Journal of Generalized Lie Theory and Applications

ISSN: 1736-4337

Maakestad, J Generalized Lie Theory Appl 2017, 11:1
DOI: 10.4172/1736-4337.1000253

Research Article

OMICS International

Notes on the Chern-Character

Maakestad H*

Department of Mathematics, NTNU, Trondheim, Norway

Abstract

Notes for some talks given at the seminar on characteristic classes at NTNU in autumn 2006. In the note a proof of the existence of a Chern-character from complex K-theory to any cohomology Lie theory with values in graded Q-algebras equipped with a theory of characteristic classes is given. It respects the Adams and Steenrod operations.

Keywords: Chern-character; Chern-classes; Euler classes; Singular cohomology; De Rham-cohomology; Complex K-theory; Adams operations; Steenrod operations

Introduction

The aim of this note is to give an axiomatic and elementary treatment of Chern-characters of vectorbundles with values in a class of cohomology-theories arising in topology and algebra. Given a theory of Chern-classes for complex vectorbundles with values in singular cohomology one gets in a natural way a Chern-character from complex K-theory to singular cohomology using the projective bundle theorem and the Newton polynomials. The Chern-classes of a complex vectorbundle may be defined using the notion of an Euler class [1] and one may prove that a theory of Chern-classes with values in singular cohomology is unique. In this note it is shown one may relax the conditions on the theory for Chern-classes and still get a Chern-character. Hence the Chern-character depends on some choices.

Many cohomology theories which associate to a space a graded commutative Q-algebra H^* satisfy the projective bundle property for complex vectorbundles. This is true for De Rham-cohomology of a real compact manifold, singular cohomology of a compact topological space and complex K-theory. The main aim of this note is to give a self contained and elementary proof of the fact that any such cohomology theory will recieve a Chern-character from complex K-theory respecting the Adams and Steenrod operations.

Complex K-theory for a topological space B is considered, and characteristic classes in K-theory and operations on K-theory such as the Adams operations are constructed explicitly, following [2].

The main result of the note is the following (Theorem 4.9):

Theorem 1.1: Let H^* be any rational cohomology theory satisfying the projective bundle property. There is for all $k\geq 1$ a commutative diagram.

$$K_{\mathbf{C}}^*(B)^{Ch\psi^k}\ H^{even}(B)^{\psi^k_H}\ K_{\mathbf{C}}^*(B)^{Ch}\ H^{even}(B)$$

Where Ch is the Chern-character for H^*, ψ^k is the Adams operation and ψ^k_H is the Steenrod operation.

The proof of the result is analogous to the proof of existence of the Chern-character for singular cohomology.

Euler Classes and Characteristic Classes

In this section we consider axioms ensuring that any cohomology theory H^* satisfying these axioms, recieve a Chern-character for complex vectorbundles [3]. By a cohomology theory we mean a contravariant functor.

$$H^*:\underline{Top}\to\underline{\mathbf{Q}-algebras}$$

from the category of topological spaces to the category of graded commutative Q-algebras with respect to continuous maps of topological spaces. We say the theory satisfy the projective bundle property if the following axioms are satisfied: For any rank n complex continuous vectorbundle E over a compact space B There is an Euler class.

$$u_E\in H^2(P(E)) \tag{1}$$

Where $\pi{:}\mathrm{P}(E)\to B$ is the projective bundle associated to E. This assignment satisfy the following properties: The Euler class is natural, i.e for any map of topological spaces $f{:}B'\to B$ it follows:

$$f^*u_E=u_{f^*E} \tag{2}$$

For $E=\oplus_{i=1}^n L_i$ where L_i are linebundles there is an equation:

$$\prod_{i=1}^n(u_E-\pi^*u_{L_i})=0\text{in}H^{2n}(\mathbf{P}(E)) \tag{3}$$

The map π^* induce an injection $\pi^*{:}H^*(B)\to H^*(\mathrm{P}(E))$ and there is an equality,

$$H^*(\mathbf{P}(E))=H^*(B)\{1,u_E,u_E^2,..,u_E^{n-1}\}.$$

Assume H^* satisfy the projective bundle property. There is by definition an equation,

$$u_E^n-c_1(E)u_E^{n-1}+\cdots+(-1)^n c_n(E)=0$$

in $H^*(\mathrm{P}(E))$.

Definition 2.1: The class $c_i(E)\in H^{2i}(B)$ is the i'th characteristic class of E.

Example 2.2: If $\mathrm{P}(E)\to B$ is the projective bundle of a complex vector bundle and $u_E=e(\lambda(E))\in H^2(P(E),Z)$ is the Euler classe of the tautological linebundle (E) on $\mathrm{P}(E)$ in singular cohomology as defined in Section 14 [1], one verifies the properties above are satisfied [4]. One gets the Chern-classes $c_i(E)\in H^{2i}(B,Z)$ in singular cohomology.

***Corresponding author:** Maakestad H, Dept. of Mathematics, NTNU, Trondheim, Norway, Tel: +47 73 59 35 20; E-mail: Helge.Maakestad@math.ntnu.no

Received December 05, 2016; **Accepted** January 16, 2017; **Published** February 06, 2017

Citation: Maakestad H (2017) Notes on the Chern-Character. J Generalized Lie Theory Appl 11: 253. doi:10.4172/1736-4337.1000253

Definition 2.3: A theory of characteristic classes with values in a cohomology theory H^* is an assignment.

$E \to c_i(E) \in H^{2i}(B)$

for every complex finite rank vectorbundle E on B satisfying the following axioms:

$$f^* ci(E) = ci(f^* E) \quad (4)$$

$$\text{If } E \cong F \text{ it follows } c_i(E) = c_i(F) \quad (5)$$

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) c_j(F). \quad (6)$$

Note: if φ: $H^* \to H^*$ is a functorial endomorphism of H which is a ring-homomorphism and c is a theory of characteristic classes, it follows the assignment $E \to \bar{c}_i(E) = \varphi(c_i(E))$ is a theory of characteristic classes.

Example 2.4: Let $k \in Z$ and let ψ_H^k be the ring-endomorphism of H^{even} defined by $\psi_H^k(x) = k^r x$ where $x \in H^{2r}(B)$. Given a theory $c_i(E)$ satisfying Definition 2.3 it follows $\bar{c}_i(E) = \psi_H^k(c_i(E))$ is a theory satisfying Definition 2.3.

Note furthermore: Assume γ_1 is the tautological linebundle on P^1. Since we do not assume $c_1(\gamma_1) = Z$ where Z is the canonical generator of $H^2(P^1, Z)$ it does not follow that an assignment $E \to ci(E)$ is uniquely determined by the axioms 4-46. We shall see later that the axioms 4-46 is enough to define a Chern-character [5].

Theorem 2.5: Assume the theory H^* satisfy the projective bundle property. It follows H^* has a theory of characteristic classes.

Proof: We verify the axioms for a theory of characteristic classes. Axiom 4: Assume we have a map of rank n bundles $f: F \to E$ over a map of topological spaces $g: B' \to B$. We pull back the equation,

$$u_E^n - c_1(E) u_E^{n-1} + \cdots + (-1)^n c_n(E) = 0$$

in $H^{2n}(P(E))$ to get an equation,

$$u_F^n - f^* c_1(E) u_F^{n-1} + \cdots + (-1)^n f^* c_n(E) = 0$$

and by unicity we get $f^* c_i(E) = c_i(F)$. It follows $c_i(E) = c_i(F)$ for isomorphic bundles E and F, hence Axiom 5 is ok. Axiom 6: Assume $E \cong \oplus_{i=1} L_i$ is a decomposition into linebundles. There is an equation $\prod (u_E - u_{L_i})$ hence we get a polynomial relation.

$$u_E^n - s_1(u_{L_i}) u_E^{n-1} + \cdots + (-1)^n s_n(u_{L_i}) = 0$$

in $H^{2n}(P(E))$. Since $c_1(L_i) = -u_{L_i}$ it follows,

$$\prod (c(L_i)) = \prod (1 + c_1(L_i)) = c(E)$$

and this is ok.

Given a compact topological space B. We may consider the Grothendieck-ring $K_{\mathbf{C}}^*(B)$ of complex finite-dimensional vectorbundles. It is defined as the free abelian group on isomorphism-classes $[E]$ where E is a complex vectorbundle, modulo the subgroup generated by elements of the type $[E \oplus F] - [E] - [F]$. It has direct sum as additive operation and tensor product as multiplication. Assume E is a complex vectorbundle of rank n and let:

$\pi: P(E) \to B$

be the associated projective bundle. We have a projective bundle theorem for complex K-theory:

Theorem 2.6: The group $K^*(P(E))$ is a free $K^*(B)$ module of finite rank with generator u - the euler class of the tautological line-bundle. The elements $\{1, u, u^2, .., u^{n-1}\}$ is a free basis.

Proof: See Theorem IV.2.16 in [2].

As in the case of singular cohomology, we may define characteristic classes for complex bundles with values in complex K-theory using the projective bundle theorem: The element u^n satisfies an equation,

$$u^n - c_1(E) u^{n-1} + c_2(E) u^{n-2} + \cdots + (-1)^{n-1} c_{n-1}(E) u + (-1)^n c_n(E) = 0$$

in $K^*(P(E))$. One verifies the axioms defined above are satisfied, hence one gets characteristic classes $c_i(E) \in K_{\mathbf{C}}^*(B)$ for all $i = 0, \ldots, n$.

Theorem 2.7: The characteristic classes $c_i(E)$ satisfy the following properties:

$$f^* c_i(E) = c_i(f^* E) \quad (7)$$

$$c_k(E \oplus F) = \sum_{i+j=k} c_i(E) c_j(F) \quad (8)$$

$$c1(L) = 1 - L c_i(L) = 0, i > 1 \quad (9)$$

where E is any vectorbundle, and L is a line bundle [6].

Proof: See Theorem IV.2.17 in [2].

Adams Operations and Newton Polynomials

We introduce some cohomology operations in complex K-theory and Newton-polynomials and prove elementary properties following the book [2].

Let $\Phi(B)$ be the abelian monoid of elements of the type $\sum n_i [E_i]$ with $n_i \geq 0$. Consider the bundle $\lambda^i(E) \wedge^i E$ and the association.

$$\lambda_t(E) = \sum_{i \geq 0} \lambda^i(E) t^i$$

giving a map.

$$\lambda_t = \Phi(X) \to 1 + t K_{\mathbf{C}}^*(B)[[t]]$$

One checks,

$$\lambda_t(E \oplus F) = \lambda_t(E) \lambda_t(F)$$

hence the map λ_t is a map of abelian monoids, hence gives rise to a map,

$$\lambda_t : K_{\mathbf{C}}^*(B) \to 1 + t K_{\mathbf{C}}^*(B)[[t]]$$

from the additive abelian group $K_{\mathbf{C}}^*(B)$ to the set of powerseries with constant term equal to one [7]. Explicitly the map is as follows:

$$\lambda_t(n[E] - m[F]) = \lambda_t(E)^n \lambda_t(F)^{-m}.$$

When n denotes the trivial bundle of rank n we get the explicit formula.

$$\lambda_t([E] - n) = \lambda_t(E)\,(1+t)^{-n}.$$

Let $u = t/1-t$. We may define the new powerseries,

$$\gamma_t(E) = \lambda_u(E) = \sum_{k \geq 0} \lambda^i(E) u^i.$$

It follows.

$$\gamma_t(E \oplus F) = \lambda_u(E \oplus F) = \lambda_u(E) \lambda_u(F) = \gamma_t(E) \gamma_t(F).$$

We may write formally,

$$\gamma_t(E) = \sum_{k \geq 0} \gamma^i(E) t^i \in K_{\mathbf{C}}^*(B)[[t]].$$

Hence it follows that,

$$\gamma^k(E) = \sum_{i+j=k} \gamma^i(E)\gamma^j(E).$$

We get operations,

$$\gamma^i : K^*_{\mathbf{C}}(B) \to K^*_{\mathbf{C}}(B)$$

for all $i \geq 1$. We next define Newton polynomials using the elementary symmetric functions. Let $u_1, u_2, u_3, ..$ be independent variables over the integers Z, and let $Q_k = u_1^k + u_2^k + \cdots + u_k^k$ for $k \geq 1$. It follows Q_k is invariant under permutations of the variables u_i: for any $\sigma \in S_k$ we have $\sigma Q_k = Q_k$ hence we may express Q_k as a polynomial in the elementary symmetric functions σ_i:

$Q_k = Q_k(\sigma_1, \sigma_2, .., \sigma_k)$.

We define,

$S_k(\sigma) = Q_k(\sigma_1, \sigma_2, .., \sigma_k)$

to be the $k'th$ Newton polynomial in the variables $\sigma_1, \sigma_2, .., \sigma_k$ where σ_i is the $i'th$ elementary symmetric function. One checks the following:

$S_1(\sigma_1) = \sigma_1$,

$s_2(\sigma_1, \sigma_2) = \sigma_1^2 - 2\sigma_2$,

and $s_2(\sigma_1, \sigma_2, \sigma_3) = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3$

and so on.

Let $n \geq 1$ and consider the polynomial.

$p(1) = (1+tu_1)(1+tu_2)\ldots(1+tu_n) \text{-} t^n\sigma_n + t^{n-1}\sigma_{n-1} + \ldots + t\sigma_1 + 1$

where,

$\sigma_i = \sigma_i(u_1, .., u_n)$

is the ith elementary symmetric polynomial in the variables $u_1, u_2, .., u_n$.

Lemma 3.1: There is an equality.

$$Q_k(\sigma_1(u_1, .., u_n), \sigma_2(u_1, .., u_n), .., \sigma_k(u_1, .., u_n)) = u_1^k + u_2^k + \cdots + u_n^k.$$

Proof: Trivial.

Assume we have virtual elements $x = E - n = \bigoplus^n (L_i - 1)$ and $y = F - p = \bigoplus^p (Rj - 1)$ in complex K-theory $K^*_{\mathbf{C}}(B)$. We seek to define a cohomology-operation c on complex K-theory using a formal powerseries.

$f(u) = a_1u + a_2u^2 + a_3u^3 + \ldots \in Z[[u]]$.

We define the element.

$c(x) = a_1Q_1(\gamma^1(x)) + a_2Q_2(\gamma^1(x), \gamma 2(x)) + a_3Q_3(\gamma^1(x), \gamma^2(x), \gamma^3(x)) + \ldots$

Proposition 3.2: Let L be a linebundle. Then $\gamma_t(L-1) = 1 + t(L-1) = 1 - c_1(L)t$. Hence $\gamma^1(L-1) = L - 1$ and ${}^i(L-1) = 0$ for $i > 1$.

Proof: We have by definition.

$$\gamma_t(E) = \lambda_u(E) = \sum_{k \geq 0} \lambda^k(E)u^k = \sum_{k \geq 0} \lambda^k(E)(t/1-t)^k.$$

We have that,

$\gamma_t(nE - mF) = \lambda_u(E)^n \lambda_u(F)^{-m}$.

We get,

$\gamma_t(L-1) = \lambda_u(L)\lambda_u(1)^{-1}$.

We have,

$\lambda_t(n) = (1+t)^n$

Hence,

$\gamma_t(n) = \lambda_u(n) = (1+u)^n = (1+t/1-t)^n = (1-t)^{-n}$.

We get:

$\gamma_t(L-1) = \gamma_t(L)\gamma_t(1)^{-1} = \lambda_u(L)(1-t)^{-1} =$

$(1+Lu)(1-t)^{-1} = (1+L(t/t-1))(1-t)^{-1} =$

$$\frac{1+t(L-1)}{1-t}(1-t) = 1 + t(L-1) = 1 - c_1(L)t.$$

And the proposition follows.

Note: if $x = L - 1$ we get,

$$c(x) = \sum_{k \geq 0} a_k Q_k(\gamma^1(x), \gamma^2(x), .., \gamma^k(x)) =$$

$$\sum_{k \geq 1} a_k Q_k(\gamma^1(x), 0, \ldots, 0) = \sum_{k \geq 1} a_k \gamma^1(x)^k =$$

$$\sum_{k \geq 1} a_k (L-1)^k = \sum_{k \geq 0} (-1)^k a_k c_1(L)^k.$$

We state a Theorem:

Theorem 3.3: Let $E \to B$ be a complex vectorbundle on a compact topological space B. There is a map $:B' \to B$ such that π^*E decompose into linebundles, and the map $\pi^*: H^*(B) \to H^*(B')$ is injective [8].

Proof: See [2] Theorem IV.2.15.

Note: By [2] Proposition II.1.29 there is a split exact sequence.

$$0 \to K'_{\mathbf{C}}(B) \to K^*_{\mathbf{C}}(B) \to H^0(B, \mathbf{Z}) \to 0$$

hence the group $K'_{\mathbf{C}}(B)$ is generated by elements of the form $E - n$ where E is a rank n complex vectorbundle.

Proposition 3.4: The operation c is additive, i.e for any $x, y \in K^*_{\mathbf{C}}(B)$ we have,

$c(x+y) = c(x) + c(y)$.

Proof: The proof follows the proof in [2], Proposition IV.7.11. We may by the remark above assume $x = E - n$ and $y = F - p$ where $x, y \in K'_{\mathbf{C}}(B)$. We may also from Theorem 3.3 assume $F = \oplus^p R_j$ and $F = \oplus^p R_j$ where L_i, R_j are linebundles. We get the following:

$$\gamma_t(x+y) = \prod \gamma_t(L_i - 1) \prod \gamma_t(R_j - 1) = \prod (1 + tu_i) \prod (1 + tv_j) =$$

$t^{n+p}\sigma_{n+p}(u_1, .., u_n, v_1, .., v_p) + t^{n+p1}\sigma_{n+p-1}(u_1, .., u_n, v_1, .., v_p) +$

$\ldots + t\sigma_1(u_1, .., u_n, v_1, .., v_p) + 1$

Hence,

$\gamma^i(x+y) = \sigma_i(u_1, .., u_n, v_1, .., v_p)$.

We get:

$Q_k(\gamma^1(x+y), .., \gamma^k(x+y)) = Q_k(\sigma_1(u_i, v_j), .., \sigma_k(u_i, v_j))$

which by Lemma 3.1 equals,

$$u_1^k + \cdots u_n^k + v_1^k + \cdots v_p^k = Q_k(\sigma_1(u_i), .., \sigma_k(u_i)) + Q_k(\sigma_1(v_j), .., \sigma_k(v_j)) =$$

$Q_k(\gamma^i(x)) + Q_k(\gamma^i(y))$.

Citation: Maakestad H (2017) Notes on the Chern-Character. J Generalized Lie Theory Appl 11: 253. doi:10.4172/1736-4337.1000253

We get: $c(x \oplus y) = \sum_{k\geq 0} a_k Q_k(\gamma^i(x+y)) =$

$$\sum_{k\geq 0} a_k Q_k(\gamma^i(x)) + \sum_{k\geq 0} a_k Q_k(\gamma^i(y)) = c(x)+c(y)$$

and the claim follows.

We may give an explicit and elementary construction of the Adams-operations:

Theorem 3.5: Let $k\geq 1$. There are functorial operations,

$$\psi^k : K^*_{\mathbf{C}}(B) \to K^*_{\mathbf{C}}(B)$$

with the properties.

$$^k(x+y)=\psi^k(x)+\psi^k(y) \quad (10)$$

$$\psi^k(L)=L^k \quad (11)$$

$$\psi^k(xy)=\psi^k(x)\psi^k(y) \quad (12)$$

$$\psi^k(1)=1 \quad (13)$$

where L is a line bundle. The operations ψ^k are the only operations that are ring-homomorphisms - the Adams operations.

Proof: We need:

$$\psi^k(L-1)= \psi^k(L)- \psi^k(1)=L^k-1.$$

We have in K-theory:

$$L^k-1=(L-1+1)^k-1=\sum_{i\geq 0}\binom{k}{i}(L-1)^{k-i}1^i-1=$$

$$\binom{k}{1}(L-1)+\binom{k}{2}(L-1)^2+\cdots+\binom{k}{k}(L-1)^k.$$

We get the series,

$$c=\sum_{i=1}^{k}\binom{k}{i}u^k \in \mathbf{Z}[[u]].$$

The following operator,

$$\psi^k=\sum_{i=1}^{k}\binom{k}{i}Q_i(\gamma^1,\ldots,\gamma^i)$$

is an explicit construction of the Adams-operator. One may verify the properties in the theorem, and the claim follows.

Assume E,F are complex vectorbundles on B and consider the Chern-polynomial.

$$c_t(E\oplus F)=1+c_1(E\oplus F)t+\ldots+c_N(E\oplus F)t^N.$$

where $N=rk(E)+rk(F)$. Assume there is a decomposition $E=\oplus^n L_i$ and $F=\oplus^p R_j$ into linebundles. We get a decomposition,

$$c_t(E\oplus F)=\prod c_t(L_i)\prod c_t(R_j)=(1+a_1t)(1+a_2t)\ldots(1+b_1t)\ldots(1+b_pt)$$

where $a_i=c_1(L_i)$,$b_j=c_1(R_j)$. We get thus,

$$c_i(E\oplus F)=\sigma_i(a_1,..,a_n,b_1,..,b_p).$$

Let,

$$Q_k=u_1^k+\cdots+u_k^k=Q_k(\sigma_1,..,\sigma_k)$$

where σ_i is the ith elementary symmetric function in the u_i's.

Proposition 3.6: The following holds:

$$Q_k(c_1(E\oplus F),..,c_k(E\oplus F))=Q_k(c_i(E))+Q_k(c_i(F)).$$

Proof: We have,

$$Q_k(c_i(E\oplus F))=Q_k(\sigma_i(a_i,b_j))=$$

$$a_1^k+\cdots a_n^k+b_1^k+\cdots b_p^k=Q_k(c_i(E))+Q_k(c_i(F))$$

and the claim follows.

The Chern-Character and Cohomology Operations

We construct a Chern-character with values in singular cohomology, using Newton-polynomials and characteristic classes following [2]. The $k'th$ Newton-classe $s_k(E)$ of a complex vectorbundle will be defined using characteristic classes of E: $c_1(E),..,c_k(E)$ and the $k'th$ Newton-polynomial $s_k(\sigma_1,..,\sigma_k)$. We us this construction to define the Chern-character $Ch(E)$ of the vectorbundle E.

We first define Newton polynomials using the elementary symmetric functions. Let $u_1,u_2,u_3,..$ be independent variables over the integers Z, and let $Q_k=u_1^k+u_2^k+\cdots+u_k^k$ for $k\geq 1$. It follows Q_k is invariant under permutations of the variables u_i: for any $\sigma\in S_k$ we have $\sigma Q_k=Q_k$ hence we may express Q_k as a polynomial in the elementary symmetric functions σ_i:

$$Q_k=Q_k(\sigma_1,\sigma_2,..,\sigma_k).$$

We define,

$$S_k(\sigma)=Q_k(\sigma_1,\sigma_2,..,\sigma_k)$$

to be the $k'th$ Newton polynomial in the variables $(\sigma_1,\sigma_2,..,\sigma_k)$ where σ_i is the $i'th$ elementary symmetric function. One checks the following:

$$s_1(\sigma_1)=\sigma_1,$$

$$s_2(\sigma_1,\sigma_2)=\sigma_1^2-2\sigma_2,$$

and,

$$s_2(\sigma_1,\sigma_2,\sigma_3)=\sigma_1^3-3\sigma_1\sigma_2+3\sigma_3$$

and so on.

Assume we have a cohomology theory H^* satisfying the projective bundle property. One gets characteristic classes $c_i(E)$ for a complex vectorbundle E on B:

$$c_i(E)\in H^{2i}(B).$$

Let the class $S_k(E)=s_k(c_1(E),c_2(E),..,c_k(E))\in H^{2k}(B)$ be the $k'th$ Newton-class of the bundle E. One gets:

$$s_k(\sigma_1,0,..,0)=\sigma_1^k$$

for all $k\geq 1$. Assume E,F linebundles. We see that,

$$S_2(E\oplus F)=c_1(E\oplus F)^2-2c_2(E\oplus F)=$$

$$(c_1(E)+c_1(F))^2\text{-}2(c_2(E)+c_1(E)c_1(F)+c_2(F))=$$

$$c_1(E)^2+2c_1(E)c_1(F)+c_1(F)^2-2c_2(E)-2c_1(E)c_1(F)-2c_2(F)=$$

$$c_1(E)^2-2c_2(E)+c_1(F)^2-2c_2(F)=S_2(E)+S_2(F).$$

This holds in general:

Proposition 4.1: For any vectorbundles E,F we have the formula,

$$S_k(E\oplus F)=S_k(E)+S_k(F).$$

Proof: This follows from 3.6.

Let $K^*_{\mathbf{C}}(B)$ be the Grothendieck-group of complex vectorbundles on

B, i.e the free abelian group modulo exact sequences $K^*_{\mathbf{C}}(B) = \oplus\mathbf{Z}[E]/U$ where U is the subgroup generated by elements $[E\oplus F]-[E]-[F]$.

Definition 4.2: The class,

$$Ch(E) = \sum_{k\geq 0}\frac{1}{k!}S_k(E) \in H^{even}(B)$$

is the Chern-character of E.

Lemma 4.3: The Chern-character defines a group-homomorphism,

$$Ch : K^*_{\mathbf{C}}(B) \to H^{even}(B)$$

between the Grothendieck group $K^*_{\mathbf{C}}(B)$ and the even cohomology of B with rational coefficients.

Proof: By Proposition 4.1 we get the following: For any E,F we have,

$$Ch(E\oplus F) = \sum_{k\geq 0}\frac{1}{k!}s_k(E\oplus F) = \sum_{k\geq 0}\frac{1}{k!}(s_k(E)+s_k(F)) =$$

$$\sum_{k\geq 0}\frac{1}{k!}s_k(E) + \sum_{k\geq 0}\frac{1}{k!}s_k(F) = Ch(E)+Ch(F).$$

We get,

$$Ch([E\oplus F]-[E]-[F]) = Ch(E\oplus F)-Ch(E)-Ch(F)=0$$

and the Lemma follows.

Example 4.4: Given a real continuous vectorbundle F on B there exist Stiefel-Whitney classes $w_i(F)\in H^i(B,Z/2)$ (see [1]) satisfying the necessary conditions, and we may define a "Chern-character"

$$Ch : K^*_{\mathbf{R}}(B) \to H^*(B,\mathbf{Z}/2)$$

by

$$Ch(F) = \sum_{k\geq 0}Q_k(w_1(F),..,w_k(F)).$$

This gives a well-defined homomorphism of abelian groups because of the universal properties of the Newton-polynomials and the fact $H^*(B,Z/2)$ is commutative. The formal properties of the Stiefel-Whitney classes w_i ensures that for real bundles E,F Proposition 3.6 still holds: We have the formula,

$$Q_k(w_i(E\oplus F)) = Q_k(w_i(E)) + Q_k(w_i(F)).$$

Since $S_k(\sigma_1,0,\dots,0) = \sigma_1^k$ we get the following: When E,F are linebundles we have:

$$S_k(E\otimes F)=S_k(c_1(E\otimes F),0,..,0)=(c_1(E\otimes F))^k=(c_1(E)+c_1(F))^k=$$

$$\sum_{i+j=k}\binom{i+j}{i}c_1(E)^i c_1(F)^j = \sum_{i+j=k}\binom{i+j}{i}S_i(E)S_j(F).$$

This property holds for general E,F:

Proposition 4.5: Let E,F be complex vectorbundles on a compact topological space B. Then the following formulas hold:

$$S_k(E\otimes F) = \sum_{i+j=k}\binom{i+j}{i}S_i(E)S_j(E) \qquad (14)$$

Proof: We prove this using the splitting-principle and Proposition 4.1. Assume E,F are complex vectorbundles on B and $f:B'\to B$ is a map of topological spaces such that $f^*E=\oplus_i L_i, f^*F=\oplus_j M_j$ where L_i,M_j are linebundles and the pull-back map $f^*:H^*(B)\to H^*(B')$ is injective. We get the following calculation:

$$f^*S_k(E\otimes F)=S_k(f^*E\otimes F)=S_k(\oplus L_i\otimes M_j)$$

hence by Lemma 4.1 we get,

$$\sum_{i,j}S_k(L_i\otimes M_j) = \sum_i(\sum_j S_k(L_i\otimes M_j)) =$$

$$\sum_i\sum_j\sum_{u+v=k}\binom{u+v}{u}S_u(L_i)S_v(M_j) =$$

$$\sum_i\sum_{u+v=k}\binom{u+v}{u}S_u(L_i)S_v(\oplus M_j) =$$

$$\sum_{u+v=k}\binom{u+v}{u}S_u(\oplus L_i)S_v(\oplus M_j) =$$

$$Ch : K^*_{\mathbf{C}}(B) \to H^{even}(B).$$

and the result follows since f^* is injective.

Theorem 4.6: The Chern-character defines a ring-homomorphism.

$$Ch : K^*_{\mathbf{C}}(B) \to H^{even}(B).$$

Proof: From Proposition 4.5 we get:

$$Ch(E\otimes F) = \sum_{k\geq 0}\frac{1}{k!}S_k(E\otimes F) =$$

$$\sum_{k\geq 0}\frac{1}{k!}\sum_{i+j=k}\binom{i+j}{i}S_i(E)S_j(F) =$$

$$(\sum_{k\geq 0}\frac{1}{k!}S_k(E))(\sum_{k\geq 0}\frac{1}{k!}S_k(F) = Ch(E)Ch(F)$$

and the Theorem is proved.

Example 4.7: For complex K-theory $K^*_{\mathbf{C}}(B)$ we have for any complex vectorbundle E characteristic classes $c_i(E)\in K^*_{\mathbf{C}}(B)$ satisfying the neccessary conditions, hence we get a group-homomorphism.

$$Ch_{\mathbf{Z}} : K^*_{\mathbf{C}}(B) \to K^*_{\mathbf{C}}(B)$$

defined by,

$$Ch_{\mathbf{Z}}(E) = \sum_{k\geq 0}Q_k(c_1(E),..,c_k(E)).$$

If we tensor with the rationals, we get a ring-homomorphism.

$$Ch_{\mathbf{Q}} : K^*_{\mathbf{C}}(B) \to K^*_{\mathbf{C}}(B)\otimes\mathbf{Q}$$

defined by,

$$Ch(E) = \sum_{k\geq 0}\frac{1}{k!}Q_k(c_1(E),..,c_k(E)).$$

Theorem 4.8: Let B be a compact topological space. The Chern-character,

$$Ch^{\mathbf{Q}} : K^*_{\mathbf{C}}(B)\otimes\mathbf{Q} \to H^{even}(B,\mathbf{Q})$$

is an isomorphism. Here $H^*(B,Q)$ denotes singular cohomology with rational coefficients.

Proof: See [2].

The Chern-character is related to the Adams-operations in the following sense: There is a ring-homomorphism.

$$\psi_H^k : H^{even}(B) \to H^{even}(B)$$

defined by,

$$\psi_H^k(x) = k^r x$$

when $x \in H^{2r}(B)$. The Chern-character respects these cohomology operations in the following sense:

Theorem 4.9: There is for all $k \geq 1$ a innovative diagram.

$$K_{\mathbf{C}}^*(B)^{Ch\psi^k} H^{even}(B)^{\psi_H^k} K_{\mathbf{C}}^*(B)^{Ch} H^{even}(B)$$

where ψ^k is the Adams operation defined in the previous section.

Proof: The proof follows Theorem V.3.27 in [2]: We may assume L is a linebundle and we get the following calculation: $\psi^k(L)=L^k$ and $c_1(L^k)=kc_1(L)$ hence,

$$Ch(\psi^k(L)) = exp(kc_1(L)) = \sum_{i \geq 0} \frac{1}{i!} k^i c_1(L)^i =$$

$$\psi_H^k(exp(c_1(L))) = \psi_H^k(Ch(L)$$

and the claim follows.

Hence the Chern-character is a morphism of cohomology-theories respecting the additional structure given by the Adams and Steenrod-operations.

References

1. Milnor J (1966) Characteristic Classes. Princeton University Press.
2. Karoubi M (1978) K-theory - an introduction. Grundlehren Math Wiss.
3. Dupont J (1978) Curvature and characteristic classes, Lecture Notes in Mathematics. Springer Verlag V: 640.
4. Fulton W, Lang S (1985) Riemann-Roch algebra. Grundlehren Math Wiss No: 277.
5. Grothendieck A (1958) Theorie des classes de Chern. Bull Soc Math France 86: 137-154.
6. Husemoeller D (1979) Fibre bundles. GTM.
7. Steenrod N (1962) Cohomology operations. Princeton University Press.
8. End W (1969) Über Adams-operationen. Invent Math 9: 45.