

On Duality of Multiobjective Rough Convex Programming Problems

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Abstract

Duality assertions are very important in optimization from the theoretical as well as from the numerical point of view. So this paper presents duality of multiobjective rough convex programming problems in rough environment when the multiobjective function is deterministic and roughness is in feasible region. Also it discussed the duality when roughness in multiobjective function and the feasible region is deterministic. The concepts and some theorems of duality in the rough environment are discussed. Also, the procedure of solution of these kind of problems described.

Keywords: Rough set; Rough function; Multiobjective rough convex programming problem; Duality

Introduction

The mathematical programming in rough environment was introduced in [1-4]. The multiobjective rough convex programming problem (MRCPP) can be classified according to existence of roughness in multiobjective function or constraints. The mathematical problems can be classified into three classes [3]. First class: MRCPP with rough feasible set and deterministic multiobjective function. Second class: problems with deterministic feasible set and rough multiobjective function. Third Class: problems with rough feasible set and rough multiobjective function. There are two solution sets of the MRCPP; surely Pareto optimal solution set and possibly Pareto optimal solution set. The aim of this paper is to present the concepts of duality of MRCPP when roughness is in the multiobjective function and the feasible region is deterministic. Also it discussed the duality when roughness is in feasible region and multiobjective is deterministic.

This paper consists of four sections. Section 2 introduces the mathematical formulation of MRCPP and presents basic concepts of rough set theory. Section 3 presents duality theorems of multiobjective convex programming problem in rough environment when the multiobjective function is rough functions and the feasible region is deterministic. Also it discussed the duality when roughness is in feasible region and multiobjective is deterministic. Section 4 includes the conclusions.

Basic Concepts

RST is a new mathematical theory introduced by Pawlak in the early 1980s to deal with vagueness or uncertainty, [5-9]. It is a very rich area for research [10-12]. RST expresses vagueness by employing a boundary region of a set and not by means of membership function, [1,3,4,7]. The basic concept of RST is the approximation of indefinable sets via definable sets or ordinary sets, [1,5,7,10]. The rough function is a new concept based on the RST [3,4,7].

Definition (Convex Rough Set) [4]:

A rough set is convex if its lower and upper approximation sets are convex.

A rough function f is a function without explicit formula but bounded by several explicit functions because it is not known precisely. It is given as result of modelling the problem due to imperfect data. For simplification of our study the rough function will be defined by two functions.

Definition (Rough Function):

A function $f(x)$ is a rough function if it is not known precisely but it is only known that the function values of $f(x)$ are bounded by two deterministic functions $f_-(x)$, $f_+(x)$ such that $f_-(x) \leq f(x) \leq f_+(x)$ at any $x \in X$ where $f_-(x)$, $f_+(x)$ are lower and upper approximation function respectively. The surely values of $f(x)$ are the values of $f_-(x)$, $f_+(x)$ at $x \in X$ such that $f_-(x) = f_+(x)$.

Definition (Convex Rough Function):

A rough function $f(x)$ is called convex rough function on the convex set X if its lower and upper approximation functions are convex.

The multiobjective convex programming problem (MCCPP) is defined as follows [13]:

$$\min(f_1(x), f_2(x), \dots, f_k(x)) \quad (1)$$

subject to

$$X = \{x \in R^n \mid g_r(x) \leq 0, r = 1, 2, \dots, m\}$$

where the functions $f_i: R^n \rightarrow R$ for $i=1, \dots, k$ and $g_r: R^n \rightarrow R$ for $r=1, \dots, m$ are assumed to be differentiable and convex.

If problem (1) is transformed into problem (2) by using the weighted method [13].

$$\min \sum_{i=1}^k w_i f_i(x) \quad (2)$$

subject to

$$X = \{x \in R^n \mid g_r(x) \leq 0, r = 1, 2, \dots, m\}$$

$$w_i \geq 0, \sum_{i=1}^k w_i = 1$$

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Received September 21, 2015; **Accepted** October 17, 2015; **Published** October 23, 2015

Citation: Elsisy MA, Osman MSA, Eid MH (2015) On Duality of Multiobjective Rough Convex Programming Problems. J Appl Computat Math 4: 263. doi:10.4172/2168-9679.1000263

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The whole Pareto optimal solution set is gotten if convexity assumption is presented but it fail to get the whole Pareto optimal solution set in of non-convex problems.

Theorem [13]:

Assume convexity if $x^* \in X^*$ (X^* is Pareto optimal solution set of (1)) then there exists $w \in W$ such that x^* solve problem (2).

Theorem [13]:

x^* a Pareto optimal solution of MCPP if there exists $w \in W$ such that x^* solves problem (2) and if either one of the following two conditions holds:

1. $W_i > 0$ for all $i=1, 2, \dots, k$.
2. x^* is a unique solution of problem (1).

From above relations, the stability set of the first kind of problem (1) according to w can be defined as

$$S(\bar{x}) = \{w \in W \mid \bar{x} \text{ is a Pareto optimal solution to problem (1)}\}.$$

The duality of multiobjective optimization problem has been studied by many authors see refs [14-16]. But the cone optimization is used to get the duality in many studies. Only some researchers study the duality in general form [16]. In this paper, a simple method is presented to get the duality.

The first dual form is used when $\bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1$ which it defined as follows:

$$\max \left(f_1(x) + \sum_{r=1}^m \mu_r g_r(x), f_2(x) + \sum_{r=1}^m \mu_r g_r(x), \dots, f_k(x) + \sum_{r=1}^m \mu_r g_r(x) \right) \quad (3)$$

subject to

$$\sum_{i=1}^k \bar{w}_i \nabla f_i(x) + \sum_{r=1}^m \mu_r \nabla g_r(x) = 0$$

$$\bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \mu_r \geq 0, r = 1, 2, \dots, m, \bar{w} \in S(\bar{x})$$

Theorem:

If there is a Pareto optimal solution \bar{x} such that $\bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \bar{w} \in S(\bar{x})$ where $S(\bar{x})$ the stability set of the first kind, then there is $\bar{\mu}_r \geq 0$ Pareto optimal solution to problem (3).

Proof:

It can be proved as the same steps as theorem 4.6 [13].

If problem (1) is transformed into problem (2) by using the weighted method. Then there are two dual form of problem (3), firstly by using Lagrange method. The theorems and concepts of convex programming problem are valid if MRCPP is transformed to convex programming problem [17] under the suitable conditions. The Lagrange function can be defined as follows:

$$\max \theta(\mu)$$

subject to $\mu \geq 0$ (4)

$$\text{where } \theta(\mu) = \inf \left\{ \sum_{i=1}^k w_i \nabla f_i(x) + \sum_{r=1}^m \mu_r \nabla g_r(x) = 0, \mu_r \geq 0 \mid x \in X \right\}$$

Secondly: The other dual form of problem (3) can be defined by differentiable properties when its Pareto optimal solution x^* is a unique optimal solution as follow:

$$\max \sum_{i=1}^k w_i f_i(x) + \sum_{r=1}^m \mu_r g_r(x) \quad (5)$$

subject to

$$\sum_{i=1}^k w_i \nabla f_i(x) + \sum_{r=1}^m \mu_r \nabla g_r(x) = 0, \mu_r \geq 0, r = 1, 2, \dots, m$$

Theorem:

If \bar{x} is a unique optimal solution of problem (2) then there is $\bar{\mu}_r \geq 0$ optimal solution to problem (5).

Proof:

It can be proved as the same steps as theorem 4.6 [13].

Duality of MRCPP

The duality of MRCPP for the two classes can be obtained by using any type of duality forms in the previous section depending on the properties of MRCPP.

Dual problem of first class of MRCPP:

As known the first class of MRCPP can be defined as:

$$\min F(x) = (f_1(x), f_2(x), \dots, f_k(x))^t$$

subject to $x \in X$ (6)

where $F: R^n \rightarrow R^k$ is a vector of convex deterministic functions, X is the feasible region which is a convex rough set such that $X_* \subseteq X \subseteq X^*$,

$$X^* = \{x \in R^n \mid G_i(x) \leq 0, i = 1, 2, \dots, A\}, X_* = \{x \in X^* \mid g_i(x) \leq 0, i = 1, 2, \dots, a\}.$$

The problem (6) can be solved in the following manner:

Firstly: Solve the following problem by using any suitable method:

$$\text{Min } F(x)$$

subject to $x \in X^*$

If \dot{C} is the Pareto optimal solution set and $\dot{F}_* = F(\dot{C})$ is the Pareto optimal set.

$$\text{Find the set } \dot{C}_1 = \dot{C} \cap X_*.$$

Definition (The surely Pareto optimal solution set):

If $\dot{C}_1 \neq \emptyset$, then \dot{C}_1 is called surely Pareto optimal solution set of problem (6) which contains all surely Pareto optimal solutions.

If $\dot{C}_1 = \dot{C}; \dot{C} \subseteq X_*$; then problem (6) has only surely Pareto optimal solutions.

Secondly: If $\dot{C}_1 = \emptyset, \dot{C} \subseteq X_{BN}$, solve the following problem:

$$\text{Min } F(x)$$

subject to $x \in X_*$

If \dot{C}_2 is the Pareto optimal solution set of the above problem and $\dot{F}^* = \{F(x) \mid x \in \dot{C}_2\}$ is the Pareto

optimal set of multiobjective functions.

Definition (The possibly Pareto optimal solution set):

If $\dot{C}_1 = \emptyset$, the set $\dot{C} \cup \dot{C}_2$ is called the possibly Pareto optimal solution set of problem (6) which contains all possibly Pareto optimal solutions.

The Pareto optimal set of the multiobjective functions $F(x)$ is between Pareto optimal set \bar{F}^* of $F(x)$ on the upper approximation set and Pareto optimal set \bar{F}^* of $F(x)$ on the lower approximation set, $\bar{F}_* \subseteq \min_{x \in X} F(x) \subseteq \bar{F}^*$.

The dual problem of the first class of MRCPP is formulated as problem (3). So the Pareto optimal set of the dual problem of the first class is between

$$\bar{\theta}_* \subseteq \max \bar{\theta}(\mu) \subseteq \bar{\theta}^*$$

where

$$\bar{\theta}_* = \begin{cases} \max \left(f_1(x) + \sum_{r=1}^A \mu_r G_r(x), \dots, f_k(x) + \sum_{r=1}^A \mu_r G_r(x) \right) \\ \text{subject to } \sum_{i=1}^k \bar{w}_i \nabla f_i(x) + \sum_{r=1}^A \mu_r \nabla G_r(x) = 0 \\ \bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \mu_r \geq 0, r = 1, 2, \dots, m, \bar{w} \in S_*(\bar{x}) \cap S^*(\bar{x}) \end{cases}$$

$$\bar{\theta}^* = \begin{cases} \max \left(f_1(x) + \sum_{r=1}^a \mu_r g_r(x), \dots, f_k(x) + \sum_{r=1}^a \mu_r g_r(x) \right) \\ \text{subject to } \sum_{i=1}^k \bar{w}_i \nabla f_i(x) + \sum_{r=1}^a \mu_r \nabla g_r(x) = 0 \\ \bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \mu_r \geq 0, r = 1, 2, \dots, m, \bar{w} \in S_*(\bar{x}) \cap S^*(\bar{x}) \end{cases}$$

Where \bar{X} and X are Pareto optimal solutions of primal problem on lower approximation set and on upper approximation set, respectively. $S_*(\bar{x})$ and $S^*(\bar{x})$ are stability sets of first kind problem on lower approximation set and on upper approximation set, respectively.

The dual problem can be solved in the following manner:

Firstly: Solve the following problem:

$$\max \left(f_1(x) + \sum_{r=1}^A \mu_r G_r(x), \dots, f_k(x) + \sum_{r=1}^A \mu_r G_r(x) \right)$$

$$\text{subject to } \sum_{i=1}^k \bar{w}_i \nabla f_i(x) + \sum_{r=1}^A \mu_r \nabla G_r(x) = 0$$

$$\bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \mu_r \geq 0, r = 1, 2, \dots, m, \bar{w} \in S_*(\bar{x}) \cap S^*(\bar{x})$$

If \bar{D} is Pareto optimal solution set and $\bar{\theta}_*$ is Pareto optimal set.

Find $\bar{D}_1 = \left\{ \mu \in \bar{D} \mid \bar{\theta}(\mu) \in \bar{\theta}^* \right\}$ which is a surely Pareto optimal solution

set if $\bar{D}_1 = \emptyset$.

Secondly: If $\bar{D}_1 = \emptyset$, solve the following problem

$$\max \left(f_1(x) + \sum_{r=1}^a \mu_r g_r(x), \dots, f_k(x) + \sum_{r=1}^a \mu_r g_r(x) \right)$$

$$\text{subject to } \sum_{i=1}^k \bar{w}_i \nabla f_i(x) + \sum_{r=1}^a \mu_r \nabla g_r(x) = 0$$

$$\bar{w}_i > 0, \sum_{i=1}^k \bar{w}_i = 1, \mu_r \geq 0, r = 1, 2, \dots, m, \bar{w} \in S_*(\bar{x}) \cap S^*(\bar{x})$$

If \bar{D}_2 is Pareto optimal solution set and $\bar{\theta}_*$ is Pareto optimal set.

The set $\bar{D} \cup \bar{D}_2$ is a possibly Pareto optimal solution set.

Dual problem of second class of MRCPP:

The second class of MRCPP is formulated as follows:

$$\text{Min } F(x)$$

$$\text{subject to } x \in X \quad (7)$$

where X is the feasible region such that $X = \{x \in R^n \mid g_i(x) \leq 0, i = 1, 2, \dots, m\}$, $F(x)$ is vector of functions which is vector rough convex functions such that $F_*(x) \leq F(x) \leq F^*(x)$, $F_*(x)$ and $F^*(x)$ are the lower and upper approximation deterministic vectors functions of $F(x)$ respectively,

$$F_*(x) = (f_{*1}(x), f_{*2}(x), \dots, f_{*k}(x))', F^*(x) = (f_1^*(x), f_2^*(x), \dots, f_k^*(x))'$$

The problem (7) can be solved in the following manner:

Firstly: Solve the following problem by using any suitable method:

$$\text{Min } F(x)$$

$$\text{subject to } x \in X$$

If \bar{C} is the Pareto optimal solution set and $\bar{F}_* = \{F_*(x) \mid x \in \bar{C}\}$ is the Pareto optimal set.

$$\text{Find } \bar{C}_1 = \left\{ \bar{x} \in \bar{C} \mid F^*(\bar{x}) \in \bar{F}_* \right\}.$$

Definition (The surely optimal solution set):

If $\bar{C}_1 \neq \emptyset$, then \bar{C}_1 is called the surely optimal solution set of the problem (7) which contains all surely optimal solutions.

If $\bar{C}_1 = \bar{C}$, the problem has only a surely Pareto optimal solution set.

Secondly: If $\bar{C}_1 = \emptyset$, solve the following problem

$$\min F^*(x)$$

$$\text{subject to } x \in X$$

If \bar{C}_2 is Pareto optimal solution set and $\bar{F}^* = \{F^*(x) \mid x \in \bar{C}_2\}$ is the Pareto optimal set.

Definition (The possibly Pareto optimal solution set):

A set $\bar{C} \cup \bar{C}_2, \bar{C}_1 = \emptyset$, is called the possibly Pareto optimal solution set of the problem (7) which contains possibly Pareto optimal solutions.

Then the Pareto optimal set of the objective function $F(x)$ is between Pareto optimal set of the lower approximation functions vector F_* and Pareto optimal set of the upper approximation functions vector F^* , $F_* \subseteq \min_{x \in X} F(x) \subseteq F^*$.

The dual problem of the second class of MRCPP is formulated as problem (5). So the optimal value of the dual problem of the second class is between

$$\bar{\theta}_* \leq \max \bar{\theta}(\mu) \leq \bar{\theta}^*$$

where

$$\bar{\theta}_* = \begin{cases} \max \sum_{i=1}^k \bar{w}_i f_i(\bar{x}) + \sum_{i=1}^k \bar{w}_i (f_i(\bar{x}) - f_i^*(\bar{x})) + \sum_{r=1}^m \mu_r g_r(\bar{x}) \\ \text{subject to } \sum_{i=1}^k \bar{w}_i \nabla f_i(\bar{x}) + \sum_{i=1}^k \bar{w}_i \nabla (f_i(\bar{x}) - f_i^*(\bar{x})) + \sum_{r=1}^m \mu_r \nabla g_r(\bar{x}) = 0 \\ \mu_r \geq 0, r = 1, 2, \dots, m \end{cases}$$

$$\theta^* = \begin{cases} \max \sum_{i=1}^k w_i f_i^*(\bar{x}) + \sum_{i=1}^k w_i (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r g_r(\bar{x}) \\ \text{subject to } \sum_{i=1}^k w_i \nabla f_i(\bar{x}) + \sum_{i=1}^k w_i \nabla (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r \nabla g_r(\bar{x}) = 0 \\ \mu_r \geq 0, r = 1, 2, \dots, m \end{cases}$$

The dual problem can be solved in the following manner [18,19]:

Firstly: Solve the following problem

$$\begin{aligned} & \max \sum_{i=1}^k w_i f_i^*(\bar{x}) + \sum_{i=1}^k w_i (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r g_r(\bar{x}) \\ & \text{subject to } \sum_{i=1}^k w_i \nabla f_i(\bar{x}) + \sum_{i=1}^k w_i \nabla (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r \nabla g_r(\bar{x}) = 0 \\ & \mu_r \geq 0, r = 1, 2, \dots, m \end{aligned}$$

If \bar{D} is the optimal solution set and θ^* is the optimal value.

Find the set $D_1 = \left\{ \mu \in \bar{D} \mid \theta^*(\mu) = \theta^* \right\}$ which is a surely optimal solution set if $D_1 \neq \emptyset$.

Secondly: If $\bar{D}_1 = \emptyset$, solve the following problem

$$\begin{aligned} & \text{Max } \sum_{i=1}^k w_i f_{s_i}(\bar{x}) + \sum_{i=1}^k w_i (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r g_r(\bar{x}) \\ & \text{subject to } \sum_{i=1}^k w_i \nabla f_i(\bar{x}) + \sum_{i=1}^k w_i \nabla (f_{s_i}(x) - f_i^*(x)) + \sum_{r=1}^m \mu_r \nabla g_r(\bar{x}) = 0 \\ & \mu_r \geq 0, r = 1, 2, \dots, m \end{aligned}$$

If the set \bar{D}_2 is the optimal solution set and θ^* is the optimal value.

The set $\bar{D} \cup \bar{D}_2$ is a possibly optimal solution set.

Conclusion

In this paper we discuss the duality of the multiobjective rough convex programming problem when the multiobjective function is deterministic and roughness is in feasible region which does not discussed before. Also it presented the duality when roughness in multiobjective function and the feasible region is deterministic. The duality the multiobjective rough nonlinear convex programming problem is defined and its optimal solution sets are characterized. The duality study is important for these kinds of problems because it helps

us for obtaining the surely optimal solution of the lower approximation function and the upper approximation function.

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