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On a correspondence between ideals and commutativity in algebraic crossed products [1]

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Abstract

In this paper we will give an overview of some recent results which display a connection between commutativity and the ideal structures in algebraic crossed products.

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1 Introduction

In the recent papers [3, 4], we have been studying a correspondence between ideals and commutativity in algebraic crossed products. Given an algebraic crossed product $\mathcal{A}_0 \rtimes_\alpha^\sigma G$, consider the following two statements:

S1: The coefficient ring $\mathcal{A}_0$ is a maximal commutative subring in $\mathcal{A}_0 \rtimes_\alpha^\sigma G$,
S2: For every non-zero two-sided ideal I in $\mathcal{A}_0 \rtimes_\alpha^\sigma G$, $I \cap \mathcal{A}_0 \neq \{0\}$.

In this paper we will give an overview of some types of crossed products for which the statements **S1** and **S2** are equivalent. We will also give an example of a situation in which these statements are not equivalent. For a general crossed product we have the following result.

Theorem 1.1 ([3]). *Let $\mathcal{A} = \mathcal{A}_0 \rtimes_\alpha^\sigma G$ be an algebraic crossed product and denote the commutant of $\mathcal{A}_0$ by $C_\mathcal{A}(\mathcal{A}_0) = \{a \in \mathcal{A} \mid ab = ba, \quad \forall b \in \mathcal{A}_0\}$. If the coefficient ring $\mathcal{A}_0$ is commutative, then $I \cap C_\mathcal{A}(\mathcal{A}_0) \neq \{0\}$ for every non-zero two-sided ideal I in the crossed product $\mathcal{A}_0 \rtimes_\sigma^\alpha G$.*

As an immediate corollary to this theorem we get that, if $\mathcal{A}_0$ is assumed to be maximal commutative in $\mathcal{A}_0 \rtimes_\alpha^\sigma G$, then $I \cap \mathcal{A}_0 \neq \{0\}$ for every non-zero two-sided ideal I in $\mathcal{A}_0 \rtimes_\alpha^\sigma G$. Hence, in a general crossed product where the coefficient ring $\mathcal{A}_0$ is commutative, **S1** always implies **S2**. As we will se in Section 2, **S2** does not always imply **S1**.

For the convenience of the reader we shall now recall the definition and the basic properties of algebraic crossed products. For more details see e.g [2]. Throughout this article all rings are assumed to be associative rings. Given a unital ring $\mathcal{R}$ we let $U(\mathcal{R})$ denote the group of multiplication invertible elements of $\mathcal{R}$.

Definition 1.1. A G-crossed system is a quadruple $\{\mathcal{A}_0, G, \sigma, \alpha\}$, consisting of a unital ring $\mathcal{A}_0$, a group G (with unit element e), a map $\sigma : G \to \mathrm{Aut}(\mathcal{A}_0)$ and a *σ-cocycle* map $\alpha : G \times G \to U(\mathcal{A}_0)$ such that for any $x, y, z \in G$ and $a \in \mathcal{A}_0$ the following conditions hold:

(i) $\sigma_x(\sigma_y(a)) = \alpha(x, y)\, \sigma_{xy}(a)\, \alpha(x, y)^{-1}$
(ii) $\alpha(x, y)\, \alpha(xy, z) = \sigma_x(\alpha(y, z))\, \alpha(x, yz)$
(iii) $\alpha(x, e) = \alpha(e, x) = 1_{\mathcal{A}_0}$

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Let $\{u_s\}_{s\in G}$ be a copy (as a set) of G. Given a G-crossed system $\{\mathcal{A}_0, G, \sigma, \alpha\}$, we denote by $\mathcal{A}_0 \rtimes_\alpha^\sigma G$ the free left $\mathcal{A}_0$-module having $\{u_s\}_{s\in G}$ as its basis and we define a multiplication on this set by

$$(a_1\, u_x)(a_2\, u_y) = a_1\, \sigma_x(a_2)\, \alpha(x,y)\, u_{xy} \tag{1.1}$$

for all $a_1, a_2 \in \mathcal{A}_0$ and $x, y \in G$ and extend it bilinearly to all of $\mathcal{A}_0 \rtimes_\alpha^\sigma G$. Each element of $\mathcal{A}_0 \rtimes_\alpha^\sigma G$ may be expressed as a formal sum $\sum_{g\in G} a_g\, u_g$ where $a_g \in \mathcal{A}_0$ and $a_g = 0_{\mathcal{A}_0}$ for all but a finite number of $g \in G$. Explicitly, the addition and multiplication of two arbitrary elements $\sum_{s\in G} a_s\, u_s, \sum_{t\in G} b_t\, u_t \in \mathcal{A}_0 \rtimes_\alpha^\sigma G$ is given by

$$\begin{aligned}
\sum_{s\in G} a_s\, u_s + \sum_{t\in G} b_t\, u_t &= \sum_{g\in G} (a_g + b_g)\, u_g \\
\left(\sum_{s\in G} a_s\, u_s\right)\left(\sum_{t\in G} b_t\, u_t\right) &= \sum_{(s,t)\in G\times G} (a_s\, u_s)(b_t\, u_t) = \sum_{(s,t)\in G\times G} a_s\, \sigma_s(b_t)\, \alpha(s,t)\, u_{st} \\
&= \sum_{g\in G} \left(\sum_{\substack{\{(s,t)\in G\times G| \\ st=g\}}} a_s\, \sigma_s(b_t)\, \alpha(s,t) \right) u_g
\end{aligned} \tag{1.2}$$

Proposition 1.1 ([2])**.** *Let $\{\mathcal{A}_0, G, \sigma, \alpha\}$ be a G-crossed system. Then $\mathcal{A}_0 \rtimes_\alpha^\sigma G$ is an associative unital ring (with the multiplication defined in* (1.1)*).*

Definition 1.2. The ring $\mathcal{A}_0 \rtimes_\alpha^\sigma G$ is called the *crossed product* of the G-crossed system $\{\mathcal{A}_0, G, \sigma, \alpha\}$.

The coefficient ring $\mathcal{A}_0$ is naturally embedded as a subring into $\mathcal{A}_0 \rtimes_\alpha^\sigma G$ via the canonical isomorphism $\iota : \mathcal{A}_0 \hookrightarrow \mathcal{A}_0 \rtimes_\alpha^\sigma G$ defined by $a \mapsto a\, u_e$. Instead of $\iota(\mathcal{A}_0)$ we will simply write $\mathcal{A}_0$.

Remark 1.1. If k is a field and $\mathcal{A}$ is a k-algebra, then so is $\mathcal{A}_0 \rtimes_\alpha^\sigma G$.

Depending on the nature of the maps σ and α we will give different names to the crossed product $\mathcal{A}_0 \rtimes_\alpha^\sigma G$. If the map α is trivial, i.e $\alpha(x,y) = 1_{\mathcal{A}_0}$ for every $(x,y) \in G \times G$, then we shall write $\mathcal{A}_0 \rtimes^\sigma G$ and refer to it as a *skew group ring.* If, on the other hand, σ is trivial, i.e. $\sigma_g = \mathrm{id}_{\mathcal{A}_0}$ for every $g \in G$, then we shall write $\mathcal{A}_0 \rtimes_\alpha G$ and refer to it as a *twisted group ring.* A crossed product where both of the maps σ and α are trivial is written as $\mathcal{A}_0 \rtimes G$ and is simply refered to as a *group ring.*

2 Group rings, skew group rings and twisted group rings

If one wants to talk about *maximal commutativity* of $\mathcal{A}_0$, it does not really make sense unless we assume that $\mathcal{A}_0$ is commutative itself, so from now on we will only consider crossed products where $\mathcal{A}_0$ is commutative. Note that if the group G is trivial, i.e. $G = \{e\}$, then **S1** and **S2** are always true. In the further discussion we will therefore assume that $G \neq \{e\}$. We will continue the further investigation by breaking it down into special cases of crossed products.

Example 2.1 (group rings)**.** Let $\mathcal{A}_0$ be a unital ring and G any (non-trivial) group and denote the group ring by $\mathcal{A}_0 \rtimes G$. Note that this corresponds to the crossed product with trivial σ and α maps. We may define the so called augmentation map $\epsilon : \mathcal{A}_0 \rtimes G \to \mathcal{A}_0$, $\sum_{s\in G} a_s\, u_s \mapsto \sum_{s\in G} a_s$, and it is straightforward to check that it is in fact a ring morphism. The kernel of this map, $\ker(\epsilon)$ is a two-sided ideal in $\mathcal{A}_0 \rtimes G$ and it is not hard to see that $\ker(\epsilon) \cap \mathcal{A}_0 = \{0\}$. This gives us an example of a non-zero two-sided ideal which has zero intersection with the coefficient ring

$\mathcal{A}_0$, i.e. **S2** is false. However, for each $s \in G$, u_s commutes with every element in $\mathcal{A}_0$ and hence **S1** is never true for a group ring (when $G \neq \{e\}$). In other words, in a group ring the two statements **S1** and **S2** are always equivalent.

For skew group rings we have the following theorems.

Theorem 2.1 ([4])**.** *If $\mathcal{A}_0 \rtimes^{\sigma} G$ is a skew group ring where the coefficient ring $\mathcal{A}_0$ is an integral domain and the group G is abelian, then the two assertions **S1** and **S2** are equivalent.*

Theorem 2.2 ([4])**.** *If $\mathcal{A}_0 \rtimes^{\sigma} G$ is a skew group ring where the coefficient ring $\mathcal{A}_0$ is commutative and G is a torsion-free abelian group, then the two assertions **S1** and **S2** are equivalent.*

Remark 2.1. Note that in the previous theorems, the action σ can be trivial, but in that case the situation is already described by Example 2.1.

Example 2.2 (the algebra associated to a dynamical system)**.** In [5, 6, 7] the authors studies crossed product algebras associated to dynamical systems. Suppose that we are given a non-empty set X and a bijection $\sigma : X \to X$. Then (X, σ) is a discrete dynamical system where the action of $n \in \mathbb{Z}$ on $x \in X$ is given by $n \mapsto \sigma^{\circ(n)}(x)$. By $\mathbb{C}^X$ we denote the algebra of functions $X \to \mathbb{C}$ under the usual pointwise operations of addition and multiplication. If we are given a subalgebra $A \subseteq \mathbb{C}^X$ such that it is invariant under σ and σ^{-1}, i.e. such that if $h \in A$ then $h \circ \sigma \in A$ and $h \circ \sigma^{-1} \in A$, then σ induces an automorphism $\tilde{\sigma} : A \to A$ defined by $\tilde{\sigma}(f) = f \circ \sigma$ by which $\mathbb{Z}$ acts on A via iterations. We may now define the skew group algebra $A \rtimes_{\tilde{\sigma}} \mathbb{Z}$.

In the current situation the coefficient algebra A is commutative and the group $(\mathbb{Z}, +)$ is clearly torsion-free and abelian, hence Theorem 2.2 is applicable. We may conclude that Theorem 2.2 is a generalization of certain parts of Corollary 4.5 in [6] and Theorem 4.5, Theorem 4.6, Corollary 4.7, Theorem 6.2 in [7].

In a twisted group ring $\mathcal{A}_0 \rtimes_{\alpha} G$, just like for group rings mentioned above, the action σ is trivial and hence for each $s \in G$ the element u_s commutes with every element in $\mathcal{A}_0$. In other words, $\mathcal{A}_0$ is never maximal commutative in a twisted group ring (when $G \neq \{e\}$).

Example 2.3 (the field of complex numbers)**.** Let $\mathcal{A}_0 = \mathbb{R}$, $G = (\mathbb{Z}_2, +)$ and define the cocycle $\alpha : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{R} \setminus \{0\}$ by $\alpha(\bar{0}, \bar{0}) = 1$, $\alpha(\bar{0}, \bar{1}) = 1$, $\alpha(\bar{1}, \bar{0}) = 1$ and $\alpha(\bar{1}, \bar{1}) = -1$. It is easy to see that $\mathbb{R} \rtimes_{\alpha} \mathbb{Z}_2 \cong \mathbb{C}$. Clearly this twisted group ring is a field and hence simple. Therefore, $\mathbb{C}$ is the only non-zero ideal and clearly $\mathbb{C} \cap \mathbb{R} \neq \{0\}$. However, as has already been mentioned, the coefficient ring $\mathbb{R}$ is not maximal commutative in $\mathbb{C}$. Example 2.3 shows that in a twisted group ring, **S1** may be false even though **S2** is true.

3 General crossed products

A crossed product $\mathcal{A}_0 \rtimes_{\alpha}^{\sigma} G$, where neither of the maps σ and α are trivial and hence not treated in the previous section, will be refered to as a *general crossed product*. For this type of crossed products we are not able to say as much as we would want to.

Theorem 3.1. *If $\mathcal{A}_0 \rtimes_{\alpha}^{\sigma} G$ is a crossed product where $\mathcal{A}_0$ is an integral domain, G is an abelian torsion-free group and α is such that $\alpha(s,t) = 1_{\mathcal{A}_0}$ whenever $\sigma_s = \mathrm{id}_{\mathcal{A}_0}$ or $\sigma_t = \mathrm{id}_{\mathcal{A}_0}$, then the two assertions **S1** and **S2** are equivalent.*

Proof. It is clear from Theorem 1.1 that **S1** $\Longrightarrow$ **S2**. Suppose that $\mathcal{A}_0$ is not maximal commutative. Since $\mathcal{A}_0$ is an integral domain, this means that there exists some $s \in G \setminus \{e\}$ such that $\sigma_s = \mathrm{id}_{\mathcal{A}_0}$. For arbitrary $g, h \in G$ we may use condition (ii) in Definition 1.1 and the assumptions we made on α to arrive at

$$\underbrace{\alpha(s,g)}_{=1_{\mathcal{A}_0}}\,\alpha(sg,h) = \underbrace{\sigma_s(\alpha(g,h))}_{=\alpha(g,h)}\,\underbrace{\alpha(s,gh)}_{=1_{\mathcal{A}_0}}$$

and since G is abelian we get $\alpha(gs,h) = \alpha(sg,h) = \alpha(g,h)$. Let I be the two-sided ideal generated by $1_{\mathcal{A}_0} + u_s$, which is an element that commutes with all of $\mathcal{A}_0$. The ideal I is obviously non-zero and furthermore, it is spanned by elements of the form $a_g\, u_g\,(1_{\mathcal{A}_0} + u_s)\, a_h\, u_h$ where $g, h \in G$ and $a_g, a_h \in \mathcal{A}_0$. We may now rewrite this expression.

$$\begin{aligned}
a_g\, u_g\,(1_{\mathcal{A}_0} + u_s)\, a_h\, u_h &= a_g\, u_g\, a_h\,(1_{\mathcal{A}_0} + u_s)\, u_h \\
&= a_g\, u_g\, a_h\, u_h + a_g\, u_g\, a_h\, u_s\, u_h \\
&= a_g\,\sigma_g(a_h)\, u_g\, u_h + a_g\,\sigma_g(a_h)\, u_g\, u_s\, u_h \\
&= a_g\,\sigma_g(a_h)\,\alpha(g,h) u_{gh} + a_g\,\sigma_g(a_h)\,\underbrace{\alpha(g,s)}_{1_{\mathcal{A}_0}}\, u_{gs}\, u_h \\
&= \underbrace{a_g\,\sigma_g(a_h)\,\alpha(g,h)}_{:=b}\, u_{gh} + a_g\,\sigma_g(a_h)\,\underbrace{\alpha(gs,h)}_{=\alpha(g,h)}\, u_{gsh} \\
&= b\, u_{gh} + b\, u_{gsh}
\end{aligned}$$

Since G is abelian, it is clear that any element of I may be written in the form

$$\sum_{t\in G}(c_t\, u_t + c_t\, u_{ts}) \tag{3.1}$$

for some $c_t \in \mathcal{A}_0$, where t only runs over a finite subset of G. By assumtion $s \neq e$ and hence $t \neq ts$ for every $t \in G$. In particular this means that every contribution from c_e to the e-graded part of the element in (3.1) comes with an equal contribution to the s-graded part. Similarly c_s:s contribution to the s-graded part equals its contribution to the c_{s^2}-graded part. Furthermore, G is assumed to be torsion-free, i.e. $s^n \neq e$ for every $n \in \mathbb{Z} \setminus \{0\}$, and hence the element in (3.1) can never be a non-zero element of degree e, which means $I \cap \mathcal{A}_0 = \{0\}$. By contra positivity we conclude that **S2** $\Longrightarrow$ **S1** and this finishes the proof. □

Remark 3.1. Note that, a twisted group ring can never fit into the conditions of Theorem 3.1, because if σ is trivial, then the conditions force α to be trivial as well.

Finite groups are clearly not torsion-free, but Example 3.1 gives an example of a situation where **S1** and **S2** are in fact equivalent for a general crossed product graded by a finite group. This raises the question whether or not Theorem 3.1 can be generalized to general crossed products graded by more general groups.

Example 3.1 (central simple algebras)**.** Let A be a finite-dimensional central simple algebra over a field F. By Wedderburn's theorem $A \cong M_i(D)$ where D is a division algebra over F and i is some integer. If K is a maximal separable subfield of D then $[K : F] = n$ where $[D : F] = n^2$. We shall assume that K is normal over F and that $[A : F] = [K : F]^2$ (see [1] for motivation). Let $\mathrm{Gal}(K/F)$ be the Galois group of K over F. For $k \in K$ and $\sigma_s \in \mathrm{Gal}(K/F)$ we shall write $\sigma_s(k)$ for the image of k under σ_s. By the Noether-Skolem theorem there is an invertible element $u_s \in A$ such that $\sigma_s(k) = u_s k u_s^{-1}$ for every $k \in K$. One can show that the u_s's are linearly independent over K. However, the linear span over K of the u_s's has dimension n^2 over F, hence must be all of A. In short $A = \{\sum_{s\in G} k_s\, u_s \mid k_s \in K\}$. If $\sigma_s, \sigma_t \in \mathrm{Gal}(K/F)$ and $k \in K$, then $u_s u_t k u_t^{-1} u_s^{-1} = u_s \sigma_t(k) u_s^{-1} = \sigma_{st}(k) = u_{st} k u_{st}^{-1}$. This says that $u_{st}^{-1}(u_s u_t) \in C_A(K) = K$, in other words $u_s u_t = f(s,t) u_{st}$ where $f(s,t) \neq 0$ is in K. Since A is an associative algebra one may verify that $f : \mathrm{Gal}(K/F) \times \mathrm{Gal}(K/F) \to K \setminus \{0\}$ is in fact a cocycle. By Theorem 4.4.1 in [1], if K is a normal extension of F with Galois group $\mathrm{Gal}(K/F)$ and f is a cocycle (factor set), then the crossed product $K \rtimes_f^\sigma \mathrm{Gal}(K/F)$ is a central simple algebra over F and hence in this situation both **S1** and **S2** are in fact true.

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References

[1] I. N. Herstein. *Noncommutative Rings.* The Carus Mathematical Monographs, No. 15, The Math. Assoc. of America, 1968.

[2] C. Năstăsescu and F. Van Oystaeyen. *Methods of Graded Rings.* Lecture Notes in Math. **1836**, Springer-Verlag, Berlin, 2004.

[3] J. Öinert and S.D. Silvestrov. Commutativity and ideals in algebraic crossed products. J. Gen. Lie Theory Appl. **2** (2008) (to be published).

[4] J. Öinert and S.D. Silvestrov. Ideals in crossed products and skew group rings. In preparation.

[5] C. Svensson, S. Silvestrov, and M. de Jeu. Dynamical sytems and commutants in crossed products. Int. J. Math. **18** (2007), 455–471.

[6] C. Svensson, S. Silvestrov, and M. de Jeu. Connections between dynamical systems and crossed products of Banach algebras by $\mathbb{Z}$. In procceding *"Operator Theory, Analysis and Mathematical Physics"*, OTAMP-2006, June 15–22, 2006, Lund, Sweden (to be published); `arXiv: math/0702118v3 (math.DS)`.

[7] C. Svensson, S. Silvestrov, and M. de Jeu. Dynamical systems associated with crossed products. In *"Operator Methods in Fractal Analysis, Wavelets and Dynamical Systems"* (to be published), Preprint `arXiv:0707.1881v2 (math.OA)`.

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