Numerical Solution of Vibrating Double and Triple-Panel Stepped Thickness Plates

Mohamed A. El-Sayad* and Ahmed M. Farag

Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria, Egypt

Abstract

The main objective of the present paper is to achieve a modified numerical method for investigating the vibration characteristics of the stepped thickness plate with many types of boundary conditions surrounding certain number of panels. The presented technique relies on dividing the entire plate into several regions of uniform thickness separated by sudden steps. Each region is divided to number of strips which are assembled and solved numerically by the Finite Strip-Transition Matrix method FSTM. A convenient basic function is applied to reduce the partial differential equation of motion of plate inside a single region into an ordinary differential one. Step continuity conditions are applied to achieve the final solution of plate. Regional rigidities of plates and mass per unit area are changed due to the change of plate thickness from a region to another. Consequently, new straining actions are occurred and then compatibility conditions become necessary to modify the nodal vector at each step. Various types of restrained boundary conditions against rotation are included in the present paper. The validity of present method is checked and the accuracy of the results is compared with those available in literature showing a good agreement.

Keywords: Numerical; Transition; Stepped; Panel; Vibration; Plate

Nomenclatures

\[ W'_i = W'_i(z,h,t) \quad \text{Regional dimension-less plate displacement in the domain of region } R_i, \quad i = 1,2,3. \]

\[ a,b \quad \text{Plate dimension in } \zeta, \eta \text{ directions respectively} \]

\[ x,y \quad \text{Plate coordinates} \]

\[ t \quad \text{Time} \]

\[ \zeta, \eta \quad \text{Plate dimension-less coordinates; } \zeta = \frac{x}{a}, \eta = \frac{y}{b} \]

\[ \beta = \frac{a}{b} \quad \text{Aspect ratio} \]

\[ h_i \quad \text{Regional thickness of plate} \]

\[ \rho \quad \text{Plate density} \]

\[ M \quad \text{Number of panel} \]

\[ k \quad \text{Parameter of homogenous sub-grade} \]

\[ (\bar{m}_i) = \frac{\rho h_i}{k} \quad \text{Plate mass per unit area} \]

\[ (D_i) = \frac{Eh_i}{(12(1-\nu^2))} \quad \text{Regional flexural rigidity of plate} \]

\[ K_u = \frac{ka^2}{D_i} \quad \text{Normalized parameter of homogenous sub-grade} \]

\[ k \quad \text{Parameter of homogenous sub-grade} \]

\[ \omega \quad \text{Natural frequency} \]

\[ \lambda_i^* = \omega^* a^*(\frac{m_i}{D_i}) \quad \text{Natural frequency parameter} \]

\[ e_{x_i}, d_{x_i} \quad \text{Integral values} \]

\[ \alpha_i = \frac{h_i}{h} \quad \text{Panel thickness ratio} \]

\[ [V_{x_i}], [V_{y_i}] \quad \text{Regional nodal vector} \]

\[ [Y_{x_i}], [Y_{y_i}] \quad \text{Regional transition matrix} \]

\[ [V_{x_i}], [V_{y_i}]_{x_i}, [V_{y_i}]_{y_i} \quad \text{Nodal vectors of strip } K_j \]

\[ \Phi \quad \text{Restrained cofficient of rotation and translation at } \eta = 0,1 \]

\[ \nu_i = \frac{h_i}{h} \quad \text{Step thickness ratio} \]

\[ V_x \quad \text{Poisson’s ratio} \]

Introduction

A certain structural optimization of a panel may be achieved by possessing a suitable variation of thickness of plate structure. The importance of types of structures increases when the aero-space and underground structures are considered. Survey of the literature on the flexural vibration of thin plate reveals that the work on this topic has been mainly confined to plates with uniform thicknesses. A relatively few studies have been published on the free vibration of isotropic plates of stepped thicknesses. During the past few decades, many researchers were devoted to mathematical modeling and numerical solution for static elastic multi-structure problems. Since vibration analysis of elastic structures plays important roles in engineering applications, this paper is concerned with Finite Strip-Transition Matrix method (FSTM) for vibration analysis of paneled stepped thickness plate.

Although, Chopra [1] has attempted an exact solution for a simply supported stepped thickness plate with two panels, Warburton [2] pointed out that continuity conditions used by Chopra [1] were incorrect. He presented a modified analytical technique for two paneled stepped plate with different properties of orthotropic. Sakata [3] proposed an approximate formula for estimating the fundamental

*Corresponding author: Mohamed A. El-Sayad, Department of Engineering Mathematics and Physics, Faculty of Engineering, Alexandria University, Alexandria, Egypt, E-mail: mel_sayad@hotmail.com

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nature frequency of an isotropic plate with stepped thickness from the natural frequencies of the isotropic plate reduced from the orthotropic one. Recently, Farag [4] applied a closed form solution for vibrating surfaces of partially restrained and clamped double-panel plates via a power matrix exponential method. Xiang and Wang [5] also studied the exact vibration solutions of stepped rectangular plates. Xiang et al. [6,7] extended these studies in cooperation with others to include the case of stepped rectangular Mindlin plate and the case of

Figure 1: Finite Strips of Triple Panel Stepped Thickness plate.

Figure 2-a: Relationships of the variation of natural frequency parameters \( \lambda_{mn} \) due to the change of thickness ratio \( \alpha_2 \) for the first three modes where \( m = 1 \) and \( n = 1, 2 \) or 3 for CCCC case.

Figure 2-b: Relationships of the variation of natural frequency parameters \( \lambda_{mn} \) due to the change of thickness ratio \( \alpha_2 \) or the first three modes where \( m = 2 \) and \( n = 1, 2 \) or 3 for CCCC case.

Figure 2-c: Relationships of the variation of natural frequency parameters \( \lambda_{mn} \) due to the change of thickness ratio \( \alpha_2 \) for the first three modes where \( m = 3 \) and \( n = 1, 2 \) or 3 for CCCC case.

Figure 3-a: Relationships of the variation of natural frequency parameters \( \lambda_{mn} \) due to the change of thickness ratio \( \alpha_2 \) for the first three modes where \( m = 1 \) and \( n = 1, 2 \) or 3 for CSCS case.

Figure 3-b: Relationships of the variation of natural frequency parameters \( \lambda_{mn} \) due to the change of thickness ratio \( \alpha_2 \) for the first three modes where \( m = 2 \) and \( n = 1, 2 \) or 3 for CSCS case.
stepped circular Mindlin plate. Gorman and Singha [8] outlined the vibration analysis of stepped cantilever plate using a supuer position method. A numerical approach for rectangular stepped plate with sides restrained against rotation has been proposed by Laura and Filipich [9]. They ensured that vibration of plate with stepped thicknesses was studied in a very few number of literature. Filipich et al. [10] used a simple polynomial coordinate function which idecificed satisified the restrained boundary conditions of plate with discontinuous thickness. Modal study via a discrete numerical approach, such as finite element method by Mukherjee and Mukhopadhyay [11], or finite strip method by Cheung and Li [12,13] introduces solution accuracy depending upon the suggested element size. Vibrations of plates with variable thickness were studied by Zanzi and Laura [14], Gutierrez et al. [15], Laura and Gutierrez [16] via different methods. Kaabi and Aksu [17,18] applied a modified method to examine the dynamic behavior of rectangular plates with bilinear variation of thickness. Cheung and Li [19] achievea a simple finite strip method to analyze the hunched continuous bridges. Vibration of orthotropic rectangular plate with free edges in the case of discontinuously varying thickness was discussed by Laura et al. [20]. They extended their studies to include two other boundary conditions different from Laura et al. [21]. Farag [22] applied Finite Strip Transition Matrix method to solve the free and forced vibration problems of uniform thickness plate. In extended work, Farag [23] analyzed the stepped thickness plate analytically by using the boundary element method. Sanches et al. [26] studied the dynamic stationary vibration analysis of stepped cantilever plate using a supper position method by Mukherjee and Mukhopadhyay [11], or finite strip method by Cheung and Li [12,13] introduces solution accuracy depending upon the suggested element size. Vibrations of plates with variable thickness were studied by Zanzi and Laura [14], Gutierrez et al. [15], Laura and Gutierrez [16] via different methods. Kaabi and Aksu [17,18] applied a modified method to examine the dynamic behavior of rectangular plates with bilinear variation of thickness. Cheung and Li [19] achievea a simple finite strip method to analyze the hunched continuous bridges. Vibration of orthotropic rectangular plate with free edges in the case of discontinuously varying thickness was discussed by Laura et al. [20]. They extended their studies to include two other boundary conditions different from Laura et al. [21]. Farag [22] applied Finite Strip Transition Matrix method to solve the free and forced vibration problems of uniform thickness plate. In extended work, Farag [23] analyzed the stepped thickness plate analytically by using the boundary element method. Sanches et al. [26] studied the dynamic stationary vibration analysis of stepped cantilever plate using a supper position method.
lines as in figure 1. The basic function of strip [23], is applied to reduce the partial differential equation of motion into an ordinary differential one. The reduced differential equation is solved numerically by the FSTM as an initial value problem under the proposed initial boundary conditions at \((\eta = 0)\). The initial nodal conditions of each strip are applied to derive the strip end nodal conditions which are used as initial nodal conditions to the next strip. All nodal straining actions of each strip are obtained until the end of the current panel is reached. Because the intermediate end of a current panel is sudden step, the initial nodal conditions of the next panel must be modified by satisfying the compatibility conditions at this step. The FSTM is applied for all strips of all panels until the final boundary end of plate is reached. Satisfying the boundary conditions at \((\eta = 1)\), one can obtain the final solution of the equation of motion of plate.

**Reduction technique**

The regional displacement of the stepped plate is:

\[
W_i(\zeta, \eta, t) = \sum_{m=1}^{M} (U_m(\zeta, \eta))(V_m(\zeta, \eta)) \sin \omega t
\]

(2)

Where \((U_m(\zeta, \eta))\) is known regional shape function satisfying the

**Table 3:** The natural frequency parameter \(\lambda_{mn}\) for 3-panels square clamped restrained plate CE

<table>
<thead>
<tr>
<th>(\alpha_c, \alpha_r)</th>
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<th>0.0</th>
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<th>0.2</th>
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<tr>
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**Table 4:** The natural frequency parameter \(\lambda_{mn}\) for 3-panels square simply supported- restrained plate SE

<table>
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<tr>
<th>(\alpha_c, \alpha_r)</th>
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<tr>
<td>(\lambda_{19})</td>
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<td>0.5, 1.0</td>
<td>(\lambda_{20})</td>
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<td>48.4637</td>
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<td>(\lambda_{21})</td>
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<td>525.1012</td>
<td>521.4784</td>
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The ordinary differential equation (4) is transformed within a region \( R_i \) to a linear system of differential equations expressed as [22]:

\[
\frac{d}{d\eta} \begin{bmatrix} V_{m1} \\ V_{m2} \\
\end{bmatrix}_i = \begin{bmatrix} A_m \end{bmatrix} \begin{bmatrix} V_{m1} \\ V_{m2} \\
\end{bmatrix}_i
\]

(5)

Where

\[
\begin{bmatrix} V_{m1} \\ V_{m2} \\
\end{bmatrix}_i = \begin{bmatrix} V_{w1} \\ V_{w2} \\
\end{bmatrix}_i
\]

(6)

And

\[
[A_m]_i = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\frac{-1}{\beta^4} (\alpha_1^2 K_{ii} - \alpha_2^2 \lambda_1^2 + d_{ii}) & 0 & (-2 \frac{\lambda_1}{\beta^2}) & 0 \\
\end{bmatrix}
\]

(7)

The general solution of the ordinary differential equation is:

\[
\begin{bmatrix} V_{m1} \\ V_{m2} \\
\end{bmatrix}_{K_i} = [Y_{K_i} \delta^s_{K_i}] \begin{bmatrix} V_{m1} \\ V_{m2} \\
\end{bmatrix}_{K_i}
\]

(8)

Where \( \delta^s_{K_i} \) is the width of strip \( K_j \) bounded by two nodal lines \( K_{j-1}, K_{j+1} \) inside region \( R_i \).

The regional Transition Matrix \( [Y_{K_i}] \) is calculated according to the Runge Kutta method such as:

\[
[Y_{K_i}] = I + \sum_{r=1}^{\infty} \left[ A_m \right]_i \frac{(\delta^s_{K_i})^r}{r!}
\]

(9)

Where \( I \) is the unit matrix?

Continuity conditions

The Magnitudes of displacement \( W \) slope \( \frac{\partial W}{\partial x} \) Moment \( M_i \) and shear \( Q_i \) must satisfy the continuity conditions at the sudden step \( S_i \) so that:
The symbols $\eta$, $i$, denote positions immediately after and before which are the components of initial $\Phi$. For all recorded modes, the results are calculated under the variation of $\alpha_i$ from 1.0 to 0.1 where $\alpha_2 = \alpha_3 = 1$. The results show that the natural frequency parameter increases by decreasing $\alpha_i$.

Table 3 shows the natural frequency parameters $\lambda_{mn}$ for the cases of 3-panels square full plate CCCC with two opposite edges clamped and other edges elastically restrained against rotation with restrained coefficient varying from 0 to $\infty$. The results are obtained for cases of panel width ratio $\ell_1 : \ell_2 : \ell_3 = 0.25 : 0.50 : 0.25$ and $\alpha_i$ varying from 1.0 to 0.25, where $\alpha_2 = \alpha_3 = 1$. The natural frequency parameter is obtained for the first three modes. The results show that the natural frequency parameter decreases by increasing the coefficient of restrained $\phi$.

This case is carried out for 3-panel square plate, SE$_m$SE$_n$, simply supported at two opposite edges and partially restrained against rotation at the other edges (Table 4). The results are obtained for panel width ratios $\ell_1 : \ell_2 : \ell_3 = 0.25 : 0.50 : 0.25$ and panel thickness ratio $1 : \alpha_1 : 1$ where $\alpha_i$ varies from 1.0 to 0.25. Table 4 shows that the natural frequency parameter increases by decreasing the restrained coefficient $\phi$. For the double panel isotropic, square, clamped supported plate at all edges CCCC the relationships of the variation of natural frequency parameters due to the change of thickness ratio $\alpha_2$ are plotted in figures 2-a,b,c. The results show that the parameter $\lambda_{mn}$ increases by decreasing the thickness ratio $\alpha_2$ for all modes.

The case of full clamped stepped rectangular plate CCCC with thickness ratio $\alpha_2 = 5$, panel width ratio $\ell_1 : (\ell_2 + \ell_3) = 1 : 2$ and aspect ratio $\beta$ of the entire plate varying from 1 to 2 is investigated as shown in figures 4-a,b,c. The results show that the natural frequency parameters increase by decreasing the thickness ratio $\alpha_2$ for all recorded modes.

**Conclusion**

The finite strip transition matrix method FSTM described herein involves a numerical solution of stepped paneled plate with classical and restrained boundary conditions. This method is a combination between the strip and transition matrix method to solve the vibration problem of stepped plates as an initial value problem. Transition matrix method is a semi analytical method relying on estimating the numerical solution of the initial value problem by means of Range Kutta method. The plate domain is divided into paneled regions consisting of strips bounded by nodal lines. Each strip is governed by the transition matrix method to solve the vibration problem of stepped plates.

The results are obtained for cases of panel width ratio $\ell_1 : (\ell_2 + \ell_3) = 1 : 2$ and aspect ratio $\beta$ of the entire plate varying from 1 to 2 is investigated as shown in figures 4-a,b,c. The results show that the parameter $\lambda_{mn}$ increases by decreasing the thickness ratio $\alpha_2$ for all recorded modes.

The finite strip transition matrix method FSTM described herein involves a numerical solution of stepped paneled plate with classical and restrained boundary conditions. This method is a combination between the strip and transition matrix method to solve the vibration problem of stepped plates as an initial value problem. Transition matrix method is a semi analytical method relying on estimating the numerical solution of the initial value problem by means of Range Kutta method. The plate domain is divided into paneled regions consisting of strips bounded by nodal lines. Each strip is governed by the transition matrix method to solve the vibration problem of stepped plates.
References