

Where $P_1,P_2,P_3\in\mathbb{R}[x,y,z]$, *and d the largest degree for each variable. We assume that* $a_0=b_0=c_0=0$

Case of polynomials of degree one. *We take*

$$P_1(x,y,z)=a_1x+a_2y+a_3z,$$
$$P_2(x,y,z)=b_1x+b_2y+b_3z,$$
$$P_3(x,y,z)=c_1x+c_2y+c_3z$$

Equation (2.5) *becomes*

$$1-\begin{vmatrix}\frac{\partial P_1(x,y,z)}{\partial x} & \frac{\partial P_1(x,y,z))}{\partial y} & \frac{\partial P_1(x,y,z)}{\partial z}\\ \frac{\partial P_2(x,y,z)}{\partial x} & \frac{\partial P_2(x,y,z)}{\partial y} & \frac{\partial P_2(x,y,z)}{\partial z}\\ \frac{\partial P_3(x,y,z)}{\partial x} & \frac{\partial P_3(x,y,z)}{\partial y} & \frac{\partial P_3(x,y,z)}{\partial z}\end{vmatrix}=0, \tag{4.3}$$

Whence

$$1-\begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}=0. \tag{4.4}$$

The polynomials P_1, P_2 *and* P_3 *are of one of this form*

$$P_1(x,y,z)=xa_1+ya_2+za_3,P_2(x,y,z)=b_2y-\frac{z}{a_1c_2},P_3(x,y,z)=c_2y. \tag{1}$$

$$P_1(x,y,z)=a_1x+a_2y+a_3z,P_2(x,y,z)=\frac{1+a_1b_3c_2}{a_1c_3}y+b_3z,P_3(x,y,z)=c_2y+c_3z \tag{2}$$

$$P_1(x,y,z)=a_1x+a_2y+a_3z,P_2(x,y,z)=b_1x+\frac{1}{a_2c_1}z,P_3(x,y,z)=c_1x \tag{3}$$

$$P_1(x,y,z)=a_1x+a_2y+a_3z,P_2(x,y,z)=\frac{-1+a_2b_3c_1}{a_2c_3}x+b_3z,P_3(x,y,z)=c_1x+c_3z \tag{4}$$

$$P_1(x,y,z)=\frac{a_2b_1c_3+b_2}{c_3x}+a_2y+a_3z,P_2(x,y,z)=b_1x+b_2y+b_3z,P_3(x,y,z)=c_3z \tag{5}$$

$$P_1(x,y,z)=\frac{1}{b_2c_3}x+a_2y+a_3z,P_2(x,y,z)=b_2y+b_3z,P_3(x,y,z)=c_3z \tag{6}$$

$$P_1(x,y,z)=a_1x+\frac{1}{b_1c_3}y+a_3z,P_2(x,y,z)=b_1x+b_3z,P_3(x,y,z)=c_3z \tag{7}$$

$$P_1(x,y,z)=a_1x+a_2y+\frac{1}{b_1c_2}z,P_2(x,y,z)=b_1x,P_3(x,y,z)=c_1x+c_2y \tag{8}$$

$$P_1(x,y,z)=a_1x+\frac{-1}{b_1c_3+a_3c_2c_3}y+a_3z,P_2(x,y,z)=b_1x,P_3(x,y,z)=c_1x+c_2y+c_3z. \tag{9}$$

$$P_1(x,y,z)=\frac{a_2b_1}{b_2}+\frac{1}{b_2c_3-b_3c_2}x+a_2y+a_3z,P_2(x,y,z)=b_1x+b_2y+b_3z,P_3(x,y,z)=\frac{b_1c_2}{b_2}x+c_2y+c_3z \tag{10}$$

$$P_1(x,y,z)=\frac{-c_3+a_2c_1c_2}{b_3c_2^2}x+a_2y+a_3z,P_2(x,y,z)=b_3z,P_3(x,y,z)=c_1x+c_2y+c_3z \tag{11}$$

$$P_1(x,y,z)=a_1x+a_2y+\frac{1}{b_1c_2-b_2c_1}z,P_2(x,y,z)=b_1x+b_2y,P_3(x,y,z)=c_1x+c_2y \tag{12}$$

$$P_1(x,y,z)=\frac{1+a_2b_1c_3-a_3b_1c_2-a_2b_3c_1+a_3b_2c_1}{b_2c_3-b_3c_2}x+a_2y+a_3z,P_2(x,y,z)=b_1x+b_2y+b_3z,P_3(x,y,z)=c_1x+c_2y+c_3z \tag{13}$$

$$P_1(x,y,z)=a_1x+\frac{b_2}{b_3}(a_3-\frac{1}{b_1c_2-b_2c_1})y+a_3z,P_2(x,y,z)=b_1x+b_2y+b_3z,P_3(x,y,z)=c_1x+c_2y+\frac{b_3c_2}{b_2}z \tag{14}$$

Particular case of polynomials of degree two. *We take one of the polynomials of degree two*

$$P_1(x,y,z)=a_1x+a_2y+a_3z$$
$$P_2(x,y,z)=b_1x+b_2y+b_3z$$
$$P_3(x,y,z)=c_1x+c_2y+c_3z+c_4x^2$$

The polynomials P_1,P_2 *and* P_3 *are of one of this form*

$$P_1(x,y,z)=\frac{a_2b_1}{b_2}+\frac{1}{b_2c_3-b_3c_2}x+a_2y+\frac{a_2b_3}{b_2}z,P_2(x,y,z)=b_1x+b_2y+b_3z,P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z. \tag{1}$$

Citation: Makhlouf A, Amri A (2015) Non-Commutative Ternary Nambu-Poisson Algebras and Ternary Hom-Nambu-Poisson Algebras. J Generalized Lie Theory Appl 9: 221. doi:10.4172/1736-4337.1000221

$$P_1(x,y,z)=a_2x+\frac{a_3b_2}{b_3}y+a_3z, P_2(x,y,z)=b_2y+b_3z, P_3(x,y,z)=c_4x^2+c_1x+c_2y+\frac{\frac{1}{a_1}+b_3c_2}{b_2}z \quad (2)$$

$$P_1(x,y,z)=a_2x+a_2y+a_3z, P_2(x,y,z)=b_2y, P_3(x,y,z)=c_4x^2+c_1x+c_2y+\frac{1}{a_1b_2}z. \quad (3)$$

$$P_1(x,y,z)=(\frac{a_2b_1}{b_3}-\frac{1}{c_2b_3})x+a_3z, P_2(x,y,z)=b_1x+b_3z, P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z \quad (4)$$

$$P_1(x,y,z)=-\frac{1}{b_3c_2}x+a_3z, P_2(x,y,z)=b_3z, P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z. \quad (5)$$

$$P_1(x,y,z)=a_1x-\frac{1}{b_1c_3}y+a_3z, P_2(x,y,z)=b_1x, P_3(x,y,z)=c_4x^2+c_1x+c_3z \quad (6)$$

$$P_1(x,y,z)=a_1x+\frac{-1}{b_1c_3}+\frac{a_3c_2}{c_3}y+a_3z, P_2(x,y,z)=b_1x, P_3(x,y,z)=c_4x^2+c_1x+c_2y+c_3z \quad (7)$$

$$P_1(x,y,z)=a_1x+a_2y+\frac{1}{b_1c_2}z, P_2(x,y,z)=b_1x, P_3(x,y,z)=c_4x^2+c_1x+c_2y \quad (8)$$

$$P_1(x,y,z)=a_1x+a_2y+a_3z, P_2(x,y,z)=\frac{(1+a_2b_3c_1)}{a_2c_3}x+b_3z, P_3(x,y,z)=c_1x+c_3z \quad (9)$$

Classification

In this section, we provide the classification of 3-dimensional ternary non-commutative Nambu-Poisson algebras. By straightforward calculations and using a computer algebra system we obtain the following result.

Theorem 5.1. *Every 3-dimensional ternary Nambu-Lie algebra is isomorphic to the ternary algebra defined with respect to basis $\{e_1, e_2, e_3\}$, by the skew-symmetric bracket defined as*

$\{e_1, e_2, e_3\}= e_1$

Moreover it defines a 3-dimensional ternary non-commutative Nambu-Poisson algebra (A, {., ., .}, μ) if and only if μ is one of the following non-commutative associative algebra defined as

$$\begin{aligned}&\mu_1(e_2,e_1)=ae_1\mu_1(e_2,e_2)=ae_2\mu_1(e_2,e_3)=ae_3\\&\mu_1(e_3,e_1)=be_1\mu_1(e_3,e_2)=be_2\mu_1(e_3,e_3)=be_3,\end{aligned} \quad (1)$$

where a, b are parameters.

The opposite algebra of (1) (2)

The multiplication which are not mentioned are equal to zero.

The first statement of the Theorem is due to Filippov [4,13]. The two families are naturally not isomorphic.

Remark 5.2. The 3-dimensional ternary Nambu-Lie algebra is endowed with a commutative Nambu-Poisson algebra structure only when the multiplication is trivial.

Using the twisting principle described in Theorem 4.1, we obtain the following 3-dimensional non-commutative ternary Hom-Nambu-Poisson algebras.

Proposition 5.3. *Any 3-dimensional ternary non-commutative Hom-Nambu-Poisson algebra $(A,\{.,.,.\}_\alpha,\mu_\alpha,\alpha)$ obtained by a twisting defined with respect to the basis $\{e_1, e_2, e_3\}=e_1$ by the ternary bracket $\{e_1,e_2,e_3\}_\alpha=ce_1$, where c is a parameter, is one of the following binary Hom-associative algebras defined by μ_{α_i} and a corresponding structure map*

$$\begin{aligned}&\mu_{\alpha_1}(e_2,e_1)=ace_1, && \mu_{\alpha_1}(e_3,e_1)=bce_1,\\&\mu_{\alpha_1}(e_2,e_2)=a(de_1+e_2), && \mu_{\alpha_1}(e_3,e_2)=b(de_1+e_2),\\&\mu_{\alpha_1}(e_2,e_3)=a(he_1+ge_2+e_3), && \mu_{\alpha_1}(e_3,e_3)=b(he_1+ge_2+e_3),\end{aligned} \quad (1)$$

With

$$\alpha_1(e_1)=ce_1,\alpha_1(e_2)=de_1+e_2,\alpha_1(e_3)=he_1+ge_2+e_3.$$

$$\begin{aligned}&\mu_{\alpha_2}(e_1,e_2)=ace_1, && \mu_{\alpha_2}(e_3,e_1)=bce_1,\\&\mu_{\alpha_2}(e_2,e_2)=a(de_1+e_2+le_3), && \mu_{\alpha_2}(e_3,e_2)=b(de_1+e_2+le_3),\\&\mu_{\alpha_2}(e_2,e_3)=a(he_1+e_3), && \mu_{\alpha_2}(e_3,e_3)=b(he_1+e_3),\end{aligned} \quad (2)$$

With

$$\alpha_2(e_1)=ce_1,\alpha_2(e_2)=de_1+e_2+le_3,\alpha_2(e_3)=he_1+e_3e_3$$

$$\begin{aligned}&\mu_{\alpha_3}(e_2,e_1)=ace_1, && \mu_{\alpha_3}(e_3,e_1)=bce_1,\\&\mu_{\alpha_3}(e_2,e_2)=a(de_1+fe_2+\frac{a}{b}(1-f)e_3), && \mu_{\alpha_3}(e_3,e_2)=bde_1+bfe_2+a(1-f)e_3,\\&\mu_{\alpha_3}(e_2,e_3)=ahe_1+b(f-1)e_2+\frac{a(b-ga)}{b}e_3), && \mu_{\alpha_3}(e_3,e_3)=bhe_1+\frac{b^2(f-1)}{a}e_2+(b-ga)e_3,\end{aligned} \quad (3)$$

With

$$\alpha_3(e_1)=ce_1,\alpha_3(e_2)=de_1+fe_2+\frac{a}{b}(1-f)e_3,\alpha_3(e_3)=he_1+\frac{b}{a}(f-1)e_2+\frac{(b-ga)}{b}e_3$$

$$\begin{aligned}&\mu_{\alpha_4}(e_1,e_2)=ace_1, && \mu_{\alpha_4}(e_2,e_3)=b(de_1+e_2),\\&\mu_{\alpha_4}(e_1,e_3)=bce_1, && \mu_{\alpha_4}(e_3,e_2)=a(he_1+ge_2+e_3),\\&\mu_{\alpha_4}(e_2,e_2)=a(de_1+e_2), && \mu_{\alpha_4}(e_3,e_3)=b(he_1+ge_2+e_3),\end{aligned} \quad (4)$$

With

$$\alpha_4(e_1)=ce_1,\alpha_4(e_2)=de_1+e_2,\alpha_4(e_3)=he_1+ge_2+e_3.$$

$$\begin{aligned}&\mu_{\alpha_5}(e_1,e_2)=ace_1, && \mu_{\alpha_5}(e_2,e_3)=b(de_1+e_2+le_3),\\&\mu_{\alpha_5}(e_1,e_3)=bce_1, && \mu_{\alpha_5}(e_3,e_2)=a(he_1+e_3),\\&\mu_{\alpha_5}(e_2,e_2)=a(de_1+e_2+le_3), && \mu_{\alpha_5}(e_3,e_3)=b(he_1+e_3)\end{aligned} \quad (5)$$

With

$$\alpha_5(e_1)=ce_1,\alpha_5(e_2)=de_1+e_2+le_3,\alpha_5(e_3)=he_1+e_3$$

$$\begin{aligned}&\mu_{\alpha_6}(e_1,e_2)=ace_1, && \mu_{\alpha_6}(e_2,e_3)=bde_1+bfe_2+a(1-f)e_3),\\&\mu_{\alpha_6}(e_1,e_3)=bce_1, && \mu_{\alpha_6}(e_3,e_2)=ahe_1-b(f-1)e_2+\frac{a(b-ag)}{b}e_3,\\&\mu_{\alpha_6}(e_2,e_2)=a(de_1+fe_2+\frac{a}{b}(1-f)e_3), && \mu_{\alpha_6}(e_3,e_3)=bhe_1-\frac{b^2(f-1)}{a}e_2+(b-ag)e_3\end{aligned} \quad (6)$$

With

$$\alpha_6(e_1)=ce_1,\alpha_6(e_2)=de_1+fe_2+\frac{a}{b}(1-f)e_3,\alpha_6(e_3)=he_1+\frac{-b}{a}(f-1)e_2+\frac{b-ag}{b}e_3z8$$

$$\begin{aligned}&\mu_{\alpha_7}(e_1,e_3)=ace_1,\\&\mu_{\alpha_7}(e_2,e_3)=a(de_1+fe_2+le_3),\\&\mu_{\alpha_7}(e_3,e_3)=a(he_1+ge_2+\frac{1+g+l}{f}e_3)\end{aligned} \quad (7)$$

With

$$\alpha_7(e_1)=ce_1,\alpha_7(e_2)=de_1+fe_2+le_3,\alpha_7(e_3)=he_1+ge_2+\frac{1+g+l}{f}e_3$$

$$\begin{aligned}&\mu_{\alpha_8}(e_1,e_3)=ace_1,\\&\mu_{\alpha_8}(e_2,e_3)=a(de_1+e_2),\\&\mu_{\alpha_8}(e_3,e_3)=a(he_1+ge_2+e_3)\end{aligned} \quad (8)$$

With

$$\alpha_8(e_1)=ce_1,\alpha_8(e_2)=de_1+e_2,\alpha_8(e_3)=he_1+ge_2+e_3.$$

$$\begin{aligned}&\mu_{\alpha_9}(e_1,e_3)=ace_1,\\&\mu_{\alpha_9}(e_2,e_3)=a(de_1-\frac{1}{g}e_3),\\&\mu_{\alpha_9}(e_3,e_3)=a(he_1+ge_2+re_3)\end{aligned} \quad (9)$$

With

$$\alpha_9(e_1)=ce_1,\alpha_9(e_2)=de_1-\frac{1}{g}e_3,\alpha_9(e_3)=he_1+ge_2+re_3$$

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