Numerical Solution of the 2-Hessian Equation by a Newton’s Algorithm

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Abstract

The elliptic 2-Hessian equation is a fully nonlinear partial differential equation that is related, for example, to intrinsic curvature for three dimensional manifolds. We solve numerically this equation with periodic boundary condition and with Dirichlet boundary condition using a Newton’s algorithm. We verify numerically, by introducing finite difference schemes, the convergence of the algorithm which is obtained in few iterations.

Keywords: Numerical solution; Boundary; Matrix; Periodic function

Introduction

We are interested by the numerical approximations of the following 2-Hessian, which is a fully nonlinear elliptic partial differential equation in 3-dimensional space [1,2]:

\[ \sum_{ij} \sigma_{ij}(D^2 \psi) = \lambda \psi \]

where \( \psi \) is a positive periodic function, \( \lambda \) is the Hessian matrix of \( \psi \), \( \lambda \) is the eigenvalues of \( \lambda \) \( \in \mathbb{R}^3 \), \( \lambda > 0 \) and \( \sigma(\lambda) > 0 \) [3]. The operator \( \lambda \) is not elliptic unless \( \lambda = 0 \) and \( \lambda \) \( \in \mathbb{R}^3 \) [4,5].

In the periodic setting, in eqn. (1) reads as follows [2]: given a positive periodic function \( f \) on \( T^3 \) ∈ \( \mathbb{R}^3 \), find a periodic function \( u : T^3 \rightarrow \mathbb{R} \) such that

\[ M[u] = \sigma(\lambda[D^2 u]) = f \]

where \( M[u] \) is the nonlinear differential operator defined by \( M[u] = \sigma(\lambda[D^2 u]) \). This equation is none other than eqn. (1) with \( \psi \) on \( T^3 \) ∈ \( \mathbb{R}^3 \). We have

\[ S_4[u] = \frac{1}{2} \int_{\Omega} \nabla u \cdot \nabla D^2 u \]

which obtained when treating computationally prescribed curvature problems.

Properties of the 2-Hessian Operator

Proposition 1.1.

We have

\[ S_4[u] = \frac{1}{2} \sum_{ij} \sigma_{ij}(D^2 u) \]

and

\[ M[u] = \sum_{ij} \sigma_{ij}(D^2 u) \]

where \( \lambda \), \( \lambda_1 \) and \( \lambda_2 \) are the eigenvalues of \( \lambda \). Therefore, by expanding in eqn. (7) and using in eqn. (2), we obtain in eqns. (5 and 6).

Algorithm of Resolution

Using a global convergence Newton method [1], to linearize the eqn. (1), the algorithm we consider reads: Given \( u_n \) loop over \( n \in \mathbb{N} \),

- Computation of \( f_n \).
- Computation of \( \theta_n \).

Then

\[ \forall T_z [u^n] : \theta_n = \frac{1}{\tau} (f_n - \lambda_n) \]

with the stabilization factor \( \tau \geq 1 \).
- Computation of \( u^{n+1} \).

Where \( T_z S_z \) for the periodic problem and \( T_z S_z \) for the Dirichlet problem. For \( n = 1 \), we obtain the classical Newton’s method.

Linearization

Let \( s \) be a parameter in \( R \). We have

\[ S_4[u + sv] = c(D^2 u + D^2 v) \]

\[ = \frac{1}{2} \int (D^2 u)^T - \int (D^2 v)^T + s (D^2 u D^2 v - trace D^2 u D^2 v) + o(s) \]

Then

\[ V S_4[u] \]

\[ = \int (D^2 u)^T - \int (D^2 v)^T \]

By expanding in eqn. (9) we obtain the linearization of \( S_4[u] \) and \( M[u] \). For \( u \in C^2 \):

\[ V S_4[u] = \frac{1}{2} \left( \int (D^2 u)^T - \int (D^2 v)^T \right) \]

and

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Let \( u \in C^2 \), \( \nabla S_\nu[u] \) and \( \nabla M_\nu[u] \) are elliptic if \( u \) is 2-admissible.

Proof. see [3].

### Numerical Experiments

We discretize the problem's domain \([0,1]^3\) by dividing the domain into a uniform grid with grid space \( h \). We denote by \( D_{\nu u} \) the centered second order finite difference discretization of the operator \( u_{\nu x} \) for \( \nu \in [x,y,z] \), and by \( D_{\nu h}^{2} \) the discretization of the Hessian matrix. That is

\[
D_{\nu h}^{2} = \begin{pmatrix}
D_{\nu u} & D_{\nu u} & D_{\nu u} \\
D_{\nu u} & D_{\nu u} & D_{\nu u} \\
D_{\nu u} & D_{\nu u} & D_{\nu u}
\end{pmatrix}
\]

The discretization of the 2-Hessian operators are then given by

\[
S_{\nu}^{2}[u] = \sigma_{\nu} \left( \lambda \left( D_{\nu h}^{2} u \right) \right) = c \left( D_{\nu h}^{2} u \right)
\]

and

\[
M_{\nu}^{2}[u] = \sigma_{\nu} \left( \lambda \left( I + D_{\nu h}^{2} u \right) \right) = c \left( I + D_{\nu h}^{2} u \right)
\]

#### Periodic problem

The numerical scheme connected to the problem in eqn. (3), obtained by using the global convergence Newton and the discrete operators described above, is given, in each iteration, by the following linear system of \((m+1)^3\) equations with \((m+1)^3\) unknowns, where \( m+2 \) is the number of discretization points given by

\[
h = \frac{1}{m+1}
\]

\[
D_{\nu h}^{2} u_{ij,k} = -4u_{ij,k} + u_{i,j+1,k} + u_{i,j-1,k} + u_{i,j,k+1} + u_{i,j,k-1} + D_{\nu u} u_{ij,k}
\]

\[
-2D_{nu} u_{ij,k} = \frac{\nu^2}{4} (\theta_{ij,k} - \frac{1}{2}u_{ij,k})
\]

\[
h \text{ for } i,j,k = 0,\ldots,m \text{ and } f_{ij,k}^\nu = M_{\nu}^{2}[u^\nu].
\]

The numerical tests are shown in the following two figures.

In Figure 1, we consider the function \( f = 1 + \sin(2\pi x) \sin(2\pi y) \sin(2\pi z) \) and we solve the problem in the tore \( T^3 \). Figure 1 shows the convergence of the error \( kL^2 \) in terms of the number of iterations where the linearized 2-Hessian equation is solved for the different values of \( m \): \( m=25 \), \( m=20 \) and \( m=15 \), and for \( r=1 \).

In Figure 2 shows the convergence of the error \( ku_{ij,k} - u_{ij,k} \) in terms of the number of iterations for \( m=15 \) and \( r=1 \).

#### Dirichlet problem

We consider the 2-Hessian equation in \( \mathbb{R}^3 \) with Dirichlet boundary conditions:
Finally, Figure 4 shows the order of convergence of the algorithm for different values of $\tau$. We remark that the order of convergence decrease very fast in terms of $\tau$. Note that in practice we have taken $\tau=1$, for which value the order of convergence is close to $2$.

**Conclusion**

We presented a Newton’s algorithm to solve the fully nonlinear 2-Hessian equation in the case where it is elliptic and the solution is smooth enough. The numerical experiments show that the convergence is very fast. Then we can solve the 2-Hessian in the cost of solving a few number of linear elliptic problems. The sparsity of the matrix of discretization allows us to solve quickly the linear problems. Moreover, the numerical tests show a good stability of this algorithm.

**References**