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On Contractions of Three-Dimensional Complex Associative Algebras

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Abstract

Contraction is one of the most important concepts that motivated by numerous applications in different fields of physics and mathematics. In this work, the contractions of complex associative algebras are considered. We focus on the variety $A_3(\mathbb{C})$ of all complex associative algebras of dimension three (including nonunital). Various contractions criteria are collected and new criteria are proposed to test the possible existence of contraction for each pair of associative algebras. One of the main tools is the use of the low-dimensional cohomology groups of these algebras. As a result, we prove that the variety $A_3(\mathbb{C})$ has seven irreducible components, two of dimension 5, four of dimension 7 and one of dimension 9.

Introduction

The notion of contractions was first introduced by Segal [1] and Inonu [2] for Lie algebras. According to references, the contractions can be divided into two major categories. The first one is more physical that deals with the applications of contractions. Another one is pure algebraic that is mainly oriented to the abstract algebraic structure and mathematical background. For associative algebras, Gabriel [3] studied the irreducible components of the algebraic variety of 4-dimensional unital associative algebras. Mazzola's paper [4] concerns unital associative algebras of dimension five. Classification of low-dimensional nilpotent rigid associative algebras and the description of the irreducible components have been treated by Makhlouf [5,6].

The main purpose of this work is to study the variety of all 3-dimensional complex associative algebras. In the paper, we deal with the algebraic point of view of the contractions. We study orbit closures of the variety of complex associative algebras of dimension three. The paper is organized as follows. Some notations on associative algebras, degeneration, rigidity and contractions of associative algebras are given in Section 2. In Section 3, we list some important invariance arguments for contractions. Calculation and collection of invariance arguments are adduced to conclude the possible existence of contractions for an arbitrary pair of associative algebras in Section 4.

Preliminaries

In this section, we recall some terminology that are used in the paper. Let $A=(V,\lambda)$ be an algebra of dimension n with an underlying vector space V over a field $\mathbb{K}$ and product $\lambda:V\times V\to V$. Let $g:(0,1]\to GL(V)$ be a continuous function. More precisely for any $t\in(0,1]$, a nonsingular linear operator g_t on V is assigned. A parameterized family of new isomorphic to $A=(V,\lambda)$ algebra structures on V is determined as follows:

$$\lambda_t(x,y)=g_t^{-1}\lambda(g_t(x),g_t(y)),\quad x,y\in V$$

Definition 2.1

If for any $x,y\in V$, the limit $\lim\limits_{t\to+0}\lambda_t(x,y)=\lambda_0(x,y)$ exists then algebraic structure λ_0 is called a contraction of λ.

A contraction B of A to algebra B is called trivial if B is abelian and improper if B is isomorphic to A. Consider an n^3-dimensional vector space $\mathrm{Hom}(V\otimes V,V)$ formed by bilinear maps $V\times V\to V$, where V is an n-dimensional vector space over an algebraically closed field $\mathbb{K}$ (char$\mathbb{K}$=0) denoted by $Alg_n(\mathbb{K})$. An algebra $A=(V,\lambda)$ is given as an element $\lambda(A)$ of $Alg_n(\mathbb{K})$ through the linear mapping $\lambda:V\otimes V\to V$. The linear reductive group $GL_n(\mathbb{K})$ acts on $Alg_n(\mathbb{K})$ by

$$(g*\lambda)(x,y)=g(\lambda(g^{-1}(x),g^{-1}(y))).$$

Under this action, two algebras A and B belong to the same orbit if and only if they are isomorphic. Moreover, we say that algebra A degenerates to algebra B, if B lies in Zariski closure of the orbit of A. This is denoted by $A\xrightarrow[deg]{}B$.

Definition 2.2

Let A be an algebra over a field $\mathbb{K}$. We call A an associative algebra if its bilinear mapping λ satisfies the following condition

$$\lambda(\lambda(x,y),z)=\lambda(x,\lambda(y,z)),\quad x,y,z\in V.$$

Let $A_n(\mathbb{K})$ be the set of all associative algebra structures on n-dimensional space over a field $\mathbb{K}$. The set $A_n(\mathbb{K})$ is an algebraic subset of the affine variety $Alg_n(\mathbb{K})$. For a fixed basis $\{e_1,e_2,\ldots,e_n\}$ of the vector space V, the multiplication table of A on this basis is given as a point $(\gamma_{ij}^k)\in\mathbb{K}^{n^3}$ as follows

$$\lambda(e_i,e_j)=\sum_{k=1}^{n}\gamma_{ij}^k e_k,\quad i,j=1,\cdots,n.$$

Let A be an associative $\mathbb{K}$-algebra, D be an A-bimodule and $\Phi:A^p\to D$ be a multilinear mapping. The set of all multilinear mappings from A^p to D is called p-dimensional cochain of A and denoted by $C^p(A,D)$. The coboundary homomorphism is a mapping $\delta^{(p)}$ from $C^p(A,D)$ to $C^{p+1}(A,D)$ given by

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$$(\delta^{(p)}\Phi)(x_1,x_2,\cdots,x_{p+1}) = x_1\Phi(x_2,\cdots,x_{p+1}) + \sum_{i=1}^{p}(-1)^i$$

$$\Phi(x_1,\cdots,x_ix_{i+1},\cdots,x_{p+1}) + (-1)^{p+1}\Phi(x_1,\cdots,x_p)x_{p+1}.$$

The kernel of the coboundary operator is denoted by $Z^p(A,D)$ whose elements are called p-cocycles with values in D. The elements of the image of $\delta^{(p-1)}$ denoted by $B^p(A,D)$ are called p-coboundaries with values in D. The quotient space:

$$H^p(A,D) = Z^p(A,D) / B^p(A,D)$$

is called the cohomology space (group) of A of degree p. In this paper, we will consider a particular case that $D=A$ as A-bimodule.

Definition 2.3

An associative algebra A is called geometrically rigid whenever its orbit is Zariski open in $A_n(\mathbb{K})$ and called algebraically rigid if the second cohomology group $H^2(A,A)$ is trivial [7].

Invariance Arguments

In this section, we list some invariance arguments which are helpful for studying the variety of a given class of algebras. Let A be an associative algebras over a field $\mathbb{K}$. We define:

- $A^k = \lambda(A^{k-1}, A)$ - the k-th degree of A, where $k\in\mathbb{N}$;
- $R(A) = \{x \in A \mid \lambda(A,x) = 0\}$ - the right annihilator of A;
- $L(A) = \{x \in A \mid \lambda(x,A) = 0\}$ - the left annihilator of A;
- $Z(A) = \{x \in A \mid \lambda(x,y) = \lambda(y,x), \forall y \in A\}$ - the center of A;
- $Aut(A) = \{d : A \to A \mid d(\lambda(x,y)) = \lambda(d(x),d(y)), \forall x,y \in A\}$ - the group of automorphisms of A;
- $SA(A)$ - the maximal abelian subalgebra of A;
- $Com(A)$ - the maximal commutative subalgebra of A;
- $O(A)$ - the orbit of A;
- $Der(A) = \{d \in End(A) \mid d(\lambda(x,y)) = \lambda(d(x),y) + \lambda(x,d(y)), \forall x,y \in A\}$ - the algebra of derivations of A;
- $r_n(A)$ -the nilpotency rank of an associative algebra A;
- $H^i(A,A)$ - the i^{th} cohomology group of A;
- $Z^2(A,A)$ - the 2-cocycles of an associative algebra A;
- $Der_{(\alpha,\beta,\gamma)}(A) = \{d \in End(A) \mid \alpha d(\lambda(x,y)) = \beta\lambda(d(x),y) + \gamma\lambda(x,d(y), \forall x,y \in A\}$ is the space of (α,β,γ)-derivations of A, for fixed $\alpha,\beta,\gamma\in\mathbb{K}$.

The following theorem is very useful to study the irreducible components of the subvariety $A_n(\mathbb{K})$ of $Alg_n(\mathbb{K})$ [8,9].

Theorem 3.1

The the following subsets of $A_n(\mathbb{K})$ are closed relative to the Zariski topology for any $r,s\in\mathbb{N}$:*0.60tw0.60tw*

- (1) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} A^r \le s\}$;
- (3) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} L(A) \ge r\}$;
- (5) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} \mathrm{Com}(A) \ge r\}$;
- (7) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} \mathrm{Aut}(A) > r\}$;
- (9) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} \mathrm{Der}(A) > r\}$;
- (11) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} H(A,A) \ge r\}$;
- (13) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} \mathrm{Der}_{(\alpha,\beta,\gamma)}(A) \ge r\}$.
- (2) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} R(A) \ge r\}$;
- (4) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} Z(A) \ge r\}$;
- (6) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} \mathrm{SA}(A) \ge r\}$;
- (8) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} O(A) < r\}$;
- (10) $\{A \in A_n(\mathbb{K}) \mid r_n(A) \le r\}$;
- (12) $\{A \in A_n(\mathbb{K}) \mid \dim_{\mathbb{K}} Z^2(A,A) \ge r\}$;

The proof of 1-4 is the same to that of Lie algebras [10,11]. For the parts 5 and 6, the proof is obtained by the following significant fact: let N be a Zariski closed subset of $A_n(\mathbb{K})$ and A_1,A_2 in $A_n(\mathbb{K})$. If A_1 lies in N and $A_1 \to A_2$ then A_2 also lies in A. More precisely, the subset N is not $GL_n(\mathbb{K})$-setwise stabilizer. However, it is B-setwise stabilizer, where B is the Borel subgroup $GL_n(\mathbb{K})$- made up of upper triangular matrices. The statements 7,8 and 9 are equivalent because of the following relation between the dimensions of $GL_n(\mathbb{K})$-- orbits, automorphism's groups and derivation algebras.

$$\dim O(A) = n^2 - \dim \mathrm{Aut}(A) = n^2 - \dim \mathrm{Der}(A)$$

The proof of 10 is directly coming behind the following fact: let A be an associative algebra. We define the lower central series:

$$A^1 \supset A^2 \supset \cdots \supset A^{k+1} \supset \cdots$$

If A is a nilpotent associative algebra then it has nilpotency rank denoted by $r_n(A)$, i.e., it is a minimal positive integer l such that $A^l=0$. It is not hard to see that if A_1 degenerates to A_2 then $\dim\ A_1^{k+1} \ge \dim A_2^{k+1}$. The proof of 11 and 12 are the same of Lie algebras [12].

The next corollary is used to reject existence of degenerations for each pair of associative algebras A and B.

Corollary 3.1

If an algebra A degenerates to an algebra B. Then the following conditions are valid:

- (1) $\dim_{\mathbb{K}} A^r \ge \dim_{\mathbb{K}} B^r$ for some r,
- (3) $\dim_{\mathbb{K}} L(A) \le \dim_{\mathbb{K}} L(B)$,
- (5) $\dim_{\mathbb{K}} \mathrm{Com}(A) \le \dim_{\mathbb{K}} \mathrm{Com}(B)$,
- (7) $\dim_{\mathbb{K}} \mathrm{Aut}(A) < \dim_{\mathbb{K}} \mathrm{Aut}(B)$,
- (11) $\dim_{\mathbb{K}} \mathrm{H}^i(A,A) \le \dim_{\mathbb{K}} \mathrm{H}^i(B,B)$,
- (11) $\dim_{\mathbb{K}} \mathrm{H}^i(A,A) \le \dim_{\mathbb{K}} \mathrm{H}^i(B,B)$,
- (13) $\dim_{\mathbb{K}} \mathrm{Der}_{(\alpha,\beta,\gamma)}(A) \le \dim_{\mathbb{K}} \mathrm{Der}_{(\alpha,\beta,\gamma)}(B)$.
- (2) $\dim_{\mathbb{K}} R(A) \le \dim_{\mathbb{K}} R(B)$,
- (4) $\dim_{\mathbb{K}} Z(A) \le \dim_{\mathbb{K}} Z(B)$,
- (6) $\dim_{\mathbb{K}} \mathrm{SA}(A) \le \dim_{\mathbb{K}} \mathrm{SA}(B)$,
- (8) $\dim_{\mathbb{K}} O(A) > \dim_{\mathbb{K}} O(B)$,
- (10) $r_n(A) \ge r_n(B)$,
- (12) $\dim_{\mathbb{K}} Z^2(A,A) \le \dim_{\mathbb{K}} Z^2(B,B)$.

In the sequel, all algebras suppose to be over the field of complex numbers $\mathbb{C}$.

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The Variety of Complex Associative Algebras of Dimension Three

In this section, we recall the complete list of non-isomorphic classes of three-dimensional complex associative algebras, which was obtained in Rikhsiboev et al. [13] to study the subvariety $A_3(\mathbb{C})$ of $Alg_3(\mathbb{C})$.

Theorem 4.1

Any 3-dimensional complex associative algebra A is isomorphic to one of the following pairwise non-isomorphic algebras.

$As_3^1: \quad e_1e_3 = e_2, \quad e_3e_1 = e_2;$

$As_3^2(\alpha): \quad e_1e_3 = e_2, \quad e_3e_1 = \alpha e_2, \quad \alpha \in \mathbb{C} \setminus \{1\};$

$As_3^3: \quad e_1e_1 = e_2, \quad e_1e_2 = e_3, \quad e_2e_1 = e_3;$

$As_3^4: \quad e_1e_3 = e_2, \quad e_2e_3 = e_2, \quad e_3e_3 = e_3;$

$As_3^5: \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_3 = e_3;$

$As_3^6: \quad e_3e_1 = e_2, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^7: \quad e_1e_2 = e_1, \quad e_2e_2 = e_2, \quad e_3e_1 = e_1, \quad e_3e_3 = e_3;$

$As_3^8: \quad e_1e_3 = e_1, \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_3 = e_3;$

$As_3^9: \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{10}: \quad e_1e_3 = e_1, \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{11}: \quad e_1e_3 = e_2, \quad e_2e_3 = e_2, \quad e_3e_1 = e_2, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{12}: \quad e_1e_1 = e_2, \quad e_1e_3 = e_1, \quad e_2e_3 = e_2, \quad e_3e_1 = e_1, \quad e_3e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{13}: \quad e_1e_1 = e_1, \quad e_2e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{14}: \quad e_1e_2 = e_1, \quad e_2e_1 = e_1, \quad e_2e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{15}: \quad e_1e_2 = e_1, \quad e_2e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{16}: \quad e_2e_1 = e_1, \quad e_2e_2 = e_2, \quad e_3e_3 = e_3;$

$As_3^{17}: \quad e_1e_1 = e_2, \quad e_3e_3 = e_3;$

$As_3^{18}:$ *Abelian.*

In Table 1, we assemble some contraction invariants of three-dimensional associative algebras that were obtained in refs. [14,15].

In Table 2 below we present the values of more contraction invariants.

Table 3 contains the values of contraction invariants applying the algorithm, which was stated in ref. [16] to describe the (α,β,γ)-derivations of three- dimensional complex associative algebras.

By using all the criteria presented above, we give in the following table all possibilities of degenerations for 3-dimensional associative algebras. The checkmark denotes that there is a degeneration $A \xrightarrow[deg]{} B$. The other symbols stand for the reason why such a degeneration is impossible. Indeed, there is more than just one reason for a non-degeneration. However, we have written down only one in the table (Table 4).

According to Table 4, the algebras $As_3^2, As_3^5, As_3^7, As_3^{12}, As_3^{13}, As_3^{15}$ and $, As_3^{16}$ and are geometrically rigid. More precisely, they are not degeneration of other 3-dimensional associative algebras structures.

IC	$\dim_{\mathbb{K}}$ (R(A))	$\dim_{\mathbb{K}}$ (L(A))	$\dim_{\mathbb{K}}$ (Aut(A))	$\dim_{\mathbb{K}}$ (Com(A))	$\dim_{\mathbb{K}}$ (Der(A))
As_3^1	1	1	4	3	4
As_3^2	1	1	4	2	4
As_3^3	1	1	3	3	3
As_3^4	2	2	3	2	3
As_3^5	1	1	4	2	4
As_3^6	2	2	3	2	3
As_3^7	0	0	2	2	2
As_3^8	1	0	3	2	3
As_3^9	0	1	3	2	3
As_3^{10}	0	0	4	3	4
As_3^{11}	2	2	2	3	2
As_3^{12}	0	0	2	3	2
As_3^{13}	0	0	0	3	0
As_3^{14}	0	0	1	3	1
As_3^{15}	1	0	2	2	2
As_3^{16}	0	1	2	2	2
As_3^{17}	1	1	2	3	2
As_3^{18}	3	3	9	3	9

Table 1: The collected contraction invariants of three-dimensional associative algebras.

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IC	$\dim_{\mathbb{K}}((A^2))$	$\dim_{\mathbb{K}}$ (Z(A))	$\dim_{\mathbb{K}}$ (SA(A))	$\dim_{\mathbb{K}}$ (O(A))	$r_3(A)$	$\dim_{\mathbb{K}}(H^1(A,A))$	$\dim_{\mathbb{K}}(H^2(A,A))$	$\dim_{\mathbb{K}}(Z^2(A,A))$
As_3^1	1	3	2	5	3	4	4	9
As_3^2	1	1	2	5	3	2	3	8
As_3^3	2	3	2	6	4	3	3	9
As_3^4	2	2	2	6	-	1	2	8
As_3^5	3	0	2	5	-	1	0	5
As_3^6	2	2	2	6	-	1	2	8
As_3^7	3	2	1	7	-	0	0	7
As_3^8	3	1	2	6	-	1	1	7
As_3^9	3	1	2	6	-	1	1	7
As_3^{10}	3	3	2	5	-	4	6	11
As_3^{11}	2	3	2	7	-	2	2	9
As_3^{12}	3	3	1	7	-	2	0	7
As_3^{13}	3	3	0	9	-	0	0	9
As_3^{14}	3	3	1	8	-	1	1	9
As_3^{15}	3	1	1	7	-	0	0	7
As_3^{16}	3	1	1	7	-	0	0	7
As_3^{17}	2	3	1	7	-	2	2	9
As_3^{18}	0	3	3	0	2	9	27	27

Table 2: The calculated contraction invariants of three-dimensional associative algebras.

IC	$\dim_{\mathbb{K}}(Der_{(1,1,0)}(A))$	$\dim_{\mathbb{K}}(Der_{(1,0,1)}(A))$	$\dim_{\mathbb{K}}(Der_{(1,0,0)}(A))$	$\dim_{\mathbb{K}}(Der_{(0,1,1)}(A))$	$\dim_{\mathbb{K}}(Der_{(0,0,1)}(A))$	$\dim_{\mathbb{K}}(Der_{(0,1,0)}(A))$
As_3^1	3	3	6	4	3	3
As_3^2	3	3	6	4	3	3
As_3^3	3	3	3	3	3	3
As_3^4	5	3	3	3	6	3
As_3^5	3	3	0	0	3	3
As_3^6	3	5	3	3	3	6
As_3^7	3	3	0	0	0	0
As_3^8	5	2	0	0	3	0
As_3^9	2	5	0	0	0	3
As_3^{10}	3	3	0	0	0	0
As_3^{11}	3	3	3	3	3	3
As_3^{12}	3	3	0	0	0	0
As_3^{13}	3	3	0	0	0	0
As_3^{14}	3	3	0	0	0	0
As_3^{15}	5	2	0	0	3	0
As_3^{16}	2	5	0	0	0	3
As_3^{17}	3	3	3	3	3	3
As_3^{18}	9	9	9	9	9	9

Table 3: Contraction invariants of three-dimensional associative algebras.

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$\overline{deg}$	As_3^1	As_3^2	As_3^3	As_3^4	As_3^5	As_3^6
As_3^1	-	$Com(A)$	$r_3(A)$	$Aut(A)$	$Z(A)$	$H^1(A,A)$
As_3^2	$Aut(A)$	-	$r_3(A)$	$H^2(A,A)$	$Z(A)$	$H^1(A,A)$
As_3^3	$\surd$	$H^2(A,A)$	-	$H^2(A,A)$	$Z^2(A,A)$	$Com(A)$
As_3^4	$R(A)$	$L(A)$	$R(A)$	-	$H^2(A,A)$	$Der_{(1,1,0)}(A)$
As_3^5	$O(A)$	$O(A)$	$Aut(A)$	$Aut(A)$	-	$Der(A)$
As_3^6	$R(A)$	$Z(A)$	$R(A)$	$Der_{(1,0,1)}(A)$	$d(A^2)$	-
As_3^7	$\surd$	$z(A)$	$\surd$	$\surd$	$Z(A)$	$\surd$
As_3^8	$Der_{(1,1,0)}(A)$	$Der_{(1,1,0)}(A)$	$Aut(A)$	$Aut(A)$	$Z(A)$	$Der(A)$
As_3^9	$Der_{(1,0,1)}(A)$	$Der_{(1,0,1)}(A)$	$Aut(A)$	$Der(A)$	$H^2(A,A)$	$Aut(A)$
As_3^{10}	$H^2(A,A)$	$Aut(A)$	$Aut(A)$	$Der(A)$	$Com(A)$	$Com(A)$
As_3^{11}	$R(A)$	$L(A)$	$R(A)$	$Com(A)$	$Z(A)$	$Com(A)$
As_3^{12}	$\surd$	$Com(A)$	$\surd$	$H^1(A,A)$	$Com(A)$	$Z(A)$
As_3^{13}	$\surd$	$Com(A)$	$\surd$	$Z(A)$	$Com(A)$	$Com(A)$
As_3^{14}	$\surd$	$Z(A)$	$\surd$	$Com(A)$	$Z(A)$	$Com(A)$
As_3^{15}	$Der_{(1,1,0)}(A)$	$Der_{(1,1,0)}(A)$	$Der_{(1,1,0)}(A)$	$\surd$	$Z(A)$	$Der_{(1,1,0)}(A)$
As_3^{16}	$Der_{(1,0,1)}(A)$	$Der_{(1,0,1)}(A)$	$Der_{(1,0,1)}(A)$	$Der_{(1,0,1)}(A)$	$Z(A)$	$\surd$
As_3^{17}	$\surd$	$Com(A)$	$\surd$	$H^1(A,A)$	$H^2(A,A)$	$H^1(A,A)$
As_3^{18}	$R(A)$	$L(A)$	$Aut(A)$	$Com(A)$	$SA(A)$	$d(A^2)$
$\overline{deg}$	As_3^7	As_3^8	As_3^9	As_3^{10}	As_3^{11}	As_3^{12}
As_3^1	$H^2(A,A)$	$O(A)$	$d^2(A)$	$R(A)$	$Der(A)$	$L(A)$
As_3^2	$R(A)$	$L(A)$	$H^2(A,A)$	$d^2(A)$	$Aut(A)$	$R(A)$
As_3^3	$Z(A)$	$Com(A)$	$R(A)$	$L(A)$	$O(A)$	$SA(A)$
As_3^4	$d^2(A)$	$Z(A)$	$H^2(A,A)$	$L(A)$	$O(A)$	$Aut(A)$
As_3^5	$R(A)$	$L(A)$	$R(A)$	$R(A)$	$Aut(A)$	$SA(A)$
As_3^6	$H^2(A,A)$	$L(A)$	$R(A)$	$L(A)$	$O(A)$	$SA(A)$
As_3^7	-	$Z(A)$	$Z(A)$	$\surd$	$Aut(A)$	$O(A)$
As_3^8	$SA(A)$	-	$R(A)$	$R(A)$	$O(A)$	$H^2(A,A)$
As	$L(A)$	$L(A)$	-	$L(A)$	$Der(A)$	$Der(A)$
As_3^{10}	$SA(A)$	$H^1(A,A)$	$O(A)$	-	$H^2(A,A)$	$Aut(A)$
As_3^{11}	$H^1(A,A)$	$H^2(A,A)$	$H^1(A,A)$	$d^2(A)$	-	$H^2(A,A)$
As_3^{12}	$H^1(A,A)$	$H^1(A,A)$	$Com(A)$	$\surd$	$Aut(A)$	-
As_3^{13}	$Com(A)$	$Z(A)$	$Der_{(1,1,0)}(A)$	$\surd$	$\surd$	$Z^2(A,A)$
As_3^{14}	$H\ (A,A)$	$Com(A)$	$Com(A)$	$\surd$	$\surd$	$H^2(A,A)$
As_3^{15}	$R(A)$	$\surd$	$Der_{(1,1,0)}(A)$	$R(A)$	$Aut(A)$	$R(A)$
As_3^{16}	$L(A)$	$L(A)$	$\surd$	$L(A)$	$Der(A)$	$L(A)$
As_3^{17}	$R(A)$	$H^2(A,A)$	$R(A)$	$L(A)$	$Aut(A)$	$Z^2(A,A)$
As_3^{18}	$O(A)$	$Z(A)$	$Der(A)$	$Z^2(A,A)$	$H^1(A,A)$	$H^2(A,A)$
$\overline{deg}$	As_3^{13}	As^{14}	As_3^{15}	As_3^{16}	As_3^{17}	As_3^{18}

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As_3^1	$H^1(A,A)$	$SA(A)$	$H^2(A,A)$	$O(A)$	$SA(A)$	√
As_3^2	$L(A)$	$Der_{(1,0,0)}(A)$	$Der_{(0,1,1)}(A)$	$SA(A)$	$H^2(A,A)$	√
As	$Aut(A)$	$Der(A)$	$Der(A)$	$Com(A)$	$H^2(A,A)$	√
As_3^4	$Der^2(A)$	$SA(A)$	$Z(A)$	$H^1(A,A)$	$O(A)$	√
As_3^5	$H^1(A,A)$	$SA(A)$	$O(A)$	$H^1(A,A)$	$Der(A)$	√
As_3^6	$Der(A)$	$Der(A)$	$H^2(A,A)$	$Z(A)$	$O(A)$	√
As_3^7	$SA(A)$	$O(A)$	$Z(A)$	$Z(A)$	$Aut(A)$	√
As_3^8	$O(A)$	$SA(A)$	$H^1(A,A)$	$H^2(A,A)$	$Aut(A)$	√
As_3^9	$Aut(A)$	$L(A)$	$H^1(A,A)$	$SA(A)$	$O(A)$	√
As_3^{10}	$O(A)$	$H^2(A,A)$	$H^2(A,A)$	$Z(A)$	$O(A)$	√
As_3^{11}	$Der(A)$	$Aut(A)$	$Z(A)$	$L(A)$	$R(A)$	√
As_3^{12}	$SA(A)$	$Aut(A)$	$Z(A)$	$Com(A)$	$Aut(A)$	√
As_3^{13}	-	√	$Der_{(1,0,1)}(A)$	$Z(A)$	√	√
As_3^{14}	$SA(A)$	-	$H^2(A,A)$	$H^1(A,A)$	√	√
As_3^{15}	$SA(A)$	$O(A)$	-	$R(A)$	$Aut(A)$	√
As_3^{16}	$SA(A)$	$Aut(A)$	$L(A)$	-	$Aut(A)$	√
As_3^{17}	$Aut(A)$	$Aut(A)$	$Z(A)$	$R(A)$	-	√
As_3^{18}	$O(A)$	$L(A)$	$Der(A)$	$Z^2(A,A)$	$H^1(A,A)$	$R(A)$

Table 4: All possibilities of degenerations of three-dimensional associative algebras.

Theorem 4.2

The rigid irreducible components of the variety $A_3(\mathbb{C})$ are generated by the algebras $As_3^2, As_3^5, As_3^7, As_3^{12}, As_3^{13}, As_3^{15}$ and As_3^{16} and with the dimensions:

$$C_1 = \overline{Orb(As_3^2)},\ \ dim_{\mathbb{C}} C_1 = 5,$$

$$C_2 = \overline{Orb(As_3^5)},\ \ dim_{\mathbb{C}} C_1 = 5,$$

$$C_3 = \overline{Orb(As_3^7)},\ \ dim_{\mathbb{C}} C_2 = 7,$$

$$C_4 = \overline{Orb(As_3^{12})},\ \ dim_{\mathbb{C}} C_3 = 7,$$

$$C_5 = \overline{Orb(As_3^{13})},\ \ dim_{\mathbb{C}} C_4 = 9,$$

$$C_6 = \overline{Orb(As_3^{15})},\ \ dim_{\mathbb{C}} C_5 = 7,$$

$$C_7 = \overline{Orb(As_3^{16})},\ \ dim_{\mathbb{C}} C_6 = 7,$$

As a consequence, $\dim_{\mathbb{C}} A_3(\mathbb{C}) = 9$.

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