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On Dynamical (B/Gd) Neutron Cancer Therapy by Accelerator Based Two Opposing Neutron Beams

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Abstract

The work reported in this paper studies the problem of irradiation of a single cancerous region in a (B/Gd) NCT setup by two opposing modulated one-speed neutron beams. The beams, which may have different pulse shapes, have different modulation frequencies and a variable relative time advance. These different frequencies and time advance are shown to be employable as three control variables in the formulation of a nonlinear optimization process that maximizes the therapeutic utility index and the ballistic index for this kind of dynamical (B/Gd)NCT.

Keywords: Accelerator Based Modulated Neutron Sources; One-Speed Neutron Diffusion; Two Opposing Neutron Beams; Dynamical NCT; Neutron-Density Waves

Introduction

In a previous paper [1] we analyzed dynamical (B/Gd) neutron cancer therapy (B/Gd) NCT, using a single modulated neutron beam, in a BNCT [2-4] and/or GdNCT [5] setup. This paper generalizes the results, obtained in [1], to the case of two opposing beams of slow neutrons, produced by accelerator sources and directed onto a cancerous region R, through two tumor-free regions Λ and Π. The beam transport is performed by means of collimators, hollow neutron guides, and possibly by solid neutron fibers, that penetrate a region Λ, to the left of R, and a region Π to its right.

An investigation of the dynamical neutronics of such a system is the main purpose of this paper. For the sake of simplicity of the analysis, we shall focus our attention on neutrons of only one speed (as in [1]) and on individual cosinusoidal time-modulation of the two opposing neutron beams, and on possible relative time advance between them. The rest of this paper is organized as follows. The next section contains the details of a one-dimensional setup formulation of the posing dynamical boundary value problem (BVP) for the modulated and time-advanced right neutron beam source, which is appropriately mapped in space and time. The section that follows addresses the solution to the BVP for this right neutron beam in the usual space-time domain with an analysis of its dissipative [1] and periodic "dispersive" components. The neutron density distribution due to the two opposing beams, in this (B/Gd) NCT setup, is reported in the fourth section. The utility and ballistic indices [1] tailored to this therapeutic setup are derived in the fifth section. Here also the therapeutic problem for a nonlinear optimization of these indices is formulated with the two modulation frequencies and relative time advance as control variables. The paper is concluded in the last section with some important remarks and suggestions for future relevant research.

One-Dimensional Problem Formulation for the Right Neutron Beam

The work in [1] studied the distribution of the neutron flux $\phi(x, t)$ that is established inside R due to the left source $S(x,t)$. The same description applies here with regard to this source in Figure 1. In addition to this flux, we shall consider a neutron flux $\psi(z,\tau)$ to be associated with the opposing right source $Ş(z,\tau)$ where z and τ are appropriately designed space and time variables. These are namely

$$z = l - x, \tag{1}$$

so as when $z = 0$, $x = l$ and when $z = l$, $x = 0$; and

$$\tau = t + \varepsilon, \tag{2}$$

so as when $\tau = 0$, $t = -\varepsilon$ and when $\tau = \varepsilon$, $t = 0$. Here ε stands for possible time advance between the cosinusoidally modulated sources of the two opposing neutron beams. Hence

$$Ş(z,\tau) = \begin{cases} Ş(\tau) = \dfrac{b_0}{2} + \displaystyle\sum_{m=1}^{\infty} b_m \cos m\varpi\tau; & z = 0^- \\ 0; & z \geq 0, \end{cases} \tag{3}$$

With $Ş(\tau)$ assumed to be a periodic function, of period Ꝥ, with even symmetry (like $S(t)$ of the opposing first neutron beam) and having a modulation frequency

$$\varpi = \frac{2\pi}{Ꝥ},$$

which can be varied freely of ω. These facts are apparent from

$$Ş(x,t) = \begin{cases} Ş(t) = \dfrac{b_0}{2} + \displaystyle\sum_{m=1}^{\infty} b_m \cos m\varpi(t+\varepsilon); & x = l^+ \\ 0; & x \leq l. \end{cases} \tag{4}$$

In this model for the neutron source, it is assumed that even before the modulation starts at $t = 0$, a steady state (stationary mode) exists with a level equaling to $\frac{b_0}{2}$ The variables of (1) and (2) in the context of composite region coupling of 2.1 in [1], happen to enable the generalization of the analytical developments of the neutron flux $\phi(x,t)$, due the left source $S(x,t)$, to $\psi(z,\tau)$ corresponding to this $Ş(z,\tau)$ right source, that stands in the diffusion equation

$$\frac{1}{v}\frac{\partial}{\partial\tau}\psi(z,\tau) - D\frac{\partial^2}{\partial z^2}\psi(z,\tau) + \Sigma_a\psi(z,\tau) = Ş(z,\tau), 0^- \leq z \leq l, \tag{5}$$

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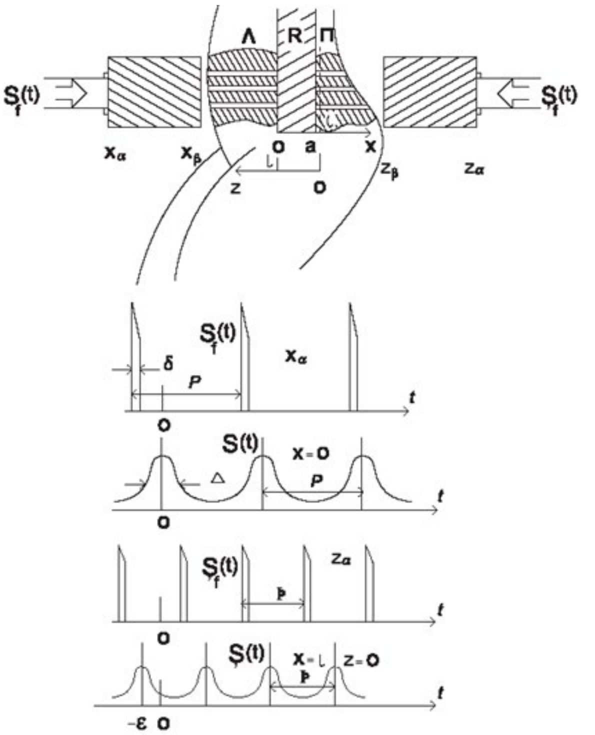

Figure 1: Sketch to illustrate the two opposing neutron beams in a dynamical (B/Gd) NCT setup.

where $(z,\tau)= v\text{Ņ}(z,\tau)$, and $\text{Ņ}(z,\tau)$ is the pertaining neutron density. The other constants are the same as in [1] or in the classical literature on reactor engineering [6]. Σ_a is the macroscopic absorption cross section of these neutrons in *R* which is loaded e.g. with B or Gd. If Σ_{aW} and Σ_{aB} are the cross sections corresponding to light water and natural boron, then the boron loading factor, [7], is

$$\varphi = \frac{\sum_a}{\sum_{aW}},\ \sum_a = \sum_{aW} + \sum_{aB}.$$

The clinically admissible value of φ which allows for the validity of the neutron diffusion model, [7], is not expected to exceed 5. The diffusion coefficient of *R* is however

$$D = \frac{1}{3\sum_{tr}},$$

with Σ_{tr} as the neutron transport cross section. The point $z = 0$ shall be the point of our $Ş(z,\tau)$ source application, $z = a$ is the physical boundary of the slab *R*, while $z = l$ shall be its extrapolated boundary; where $l - a = \frac{0.667}{\sum_{tr}}$

Furthermore, the diffusion length *L* of neutrons in *R*, satisfies: $L = \sqrt{\frac{1}{3\sum_{trW}\sum_{aW}\varphi}}$, and for rather small cancers, it may be assumed that the cancerous slab width satisfies $a \precsim$ (2-3) *L*.

The source term in (3) is discontinuous in *z* and is generated by a moderator attached to an accelerator target that can be (ϖ, ε) modulated in time. The accelerator generates pulses of fast neutron emissions $Ş_f(t)$ at z_α of pulse width δ in the range $10\ \mu s < \delta < 1000\ \mu s$. At the end of the moderator, i.e. at z_β, the thermal neutron source becomes $Ş_T(t)$. This same source of thermal neutrons is transported by a system of hollow neutron guides or solid neutron fibers, [1], through Π to emerge at $z = 0$ with a reduced amplitude represented by $Ş(t)$, as sketched in Figure 1.In this setup, the ratio $\frac{\delta}{Þ} \to 0$, when Þ→∞, i.e. when ϖ→0. Moreover, to distinguish a dynamical therapy from a steady state of therapy, Þ should exceed the life time $T_0 = \frac{1}{v\sum_{aW}\varphi}$ of thermal neutrons in *R*, i.e. T_0<Þ<<∞ and

$$\frac{2\pi}{T_0} > \varpi >> 0 \tag{6}$$

a condition that needs to be satisfied by ω as well.

Obviously, for any c ϵ(-∞,∞),

$$b_m = \frac{1}{T}\int_c^{c+2T} Ş(\tau)\cos m\varpi\tau\, d\tau = \frac{2}{T}\int_0^T Ş(\tau)\cos m\varpi\tau\, d\tau \tag{7}$$

Composite region coupling by a source at the common boundary

As in [1], it would be assumed that the neutron flux intensity $Į(0^+,\tau)$ at $z = 0^+$ should satisfy:

$$Į(0^+,\tau) = \chi_{R;\Pi} Ş(z,\tau) \tag{8}$$

Where

$$\chi_{R;\Pi} = \frac{\varrho_\Pi}{\varrho_R + \varrho_\Pi}, \tag{9}$$

is the coupling factor between the *R* region and the adjacent Π region, whose physical boundary is at *z*=*b* (i.e. x=*l*-*b*) and extrapolated boundary is at $z = l_\Pi$ (i.e. $x = l - l_\Pi$).

Here ϱ_Π is the albedo [8] for Π, $\varrho_\Pi = \left[1 - 2\frac{D_\Pi}{L_\Pi} ctanh \frac{b}{L_\Pi}\right]$, and $\chi_{R;\Pi}$ accounts for boosting up $Į(0^+,\tau)$ via reflection of neutrons from region Π to region *R*.

To simplify notation in the forthcoming analysis, as in [1], we shall throughout consider:

$$\mathring{a}_m = \chi_{R;\Lambda} a_m,\ m = 0,1,2,3,\ldots.$$

$$\tilde{b}_m = \chi_{R;\Pi} b_m,\ m = 0,1,2,3,\ldots. \tag{10}$$

The PDE (5) is subjected to satisfaction of (i) a zero flux at the extrapolated boundary *z*=*l*, and (ii) modulated neutron flux intensity at *z*=0 (instead of at $z=0^-$ in a physically acceptable ad-hock sense). Accordingly, for *z*≥0,the posing mixed-type BVP becomes

$$\frac{1}{v}\frac{\partial}{\partial\tau}\psi(z,\tau) - D\frac{\partial^2}{\partial z^2}\psi(z,\tau) + \sum_a \psi(z,\tau) = 0,$$

$$\text{(i)}\ \ \psi(l,\tau) = 0,$$

$$\text{(ii)}\ \ \frac{\partial}{\partial z}\psi(z,\tau)|_{z=0} = -\frac{\tilde{b}_0}{2D} - \sum_{m=1}^{\infty}\frac{\tilde{b}_m}{D}\cos m\varpi\tau,$$

$$\text{(iii)}\ \ \psi(z,0) = \varnothing(z), \tag{11}$$

where Ø(z) satisfies the auxiliary ordinary BVP:

$$\frac{d^2}{dz^2}\phi(z) - \frac{\sum_a}{D}\phi(z) = 0,$$

(i) $\phi(l)=0$,

(ii) $\frac{d}{dz}\phi(z)|_{z=0}=-\frac{\tilde{b}_0}{2D}$, (12)

with the condition (ii) as the natural temporal boundary condition corresponding to the m=0 mode. This implies that even before the source modulation starts at τ=0, a steady state source exists with a level equalling to $\frac{\tilde{b}_0}{2}$.

Solution of the auxiliary BVP

If $\mu=\sqrt{\frac{\sum_a}{D}}$, the solution to (12) is

$$\varnothing(z)=\frac{\tilde{b}_0}{2D\mu}\frac{\sinh\mu(l-z)}{\cosh\mu l} \tag{13}$$

with $\varnothing(0)=\frac{\tilde{b}_0}{2D\mu}\tan h\mu l$, while $\varnothing(l)=0$.

Solution of the dynamical BVP

Each $m\geq1$ temporal mode of the dynamical BVP solution shall be denoted as $\psi_m(z,\tau)$ and should satisfy a corresponding modal partial BVP viz

$$\frac{\partial}{\partial\tau}\psi_m(z,\tau)-vD\frac{\partial^2}{\partial z^2}\psi_m(z,\tau)+v\sum_a\psi_m(z,\tau)=0,\ z\geq0,$$

(i) $\psi_m(l,\tau)=0$,

(ii) $\frac{\partial}{\partial z}\psi_m(z,\tau)|_{z=0}=-\frac{\tilde{b}_m}{D}\cos m\varpi\tau$,

(iii) $\psi_m(z,0)=\varnothing_m(z)=0$. (14)

With respect to BC(iii), $\varnothing(z)=\varnothing_0(z)$; hence $\varnothing_m(z)=0,\ \forall\ m\geq1$.

This BVP happens to quite simplify by application of the Laplace transformation in the τ-domain. So after invoking the Laplace transform pair $\psi_m(z,\tau)\leftrightarrow\bar{\psi}_m(z,s)$, it is possible to write

$$\frac{\partial}{\partial\tau}\psi_m(z,\tau)\leftrightarrow s\bar{\psi}_m(z,s), \tag{15}$$

which leads to the ODE of the following modal ordinary BVP

$$s\bar{\psi}_m(z,s)-vD\frac{d^2}{dz^2}\bar{\psi}_m(z,s)+v\sum_a\bar{\psi}_m(z,s)=0,\ z\geq0, \tag{16}$$

(i) $\bar{\psi}_m(l,s)=0$,

(ii) $\frac{d}{dz}\bar{\psi}_m(z,s)|_{z=0}=-\frac{\tilde{b}_m}{D}\frac{s}{s^2+m^2\varpi^2}$.

Utilization of (15) in (16), after assuming:

$$\alpha^2=\left(s+v\sum_a\right)/vD$$

leads to

$$\bar{\psi}_m(z,s)=\frac{\tilde{b}_m}{D}\frac{\sinh(l-z)\alpha}{\acute{a}\cosh l\alpha}\frac{s}{s^2+m^2\varpi^2}. \tag{17}$$

After adoption of the notation

$$\bar{F}(z,s)=\frac{\sinh(l-z)\alpha}{\alpha\cosh l\alpha}\leftrightarrow F(z,\tau), \tag{18}$$

we may utilize the real convolution property of the Laplace transformation to write:

$$\bar{\psi}_m(z,s)=\frac{\tilde{b}_m}{D}\bar{F}(z,s)\frac{s}{s^2+m^2 s\varpi^2}$$

$$\leftrightarrow\psi_m(z,\tau)=\frac{\tilde{b}_m}{D}\int_0^\tau F(z,\tau-\varsigma)\cos m\varpi\varsigma\, d\varsigma. \tag{19}$$

F (z,τ) can be derived from existing tables. [9] of Laplace transform pairs to be

$$F(z,\tau)=\frac{2vD}{l}\sum_{n=1}^{\infty}(-1)^{n-1}e^{-\beta_n\tau}\sin(2n-1)\frac{\pi}{2}\frac{(l-z)}{l}, \tag{20}$$

with

$$\beta_n=vD(2n-1)^2\frac{\pi^2}{4l^2}+v\sum_a \tag{21}$$

Apparently

$$\sin(2n-1)\frac{\pi}{2}\frac{(l-z)}{l}=(-1)^{n-1}\cos(2n-1)\frac{\pi}{2}\frac{z}{l}=(-1)^{n-1}Q_n(z), \tag{22}$$

and that allows for rewriting

$$F(z,\tau)=\frac{2vD}{l}\sum_{n=1}^{\infty}e^{-\beta_n\tau}Q_n(z), \tag{23}$$

Then

$$\psi_m(z,\tau)=2\frac{v\tilde{b}_m}{l}\sum_{n=1}^{\infty}\frac{\{\beta_n\cos m\varpi\tau+m\varpi\sin m\varpi\tau\}}{(\beta_n^2+m^2\varpi^2)}Q_n(z)$$
$$-2\frac{v\tilde{b}_m}{l}\sum_{(n=1)}^{\infty}\frac{\beta_n}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\tau}Q_n(z). \tag{24}$$

By summing up over all the m temporal harmonics we arrive at the posing BVP solution $\psi(z,\tau)=\sum_{m=0}^{\infty}\psi_m(z,\tau)$ which upon division by v yields the associated, with $Ş(z,\tau)$, neutron density distribution

$$N(z,\tau)=\sum_{m=0}^{\infty}N_m(z,\tau)=2\frac{1}{l}\sum_{m=1}^{\infty}\tilde{b}_m\sum_{n=1}^{\infty}\left\{\frac{\beta_n\cos m\varpi\tau+m\varpi\sin m\varpi\tau}{(\beta_n^2+m^2\varpi^2)}\right\}Q_n(z)$$
$$-2\frac{1}{l}\sum_{m=1}^{\infty}\tilde{b}_m\sum_{n=1}^{\infty}\frac{\beta_n}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\tau}Q_n(z)+\frac{1}{v}\varnothing(z), \tag{25}$$

which decomposes, like $N(x,t)$, into the three superimposed distinct effects viz

$$N(z,\tau)=N_\rho(z,\tau)+N_\sigma(z,\tau)+N_\gamma(z) \tag{26}$$

where $N_\rho(z,\tau)$ is periodic, $N_\rho(z,\tau)$ is dissipative and $N_\gamma(z)$ is stationary.

The Neuron Sensity Ditribution From The Right Beam In The xt –Domain

Back substitution of the maps (1) and (2) for z and τ in (29) invokes

$$Q_n(z)=\cos(2n-1)\frac{\pi}{2}\frac{z}{l}=\cos(2n-1)\frac{\pi}{2}\frac{(l-x)}{l}$$
$$=(-1)^{(n-1)}\sin(2n-1)\frac{\pi}{2}\frac{x}{l}=(-1)^{(n-1)}G_n(x), \tag{27}$$

$$e^{-\beta_n\tau}=e^{-\beta_n\varepsilon}e^{-\beta_n t}, \tag{28}$$

Sinhμ(1-z)=sinhμx, (29)

$\{\beta_n\cos\varpi\tau+m\varpi\sin\varpi\tau\}=\beta_n\cos\varpi(t+\varepsilon)+m\varpi\sin m\varpi(t+\varepsilon)$

$=\{[\beta_n\cos m\varpi\varepsilon+m\varpi\sin m\varpi\varepsilon]\cos m\varpi t+$

$[-\beta_n\sin m\varpi\varepsilon+m\varpi\cos m\varpi\varepsilon]\sin m\varpi t\}$

$=\{\Xi_{mn}^{+}(\varpi,\varepsilon)\cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\sin m\varpi t\}$. (30)

to yield

$$Ņ(x,t)=\sum_{m=0}^{\infty}Ņ_m(x,t)$$

$$=2\frac{1}{l}\sum_{m=1}^{\infty}\tilde{b}_m\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\{\Xi_{mn}^{+}(\varpi,\varepsilon)cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\sin m\varpi t\}}{(\beta_n^{2}+m^2\varpi^2)}G_n(x)$$

$$-2\frac{1}{l}\sum_{m=1}^{\infty}\tilde{b}_m\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\beta_n}{(\beta_n^{2}+m^2\varpi^2)}e^{-\beta_n(t+\varepsilon)}G_n(x)$$

$$+\frac{\tilde{b}_0}{2Dv\mu}\frac{sinh\,\mu x}{\cos h\mu l} \quad (31)$$

The periodic component of the space –time transient $Ņ(x,t)$ is

$$Ņ_\rho(x,t)=2\frac{1}{l}\sum_{m=1}^{\infty}\tilde{b}_m\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\{\Xi_{mn}^{+}(\varpi,\varepsilon)cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\sin m\varpi t\}}{(\beta_n^{2}+m^2\varpi^2)}G_n(x). \quad (32)$$

Distinctively, the stationary component

$$Ņ_\gamma(x)=\frac{\tilde{b}_0}{2Dv\mu}\frac{sinh\,\mu x}{\cos h\mu l} \quad (33)$$

appears to be explicitly independent of the modulation frequency ϖ of the neutron source or its advance ε.

The dissipative neutron density wave

The dynamical component

$$Ņ_\sigma(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\hat{M}_{mn}(x,t) \quad (34)$$

with

$$\hat{M}_{mn}(x,t)=-2\frac{\tilde{b}_m}{l}(-1)^{n-1}\frac{\beta_n}{(\beta_n^{2}+m^2\varpi^2)}e^{-\beta_n(t+\varepsilon)}G_n(x), \quad (35)$$

vanishes asymptotically with time and increasingly faster with increasing Σa, i.e. by increasing the B/Gd content of *R*.

Apart from the implicit dependence of its *bm*'s on ϖ, $Ņ_\sigma(x,t)$ itself is rather sensitive to variations in ϖ. However, when $\omega\to\infty$ all the *bm*'s are zeros and the entire $Ņ_\sigma(x,t)$ disappears.

The periodic neutron-density wave

The dispersive term of

$$Ņ(x,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}Ņ_{mn}(x,t), \quad (36)$$

with an *mn*- double indexed modal term

$$Ņ_{mn}(x,t)=2\frac{\tilde{b}_m}{l}(-1)^{n-1}\frac{\{\Xi_{mn}^{+}(\varpi,\varepsilon)cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\sin m\varpi t\}}{(\beta_n^{2}+m^2\varpi^2)}G_n(x) \quad (37)$$

This term is indexed in *m* temporally, following the harmonic resolution of the external neutron source $Ş(\tau)$ with a modulation frequency ϖ and relative delay ε from $S(t)$; while its *n* index refers to the spatial modes of the finite slab *R*.

$$\{\Xi_{mn}^{+}(\varpi,\varepsilon)\ cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)sin m\varpi t\}=\{\beta_n cos m\varpi\tau+m\varpi sin m\tau\}$$

$$=\aleph_{mn}sin\left[m\varpi\tau+\theta_{mn}(\varpi)\right],$$

with $\aleph_{mn}=\sqrt{\beta_n^{2}+m^2\varpi^2}$ and

$$\theta_{mn}(\varpi)=\tan^{-1}\frac{\beta_n}{m\varpi} \quad (38)$$

which approximates with

$$\theta_{mn}(\varpi)=\frac{\beta_n}{m\varpi}, as\varpi\to\infty. \quad (39)$$

Clearly then

$$\{\Xi_{mn}^{+}(\varpi,\varepsilon)\ cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\ sin m\varpi t\}=\aleph_{mn}sin\left[m\varpi(t+\varepsilon)+\theta_{mn}(\varpi)\right]=\aleph_{mn}sin\left[m\varpi t+\theta_{mn}(\varpi)+m\varpi\varepsilon\right],$$

which in view of

$$\Omega_{mn}(\varpi,\varepsilon)]=\theta_{mn}(\varpi)+m\varpi\varepsilon=\tan^{-1}\frac{\beta_n}{m\varpi}+m\varpi\varepsilon \quad (40)$$

becomes

$$\{\Xi_{mn}^{+}(\varpi,\varepsilon)\ cos m\varpi t+\Xi_{mn}^{-}(\varpi,\varepsilon)\ sin m\varpi t\}=\aleph_{mn}sin\left[m\varpi t+\Omega_{mn}(\varpi,\varepsilon)\right].$$

Then it is possible to rewrite (41) as:

$$Ņ_{mn}(x,t)=-2\frac{\tilde{b}_m}{l}(-1)^{n-1}\frac{1}{\sqrt{\beta_n^{2}+m^2\varpi^2}}sin\left[m\varpi t+\Omega_{mn}(\varpi,\varepsilon)\right]G_n(x), \quad (41)$$

which is a standard harmonic vibration

$$Ņ_{mn}(x,t)=Ȩ_{mn}(\varpi)\Pi_{mn}(\varpi,t,\varepsilon)G_n(x),$$

of a fixed (with varying ϖ) shape function $G_n(x)=sin(2n-1)\frac{\pi}{2}\frac{x}{l}$, but with a varying (with varying ϖ and ε) amplitude $Ȩ_{mn}(\varpi)\Pi_{mn}(\varpi,t,\varepsilon)$, whose $\Pi_{mn}(\varpi,t,\varepsilon)=\sin[m\varpi t+\Omega_{mn}(\varpi,\varepsilon)]$ factor is also time-dependent, while $Ȩ_{mn}(\varpi)=-2\frac{\tilde{b}_m}{l}(-1)^{n-1}\frac{\tilde{b}_m}{\sqrt{\beta_n^{2}+m^2\varpi^2}}$.

$G_n(x)$ is a standing (in time) wave with a number of nodes ¢(*n*) that depends on *n*. Obviously, there is always, and for all *n*, a node at $x = 0$; and a summary of ¢(*n*) for the first 10 *n*'s is given in Table 1.

The trigonometric identity 2sinAsinB = cos(A-B) – cos(A+B) allows for a travelling wave representation for $Ņ_{mn}(x,t)$, which is namely

$$Ņ_{mn}(x,t)=Ȩ_{mn}(\varpi)\Pi_{mn}(\varpi,t,\varepsilon)G_n(x)=\mathfrak{A}_{mn}(x,\varpi,t,\Omega_{mn})+B_{mn}(x,-t,\varpi-\Omega_{mn}), \quad (42)$$

where

n	Node Location	¢(*n*)
1	0	1
2	2l/3	1
3	2l/5	1
4	2l/7	1
5	2l/3, 2l/9	2
6	2l/11	1
7	2l/13	1
8	2l/3, 2l/5, 2l/15	3
9	2l/17	1
10	2l/19	1

Table 1: Nodes for some $G_n(x)$.

$$\mathfrak{A}_{mn}\left(x,t,\varpi,\Omega_{mn}\right)=-\frac{1}{2}E_{mn}\left(\varpi\right)\cos\left[(2n-1)+m\varpi t+\Omega_{mn}\left(\varpi,\varepsilon\right)\right], \quad (43)$$

$$B_{mn}\left(x,-t,\varpi,-\Omega_{mn}\right)=\frac{1}{2}E_{mn}\left(\varpi\right)\cos\left[(2n-1)\frac{\pi}{2}\frac{x}{l}-m\varpi\tau-\Omega_{mn}\left(\varpi,\varepsilon\right)\right]. \quad (44)$$

Relation (42) is a d'Alembert-like modal solution, with $\mathfrak{U}_{mn}$ and B_{mn} as two different modal waves, one progressing to the left, while the other is progressing to the right of the *x*-axis. Summing up over all *mn*-indices defines two periodic neutron density travelling waves,

$$\mathfrak{U}_{\rho}\left(x,t\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mathfrak{U}_{mn}\left(x,t,\varpi,\ \Omega_{mn}\right),$$

and

$$B_{\rho}\left(x,t\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}B_{mn}\left(x,-t,\ \varpi,\ -\Omega_{mn}\right), \quad (45)$$

and allows for the representation of (41) as the periodic neutron-density wave

$$\text{Ņ}_{\rho}\left(x,t\right)=\mathfrak{U}_{\rho}\left(x,t\right)+B_{\rho}\left(x,t\right). \quad (46)$$

Definition [1] **1.** For the establishment of a standing neutron-density wave $\text{N}_{\rho}(x,t)$, the following condition

$$\mathfrak{U}_{mn}\left(x,t,\varpi,\ \Omega_{mn}\right)=B_{mn}\left(x,-t,\ \varpi,\ -\Omega_{mn}\right),\forall mn-mode, \quad (47)$$

must be satisfied in (45). However, if $U_{mn}\neq B_{mn}$ in (45), then $N_{\rho}(x, t)$ of (46) is a quasi-standing periodic neutron-density wave.

Interestingly, by looking back at relation (40) for $\Omega_{mn}\left(\varpi,\varepsilon\right)$ and at relation (35) for β_n we can easily see that for large *n*, small *l*, ϖ and *m*, the argument $\frac{\beta_n}{m\varpi}$ grows in magnitude towards ∞, a situation that can also be boosted by increasing $\sum_a(or\varphi)$, i.e. by increasing the B/Gd content of *R*. Hence it is possible to write:

$$\Omega_{mn}(\varpi,\varepsilon)\approx\begin{cases}\frac{\pi}{2},\ as\,\varpi\to 0(\text{pulsed a periodic source}),\\ \frac{\beta_n}{m\varpi}+m\varpi\varepsilon,\ as\,\varpi\to\infty(stationary\,source);\to\frac{\pi}{2},\ as\,\varpi\to\frac{\pi}{2m\varepsilon},\end{cases} \quad (48)-(49)$$

Now upon assuming the satisfaction of (48) and/or (49), we my invoke the elementary trigonometry $cos(\text{A}+\frac{\pi}{2})=-\sin A$ and $cos(\text{A}-\frac{\pi}{2})=\sin A$ to demonstrate that the expression (42) for $\text{Ņ}_{mn}(x,t)$, transforms to a modal dispersive neutron-density standing wave.

$$\Upsilon_{mn}\left(x,t\right)=\Psi_{mn}\left(x,t,\varpi\right)+\Psi_{mn}\left(x,-t,\varpi\right) \quad (50)$$

where

$$\Psi_{mn}\left(x,\pm t,\varpi\right)=\frac{1}{2}E_{mn}\left(\varpi\right)\sin\left[(2n-1)\frac{\pi}{2}\frac{x}{l}\pm m\varpi t\right], \quad (51)$$

are the associated modal travelling periodic neutron-density waves. Interestingly, $\Upsilon_{mn}\left(x,t\right)$ happens to be explicitly independent of ε. Here we may also define the periodic neutron-density travelling waves

$$\Psi_{\lambda}\left(x,\pm t,\varpi\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Psi_{mn}\left(x,\pm t,\varpi\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Upsilon_{mn}\left(x,t\right)$$

and conclude that

$$\Upsilon_{\lambda}\left(x\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\Upsilon_{mn}\left(x,t\right)=\Psi_{\lambda}\left(x,+t,\varpi\right)+\Psi_{\lambda}\left(x,-t,\varpi\right)=\pm\Upsilon\left(x\right),\forall\,t,$$

is a periodic neutron-density standing wave.

The Neutron-Density Distribution From The Two Opposing Beams

Consider now the total neutron flux $f(x,t)$ inside *R* resulting from the combined application of both left and right beam sources $S(x, t)$ and $\text{Ş}(x, t)$, respectively, i.e.

$$f(x,t)=\phi(x,t)+\psi(x,t). \quad (52)$$

The pertaining neutron density distribution is

$$g(x,t)=\text{N}(x,t)+\text{Ņ}(x,t), \quad (53)$$

and substitution of relations (30), of [1], and (31) for $\text{Ņ}(x, t)$ leads to

$$\text{g}\left(x,t\right)=\sum_{m=0}^{\infty}\left[N_m\left(x,t\right)+\text{Ņ}_m\left(x,t\right)\right]$$

$$=\frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\mathring{a}_m\left[\beta_n\cos m\omega t+m\omega\sin m\omega t\right]}{\left(\beta_n^2+m^2\omega^2\right)}Q_n\left(x\right)\right.$$

$$\left.+\frac{(-1)^{n-1}\tilde{b}_m\left[\Xi_{mn}^{+}\left(\varpi,\varepsilon\right)\cos m\varpi t+\Xi_{mn}^{-}\left(\varpi,\varepsilon\right)\sin m\varpi t\right]}{\left(\beta_n^2+m^2\varpi^2\right)}G_n\left(x\right)\right\}$$

$$-\frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\mathring{a}_m\beta_n}{\left(\beta_n^2+m^2\omega^2\right)}Q_n\left(x\right)+\frac{(-1)^{n-1}\tilde{b}_m\beta_n}{\left(\beta_n^2+m^2\varpi^2\right)}e^{-\beta_n\varepsilon}G_n\left(x\right)\right]e^{-\beta_n t}$$

$$+\frac{1}{2D\nu\mu\cosh\mu l}\left[\mathring{a}_0\sinh\mu\left(l-x\right)+\tilde{b}_0\sinh\mu x\right]. \quad (54)$$

The first term (with figurative brackets), in the relation above, is the periodic part of the neutron-density wave generated inside *R* by the two opposing neutron beams. It can also be represented as

$$\text{g}_\rho\left(x,t\right)=\frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\frac{\mathring{a}_m\sin\left[m\omega t+\theta_{mn}\left(\omega\right)\right]}{\sqrt{\beta_n^2+m^2\omega^2}}\cos\left(2\text{n}-1\right)\frac{\pi}{2}\frac{x}{l}\right.$$

$$\left.+\frac{(-1)^{n-1}\tilde{b}_m\sin\left[m\varpi t+\Omega_{mn}\left(\varpi,\varepsilon\right)\right]}{\sqrt{\beta_n^2+m^2\varpi^2}}\sin\left(2\text{n}-1\right)\frac{\pi}{2}\frac{x}{l}\right\}, \quad (55)$$

which is explicitly a weighted sum of two quasi-standing waves having distinct frequencies ω and ϖ. With respect to time periodicity, $g_{\rho}(x, t)$ can be periodic only when the ratio ω/ϖ is commensurate.

The dissipative part of this wave is:

$$\text{g}_\sigma\left(x,t\right)=-\frac{2}{l}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\frac{\mathring{a}_m\beta_n}{\left(\beta_n^2+m^2\omega^2\right)}\cos\left(2\text{n}-1\right)\frac{\pi}{2}\frac{x}{l}\right.$$

$$\left.+\frac{(-1)^{n-1}\tilde{b}_m\beta_n}{\left(\beta_n^2+m^2\varpi^2\right)}e^{-\beta_n\varepsilon}\sin\left(2\text{n}-1\right)\frac{\pi}{2}\frac{x}{l}\right]e^{-\beta_n t}, \quad (56)$$

while the stationary part is

$$\text{g}_\lambda\left(x\right)=\frac{1}{2D\nu\mu\cosh\mu l}\left[\mathring{a}_0\sinh\mu\left(l-x\right)+\tilde{b}_0\sinh\mu x\right], \quad (57)$$

Utility and Ballistic Indices for Neutrons From the Two Opposing Beams

The neutron-density wave utility index

For the opposing beams arrangement, the ratio

$\zeta(\omega,x,t)=\dfrac{N_\gamma(x)+N_{\acute{o}}(x,t)}{N_{\tilde{n}}(x,t)}$, introduced in [1], becomes

$$\tilde{\zeta}(\omega,\varpi,\varepsilon,x,t)=\frac{g_\gamma(x)+g_\sigma(x,t)}{g_\rho(x,t)}$$

$$=\frac{g_\gamma(x)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[M_{mn}(x,t)+\hat{M}_{mn}(x,t)\right]}{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[N_{mn}(x,t)+\underset{\sim}{N}_{mn}(x,t)\right]},\qquad(58)$$

which is additionally dependent on ϖ and ε, in addition to ω.

Inside (58), let us contemplate the following modal component ratio, which undoubtedly affects $\tilde{\zeta}(\omega,\varpi,\varepsilon,x,t)$ in some indirect sense.

$$\frac{M_{mn}(x,t)+\hat{M}_{mn}(x,t)}{N_{mn}(x,t)+\underset{\sim}{N}_{mn}(x,t)}=$$

$$-\frac{\left[\dfrac{\beta_n}{(\beta_n^2+m^2\omega^2)}+\dfrac{\tilde{b}_m}{\mathring{a}_m}(-1)^{n-1}\dfrac{\beta_n}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\varepsilon}\,tan(2n-1)\dfrac{\pi}{2}\dfrac{x}{l}\right]e^{-\beta_n t}}{\left[\dfrac{\sin[m\omega t+\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}+\dfrac{\tilde{b}_m}{\mathring{a}_m}(-1)^{n-1}\dfrac{\sin[m\varpi t+\Omega_{mn}(\varpi,\varepsilon)]}{\sqrt{\beta_n^2+m^2\varpi^2}}tan(2n-1)\dfrac{\pi}{2}\dfrac{x}{l}\right]}.\qquad(59)$$

Since the center of R (i.e. $x=l/2$)is the most critical point, with respect to effectiveness of this kind of neutron therapy, then there is a need to focus attention on the amplitude $\dfrac{M_{mn}(x,0)+\hat{M}_{mn}(x,0)}{N_{mn}(x,0)+\underset{\sim}{N}_{mn}(x,0)}$ at $x=l/2$, which in view of $(-1)^{n-1}\,tan(2n-1)\dfrac{\pi}{4}=1$ writes as

$$\frac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}=$$

$$-\frac{\left[\dfrac{\beta_n}{(\beta_n^2+m^2\omega^2)}+\dfrac{\tilde{b}_m}{\mathring{a}_m}\dfrac{\beta_n}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\varepsilon}\right]}{\left[\dfrac{\sin[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}+\dfrac{\tilde{b}_m}{\mathring{a}_m}\dfrac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sqrt{\beta_n^2+m^2\varpi^2}}\right]}\qquad(60)$$

which is obviously the same as

$$\frac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}=$$

$$-\frac{\beta_n\,csec[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}\frac{\left[1+\dfrac{\tilde{b}_m}{\mathring{a}_m}\dfrac{(\beta_n^2+m^2\omega^2)}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\varepsilon}\right]}{\left[1+\dfrac{\tilde{b}_m}{\mathring{a}_m}\dfrac{\sqrt{\beta_n^2+m^2\omega^2}}{\sqrt{\beta_n^2+m^2\varpi^2}}\dfrac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sin[\theta_{mn}(\omega)]}\right]}.\qquad(61)$$

Due to symmetry of the two opposing beams therapeutic setup, ϖ and ω are free to satisfy

$$\varpi \gtrless \omega.\qquad(62)$$

Therefore we may assume in the remaining part of this paper, without loss of generality, that

$$\varpi>\omega,\qquad(63)$$

and write

$$\frac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}\approx-\frac{\beta_n\,csec[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}$$

$$\times\left[1+\frac{\tilde{b}_m}{\mathring{a}_m}\frac{(\beta_n^2+m^2\omega^2)}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\varepsilon}\right]\left[1-\frac{\tilde{b}_m}{\mathring{a}_m}\frac{\sqrt{\beta_n^2+m^2\omega^2}}{\sqrt{\beta_n^2+m^2\varpi^2}}\frac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sin[\theta_{mn}(\omega)]}\right].\qquad(64)$$

The last relation can in general be approximated to

$$\frac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}\approx$$

$$-\frac{\beta_n\,csec[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}\left[1-\left(\frac{\tilde{b}_m}{\mathring{a}_m}\right)^2\left(\frac{\beta_n^2+m^2\omega^2}{\beta_n^2+m^2\varpi^2}\right)^{3/2}\frac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sin[\theta_{mn}(\omega)]}e^{-\beta_n\varepsilon}\right].\qquad(65)$$

Which for $m^2\varpi^2>\beta_n^2$ reduces to

$$\frac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}\approx$$

$$-\frac{\beta_n\,csec[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}\left[1-\left(\frac{\tilde{b}_m}{\mathring{a}_m}\right)^2\left(\frac{\omega}{\varpi}\right)^3\frac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sin[\theta_{mn}(\omega)]}e^{-\beta_n\varepsilon}\right].\qquad(66)$$

Proposition 1. In the two opposing beams NCT setup, if $\varpi>\omega$, the negativity of the modal amplitude $\dfrac{M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)}{N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)}$ requires, when

$$C_{mn}(\omega,\varpi,\varepsilon)=\left(\frac{\tilde{b}_m}{\mathring{a}_m}\right)^2\frac{\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sin[\theta_{mn}(\omega)]}e^{-\beta_n\varepsilon},\qquad(67)$$

that

$$\varpi>\sqrt[3]{C_{mn}(\omega,\varpi,\varepsilon)}\;\omega,\forall m,n\,and\,\varepsilon.\qquad(68)$$

Realizing that the expectation of $C_{mn}\approx 1$-2, i.e. $\sqrt[3]{C_{mn}}\approx 1-1.26$, then the rather loose conditions for the above proposition indicate that (68) can be conceived as a possible mathematical consistency constraint. Certainly, more important than this weak proposition is the neutron density therapeutic utility index

$$\tilde{\eta}(\omega,\varpi,\varepsilon)=\tilde{\zeta}(\omega,\varpi,\varepsilon,l/2,0)$$

$$=\frac{g_\gamma(l/2)+\sum_{m=1}^{M}\sum_{n=1}^{N}\left[M_{mn}(l/2,0)+\hat{M}_{mn}(l/2,0)\right]}{\sum_{m=1}^{M}\sum_{n=1}^{N}\left[N_{mn}(l/2,0)+\underset{\sim}{N}_{mn}(l/2,0)\right]},\qquad(69)$$

for this two-opposing beams setup. It rewrites in detail as the relation

$$\tilde{\eta}(\omega,\varpi,\varepsilon)=\left\{\frac{1}{2D\nu\mu\cos h\mu l}\left[\mathring{a}_0\sin h\mu\frac{l}{2}+\tilde{b}_0\sin h\mu\frac{l}{2}\right]\right.$$

$$-\frac{2}{l}\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\cos(2n-1)\frac{\pi}{4}\right]\beta_n\left[\frac{\mathring{a}_m}{(\beta_n^2+m^2\omega^2)}+\frac{\tilde{b}_m}{(\beta_n^2+m^2\varpi^2)}e^{-\beta_n\varepsilon}\right]$$

$$\left./\left\{\frac{2}{l}\sum_{m=1}^{M}\sum_{n=1}^{N}\left[\cos(2n-1)\frac{\pi}{4}\right]\left[\frac{\mathring{a}_m\sin[\theta_{mn}(\omega)]}{\sqrt{\beta_n^2+m^2\omega^2}}+\frac{\tilde{b}_m\sin[\Omega_{mn}(\varpi,\varepsilon)]}{\sqrt{\beta_n^2+m^2\varpi^2}}\right]\right\}\right.\qquad(70)$$

which exhibits a nonlinear dependence on ω, ϖ , and ε, for every set {Σa , D, l} of parameters of R. A dependence that is different of course in (64) or in proposition 1. In particular, it is apparent, from the $[cos(2n-1)\dfrac{\pi}{4}]$ factors, that the numerators and denominators in η(ω,ϖ,ε) undergo a cyclic behaviour, of period 4, with varying *n*.

Remark 1. It should be noted, moreover, that a hoped for applicability of conclusions based on (65)-(66) to (70) can obviously only be restricted to very small M and N. For instance, an interesting situation emerges from (68) when the ratio $\frac{\varpi}{\omega}=\sqrt[3]{C_{mn}(\omega,\varpi,\varepsilon)}$ $\forall m, n$ and ε, is commensurate, i.e. when $\frac{\varpi}{\omega}=k\in\mathbb{N}$. This corresponds to a situation when ϖ-ω = (k-1)ω, especially if ϖ is close to ω. Is this going to be some kind of tenuous beat effect [10], for the neutron-density waves? Admittedly, a yet not understandable and perhaps fine effect, suggested for the first time in this work. The likely impact of such an effect on $\tilde{\eta}(\omega,\varpi,\varepsilon)$ is another pending open question.

The neutron ballistic index

The neutron therapeutic ballistic index α̃(ω, ϖ, ε), tailored to the present setup of two opposing neutron beams, happens to accept the same Definition 6.1 of [1] for α(ω), but with each N(*x*, *t*) replaced with a corresponding *g*(*x*, *t*) of (54), viz

$$\tilde{\alpha}(\omega,\varpi,\varepsilon)=\frac{v\sum_a\left|\int_0^q\int_0^l[g_\gamma(x)+g_\delta(x,t)]dxdt\right|}{vD\left|\int_0^q\frac{\partial}{\partial x}\{g_\gamma(x)+g_\delta(x,t)\}|_{x=0}dt\right|+vD\left|\int_0^q\frac{\partial}{\partial x}\{g_\gamma(x)+g_\delta(x,t)\}|_{x=l}dt\right|},\tag{71}$$

with

$$q=max\{P,Þ\}.\tag{72}$$

By invoking the same integral facts listed between (57) and (58) of [1], we can write

$$v\sum_a\int_0^q\int_0^l g_\gamma(x)\,dxdt=\frac{q}{2}\frac{\sum_a}{D}\frac{\left(\mathring{a}_0+\tilde{b}_0\right)}{\mu^2}(1-sech\,\mu l),$$

$$\frac{\partial}{\partial x}g_\gamma(x)|_{x=0}=\frac{1}{2vD}\,(\tilde{b}_0\,sech\,\mu l-\mathring{a}_0),$$

$$\frac{\partial}{\partial x}g_\gamma(x)|_{x=l}=-\frac{1}{2vD}\,(\mathring{a}_0\,sech\,\mu l-\tilde{b}_0),$$

$$\int_0^l Q_n(x)\,dx=\frac{(-1)^{n-1}2l}{(2n-1)\pi},$$

$$\int_0^q e^{-\beta_n t}dt=\frac{1}{\beta_n}[1-e^{-\beta_n q)}],$$

$$\frac{\partial}{\partial x}Q_n(x)|_{x=0}=0,$$

$$\frac{\partial}{\partial x}Q_n(x)|_{x=l}=(2n-1)\frac{\pi}{2l}(-1)^n,$$

together with

$$\int_0^l G_n(x)\,dx=\frac{2l}{(2n-1)\pi},$$

$$\frac{\partial}{\partial x}G_n(x)|_{x=0}=(2n-1)\frac{\pi}{2l},$$

$$\frac{\partial}{\partial x}G_n(x)|_{x=l}=0,$$

and,

$$\hbar_{mn}(\omega)=(-1)^{n-1}\frac{1}{(\beta_n^{\,2}+m^2\,\omega^2)}[1-e^{(-\beta_n q)}.\tag{73}$$

Substitution of these relations in (71) transforms it to:

$$\tilde{\alpha}(\omega,\varpi,\varepsilon)\approx\left\{\left|\frac{q}{2}\frac{\sum_a}{D}\frac{\left(\mathring{a}_0+\tilde{b}_0\right)}{\mu^2}(1-sech\,\mu l)\right.\right.$$
$$\left.\left.-\frac{4}{\pi}v\sum_a\sum_{m=1}^{M}\sum_{n=1}^{N}\frac{1}{(2n-1)}\left[\mathring{a}_m\hbar_{mn}(\omega)+\tilde{b}_m e^{-\beta_n\varepsilon}\hbar_{mn}(\varpi)\right]\right|\right\}$$
$$/\left\{\left|\frac{q}{2}(\mathring{a}_0\,sech\mu l-\tilde{b}_0)-vD\frac{\pi}{l^2}\sum_{m=1}^{M}\sum_{n=1}^{N}(2n-1)\mathring{a}_m\hbar_{mn}(\omega)\right|\right.$$
$$\left.+\left|\frac{q}{2}\left(\tilde{b}_0\,sech\,\mu l-\mathring{a}_0\right)-vD\frac{\pi}{l^2}\sum_{m=1}^{M}\sum_{n=1}^{N}(2n-1)\tilde{b}_m e^{-\beta_n\varepsilon}\hbar_{mn}(\varpi)\right|,\right\}\tag{74}$$

which is a unique rational nonlinear function of three independent temporal variables (ω,ϖ,ε) for every set {Σ_a,D,*l*} of composition and geometric parameters of the region *R*. Obviously the existence of a critical point $\wp(\boldsymbol{\omega}^\wedge)\triangleq(\omega^\wedge,\varpi^\wedge,\varepsilon^\wedge)$, or points, for which

$$\frac{\partial}{\partial\omega}\bar{\alpha}\,|_\wp=\frac{\partial}{\partial\varpi}\bar{\alpha}\,|_\wp=\frac{\partial}{\partial\varepsilon}\bar{\alpha}\,|_\wp=0,\tag{75}$$

does not guarantee the existence of a local unconstrained maximum for $\bar{\alpha}\,(\omega,\varpi,\varepsilon)$ at $\wp=\wp(\varpi^\wedge)$. The existence of such a maximum requires additionally that all the eigenvalues of the Hessian matrix $\mathcal{H}(\bar{\alpha}(\wp))$ to be negative [11]. Moreover if $\mathcal{H}(\bar{\alpha}(\wp))$ turns out to have mixed sign eigenvalues, then $\wp$ is a saddle point.

In distinction from the therapeutic optimization problem of [1], we have here a control variable vector $\boldsymbol{\omega}$ for the (B/Gd) NCT nonlinear optimization problem of the two opposing neutron beams with a an optimal control vector $\boldsymbol{\omega}^*=(\omega^*,\varpi^*,\varepsilon^*)\triangleq\Im(\boldsymbol{\omega}^*)$ satisfying

$$\left.\begin{aligned}&\underset{\omega}{Maximize}\quad\tilde{\eta}(\boldsymbol{\omega}),\text{of }(70),\\&Subject\ to:\ \|\nabla\tilde{\alpha}(\boldsymbol{\omega})\|\le æ,\\&\varepsilon\omega<2\pi,\\&\frac{2\pi}{T_0}\ge\varpi\ge\sqrt[3]{C_{mn}(\omega,\varpi,\varepsilon)}\ \ \omega>>0,\end{aligned}\right\}\tag{76}$$

Where $æ$ is the tolerance associated with truncating the infinite harmonic sums in (74) after the M and N numbers, and $\|\nabla\tilde{\alpha}(\boldsymbol{\omega})\|$ is some norm of the gradient $\nabla\tilde{\alpha}(\boldsymbol{\omega})$ vector in the ω–space. The search for such a point ℑ, for the optimal **ω***, is expected to be carried out numerically via conjugate gradients or variable metric methods [12], which both require computations of the gradient ∇α̃(ℑ). The nonlinear inequality constraint $\varepsilon<\frac{2\pi}{2}$ means that the time advance should not exceed the period of the modulated reference neutron beam, otherwise if $\varepsilon=k\frac{2\pi}{\omega}$, k = 1, 2, 3, ...it becomes redundant. Although a direct solution of (71) is possible analytically (essentially due to simplicity of (74)), but the emerging expressions turn out to become formidably long to be put down in writing. This situation happens to simplify considerably when ω for the left beam is fixed as ω° (in a special optimization process) to reduce the size of the control vector (ω,ϖ,ε) to $(\omega^\circ,\varpi,\varepsilon)\triangleq(\varpi,\varepsilon)$.

Remark 2. In some cancer patients, transport of thermal neutrons by neutron guides or neutron optical fibres through the regions Λ and Π (with respective thicknesses l_Λ and l_Π and neutron macroscopic removal cross sections Σ_Λ and Σ_Π), may turn out to be medically unfeasible. As an approximate substitute to solving the composite RU(ΛUΠ) regional neurtronics problem in such cases, one can simply assume a planar attenuation towards *R*, of all (or some of) the sources $S(x,\ t)$ and $Ş(x,\ t)$ respectively to $S^{\looparrowright}(x,t)=\mathrm{V}_\Lambda S(x,t)=e^{-\Sigma_\Lambda l_\Lambda}S(x,t)$ and $Ş^{\looparrowright}(x,t)=\mathrm{V}_\Pi Ş(x,t)=e^{-\Sigma_\Pi l_\Pi}Ş(x,t)$.

As a result, the entire analysis, reported in this paper, should hold true if we replace each affected $\mathring{a}_m=\chi_{R;\Lambda}\mathrm{a}_m$ and $\tilde{b}_m=\chi_{R;\Pi}b_m$ respectively with $\breve{\mathrm{a}}_\mathrm{m}=\mathrm{V}_\Lambda\ \mathring{a}_m$ and $\breve{b}_\mathrm{m}=\mathrm{V}_\Pi\ \tilde{b}_m$.

Concluding Remarks

The basic result of this paper has been its demonstration, for the first time, that (ω,ϖ,ε) is employable as a control vector in the formulation of a nonlinear optimization process that can maximize the therapeutic utility index $\tilde{\eta}$ and the ballistic index $\tilde{\alpha}$ for this kind of dynamical (B/Gd) NCT.

Possible polarization of neutrons in both regions Λ and Π has the same qualitative impact on the migration of neutrons in *R* of a single region. Namely,

(i)Increasing the albedo ϱ_Λ, which increases $\chi_{R;\Lambda}$, plus increasing the albedo ϱ_Π, which increases $\chi_{R;\Pi}$. Accordingly a $\vec{\chi}_{R;\Lambda}$ and $\vec{\chi}_{R;\Pi}$, corresponding to polarizing Λ and Π, becomes conceivable in distinction from $\chi_{R;\Lambda}$ and $\chi_{R;\Pi}$ respectively.

(ii)Additive (possibly duplicated) decreasing in the value of *D* in *R*, to $\vec{D}$, corresponding to combined neutron polarization in Λ and Π.

Clearly Remark 6.1 of [1] remains to hold here as well. Indeed, since $\tilde{\alpha}(\omega,\varpi,\varepsilon)$ is apparently sensitive to variations in *D* then, the increased difference $(D-\vec{D})$, due to neutron polarization in both Λ and Π, can produce more significant difference in $\tilde{\eta}(\omega,\varpi,\varepsilon)$, across the *R* region, especially at high ϖ modulation frequencies.

At this point, it should be noted that the analysis of the rather simple present setup with two opposing modulated neutron beams admits a welter of approximations; and that is not unusual when it comes to analysis of wave-like phenomena in composites. The therapeutic B/Gd) NCT setup for a two-dimensional finite cancerous *R* with two non-opposing modulated neutron beams poses, by far, a more complicated problem than the present one. Carrying the present analysis further over tensored two-dimensional diffusion fields, we plan to deal with such a problem in a future paper.

Finally, the optimal control vector (ω*,ϖ*,ε*) of this paper is hoped to lead to an improved ballistic index $\tilde{\alpha}(\omega,\varpi,\varepsilon)$ and utility index $\tilde{\eta}(\omega,\varpi,\varepsilon)$ of the two opposing neutron beams setup over the respective indices α(ω) and η(ω) of the one beam setup. The same objective can possibly also be achieved via a clinically feasible profiling of the (B/Gd) uptake inside the tumor. Indeed, a spatially-dependent $\Sigma_a(x)$, instead of the present constant Σ_a, that is maximal at $x = l/2$ and minimal at $x = 0$ and $x = l$, could turn out to be ideal for boosting up the ballistic index of this NCT. Clearly then, research into the associated pharmacology and biochemistry of B/Gd distributed uptakes, and into histology of pertaining tissue-engineered tumors, could be not only relevant, but even quite essential.

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