

On jets, extensions and characteristic classes I

Helge MAAKESTAD

Oslo, Norway

E-mail: h_maakestad@hotmail.com

Abstract

In this paper, we give general definitions of non-commutative jets in the local and global situation using square zero extensions and derivations. We study the functors $\text{Exan}_k(A, I)$, where A is any k -algebra, and I is any left and right A -module and use this to construct affine non-commutative jets. We also study the Kodaira-Spencer class $\text{KS}(\mathcal{L})$ and relate it to the Atiyah class.

2000 MSC: 14F10, 14F40

1 Introduction

In this paper, we give general definitions of non-commutative jets in the local and global situation using square zero extensions and derivations. We study the functors $\text{Exan}_k(A, I)$, where A is any k -algebra, and I is any left and right A -module and use this to construct affine non-commutative jets. In the final section of the paper, we define and prove basic properties of the Kodaira-Spencer class $\text{KS}(\mathcal{L})$ and relate it to the Atiyah class.

2 Jets, liftings, and small extensions

We give an elementary discussion of structural properties of square zero extensions of arbitrary associative unital k -algebras. We introduce for any k -algebra A and any left and right A -module I the set $\text{Exan}_k(A, I)$ of isomorphism classes of square zero extensions of A by I and show it is a left and right module over the center $C(A)$ of A . This structure generalize the structure as left $C(A)$ -module introduced in [3]. We also give an explicit construction of $\text{Exan}_k(A, I)$ in terms of cocycles. Finally, we give a direct construction of non-commutative jets and generalized Atiyah sequences using derivations and square zero extensions.

Let in the following k be a fixed base field, and let

$$0 \longrightarrow I \xrightarrow{i} B \xrightarrow{p} A \longrightarrow 0$$

be an exact sequence of associative unital k -algebras with $i(I)^2 = 0$. Assume s is a map of k -vector spaces with the following properties:

$$s(1) = 1,$$

and

$$p \circ s = \text{id}.$$

Such a section always exists since B and A are vector spaces over the field k . Note: s gives the ideal I a left and right A -action.

Lemma 2.1. *There is an isomorphism:*

$$B \cong I \oplus A$$

of k -vector spaces.

Proof. Define the following maps of vector spaces: $\phi : B \rightarrow I \oplus A$ by $\phi(x) = (x - sp(x), p(x))$ and $\psi : I \oplus A \rightarrow B$ by $\psi(u, x) = u + s(x)$. It follows that $\psi \circ \phi = \text{id}$ and $\phi \circ \psi = \text{id}$ and the claim of the proposition follows. \square

Define the following element:

$$\tilde{C} : A \times A \longrightarrow I,$$

by

$$\tilde{C}(x \times y) = s(x)s(y) - s(xy).$$

It follows that $\tilde{C} = 0$ if and only if s is a ring homomorphism.

Lemma 2.2. *The map \tilde{C} gives rise to an element $C \in \text{Hom}_k(A \otimes_k A, I)$.*

Proof. We easily see that $\tilde{C}(x+y, z) = \tilde{C}(x, z) + \tilde{C}(y, z)$ and $\tilde{C}(x, y+z) = \tilde{C}(x, y) + \tilde{C}(x, z)$ for all $x, y, z \in A$. Moreover, for any $a \in k$, it follows that

$$\tilde{C}(ax, y) = \tilde{C}(x, ay) = a\tilde{C}(x, y).$$

Hence we get a well-defined element $C \in \text{Hom}_k(A \otimes_k A, I)$ as claimed. \square

Define the following product on $I \oplus A$:

$$(u, x) \times (v, y) = (uy + xv + C(x, y), xy). \quad (2.1)$$

We let $I \oplus^C A$ denote the abelian group $I \oplus A$ with product defined by (2.1).

Proposition 2.3. *The natural isomorphism:*

$$B \cong I \oplus A$$

of vector spaces is a unital ring isomorphism if and only if the following holds:

$$xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z = 0$$

for all $x, y, z \in A$.

Proof. We have defined two isomorphisms of vector spaces ϕ, ψ :

$$\phi(x) = (x - sp(x), p(x)),$$

and

$$\psi(u, x) = u + s(x).$$

We define a product on the direct sum $I \oplus A$ using ϕ and ψ :

$$(u, x) \times (v, y) = \phi(\psi(u, x)\psi(v, y)) = \phi((u + s(x))(v + s(y)))$$

$$\begin{aligned}
&= \phi(uv + us(y) + s(x)v + s(x)s(y)) \\
&= (us(y) + s(x)v + s(x)s(y) - s(xy), xy) \\
&= (uy + xv + C(x, y), xy).
\end{aligned}$$

Here, we define

$$uy = us(y),$$

and

$$xv = s(x)v.$$

One checks that

$$\phi(1) = (1 - sp(1), 1) = (0, 1) = \mathbf{1},$$

and

$$\mathbf{1}(u, x) = (u, x)\mathbf{1} = (u, x)$$

for all $(u, x) \in I \oplus A$. It follows that the morphism ϕ is unital. Since $C(x + y, z) = C(x, z) + C(y, z)$ and $C(x, y + z) = C(x, y) + C(x, z)$ the following holds:

$$(u, x)((v, y) + (w, z)) = (u, x)(v, y) + (u, x)(w, z),$$

and

$$((v, y) + (w, z))(u, x) = (v, y)(u, x) + (w, z)(u, x).$$

Hence, the multiplication is distributive over addition. Hence for an arbitrary section s of p of vector spaces mapping the identity to the identity, it follows the multiplication defined above always has a left and right unit and is distributive. We check when the multiplication is associative:

$$((u, x)(v, y))(w, z) = (uyz + xvz + xyw + C(x, y)z + C(xy, z), xyz).$$

Also,

$$(u, x)((v, y)(w, z)) = (uyz + xvz + xyw + xC(y, z) + C(x, yz), xyz).$$

It follows that the multiplication is associative if and only if the following equation holds for the element C :

$$xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z = 0$$

for all $x, y, z \in A$. The claim follows. \square

Let

$$xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z = 0 \tag{2.2}$$

be the *cocycle condition*.

Definition 2.4. Let $\text{exan}_k(A, I)$ be the set of elements $C \in \text{Hom}_k(A \otimes_k A, I)$ satisfying the cocycle condition (2.2).

Proposition 2.5. Equation (2.2) holds for all $x, y, z \in A$:

Proof. We get,

$$\begin{aligned} xC(y, z) &= s(x)s(y)s(z) - s(x)s(yz), \\ C(xy, z) &= s(xy)s(z) - s(xyz), \\ C(x, yz) &= s(x)s(yz) - s(xyz), \end{aligned}$$

and

$$C(x, y)z = s(x)s(y)s(z) - s(xy)s(z).$$

We get

$$\begin{aligned} &xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z \\ &= s(x)s(y)s(z) - s(x)s(yz) - s(xy)s(z) + s(xyz) \\ &\quad + s(x)s(yz) - s(xyz) - s(x)s(y)s(z) + s(xy)s(z) \\ &= 0, \end{aligned}$$

and the claim follows. \square

Corollary 2.6. The morphism $\phi : B \rightarrow I \oplus^C A$ is an isomorphism of unital associative k -algebras.

Proof. This follows from Proposition 2.5 and Proposition 2.3. \square

Hence, there is always a commutative diagram of exact sequences:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow = & & \downarrow \cong & & \downarrow = \\ 0 & \longrightarrow & I & \xrightarrow{i} & I \oplus^C A & \xrightarrow{p} & A \longrightarrow 0 \end{array}$$

where the middle vertical morphism is an isomorphism associative unital k -algebras.

Define the following left and right A -action on the ideal I :

$$xu = s(x)u, \quad ux = us(x),$$

where s is the section of p and $x \in A$, $u \in I$. Recall $I^2 = 0$.

Proposition 2.7. The actions defined above give the ideal I a left and right A -module structure. The structure is independent of choice of section s .

Proof. One checks that for any $x, y \in A$ and $u, v \in I$, the following holds:

$$(x + y)u = xu + yu, \quad x(u + v) = xu + xv, \quad 1u = u.$$

Also,

$$(xy)u - x(yu) = s(xy)u - s(x)s(y)u = (s(xy) - s(x)s(y))u = 0,$$

since $I^2 = 0$. It follows that $(xy)u = x(yu)$, hence I is a left A -module. A similar argument prove I is a right A -module. Assume t is another section of p . It follows that

$$s(x)u - t(x)u = (s(x) - t(x))u = 0,$$

since $I^2 = 0$. It follows that $s(x)u = t(x)u$. Similarly, $us(x) = ut(x)$ hence s and t induce the same structure of A -module on I and the proposition is proved. \square

We have proved the following theorem: let A be any associative unital k -algebra and let I be a left and right A -module. Let $C : A \otimes_k A \rightarrow I$ be a morphism satisfying the cocycle condition (2.2).

Theorem 2.8. *The exact sequence:*

$$0 \longrightarrow I \longrightarrow I \oplus^C A \longrightarrow A \longrightarrow 0$$

is a square zero extension of A with the module I . Moreover, any square zero extension of A with I arise this way for some morphism $C \in \text{Hom}_k(A \otimes_k A, I)$ satisfying equation (2.2).

Proof. The proof follows from the discussion above. \square

Let

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

with $i : I \rightarrow E$ and $p : E \rightarrow A$ and

$$0 \longrightarrow J \longrightarrow F \longrightarrow B \longrightarrow 0$$

with $j : J \rightarrow F$ and $q : F \rightarrow B$ be square zero extensions of associative k -algebras A, B with left and right modules I, J . This means the sequences are exact and the following holds $i(I)^2 = j(J)^2 = 0$. A triple (w, u, v) of maps of k -vector spaces giving rise to a commutative diagram of exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{i} & E & \xrightarrow{p} & A & \longrightarrow & 0 \\ & & \downarrow w & & \downarrow u & & \downarrow v & & \\ 0 & \longrightarrow & J & \xrightarrow{j} & F & \xrightarrow{q} & B & \longrightarrow & 0 \end{array}$$

is a morphism of extensions if u and v are maps of k -algebras and w is a map of left and right modules. This means

$$w(x + y) = w(x) + w(y), \quad w(ax) = v(a)w(x), \quad w(xa) = w(x)v(a)$$

for all $x, y \in I$ and $a \in A$.

We say two square zero extensions:

$$0 \longrightarrow I \longrightarrow E \longrightarrow A \longrightarrow 0$$

and

$$0 \longrightarrow I \longrightarrow F \longrightarrow A \longrightarrow 0$$

are *equivalent* if there is an isomorphism $\phi : E \rightarrow F$ of k -algebras making all diagrams commute.

Definition 2.9. Let $\text{Exan}_k(A, I)$ denote the set of all isomorphism classes of square zero extensions of A by I .

Theorem 2.10. Let $C(A)$ be the center of A . The set $\text{exan}_k(A, I)$ is a left and right module over $C(A)$. Moreover, there is a bijection:

$$\text{Exan}_k(A, I) \cong \text{exan}_k(A, I)$$

of sets.

Proof. We first prove that $\text{exan}_k(A, I)$ is a left and right $C(A)$ -module. Let $C, D \in \text{exan}_k(A, I)$. This means $C, D \in \text{Hom}_k(A \otimes_k A, I)$ are elements satisfying the cocycle condition (2.2). Let $a, b \in C(A) \subseteq A$ be elements. Define aC, Ca as follows:

$$(aC)(x, y) = aC(x, y),$$

and

$$(Ca)(x, y) = C(x, y)a.$$

We see

$$\begin{aligned} x(aC)(y, z) - (aC)(xy, z) + (aC)(x, yz) - (aC)(x, y)z \\ = a(xC(y, z) - C(xy, z) + C(x, yz) - C(x, y)z) = a(0) = 0, \end{aligned}$$

hence $aC \in \text{exan}_k(A, I)$. Similarly, one proves $Ca \in \text{exan}_k(A, I)$ hence we have defined a left and right action of $C(A)$ on the set $\text{exan}_k(A, I)$. Given $C, D \in \text{exan}_k(A, I)$ define

$$(C + D)(x, y) = C(x, y) + D(x, y).$$

One checks that $C + D \in \text{exan}_k(A, I)$ hence $\text{exan}_k(A, I)$ has an addition operation. One checks the following hold:

$$\begin{aligned} a(C + D) &= aC + aD, & (C + D)a &= Ca + Da, \\ (a + b)C &= aC + bC, & C(a + b) &= Ca + Cb, \\ a(bC) &= (ab)C, & C(ab) &= (Ca)b, & 1C &= C1 = C, \end{aligned}$$

hence the set $\text{exan}_k(A, I)$ is a left and right $C(A)$ -module. Define the following map: let $[B] = [I \oplus^C A] \in \text{Exan}_k(A, I)$ be an equivalence class of a square zero extension. Define

$$\phi : \text{Exan}_k(A, I) \longrightarrow \text{exan}_k(A, I)$$

by

$$\phi[B] = \phi[I \oplus^C A] = C.$$

We prove this gives a well-defined map of sets. Assume $[I \oplus^C A]$ and $[I \oplus^D A]$ are two elements in $\text{Exan}_k(A, I)$. Note: we use brackets to denote isomorphism classes of extensions. The two extensions are equivalent if and only if there is an isomorphism:

$$f : I \oplus^C A \longrightarrow I \oplus^D A$$

of k -algebras such that all diagrams are commutative. This means that

$$f(u, x) = (u, x)$$

for all $(u, x) \in I \oplus^C A$. We get

$$f((u, x)(v, y)) = f(u, x)f(v, y).$$

This gives the equality:

$$(uy + xv + C(x, y), xy) = (uy + xv + D(x, y), xy)$$

for all $(u, x), (v, y) \in I \oplus^C A$. Hence, $\phi[I \oplus^C A] = C = D = \phi[I \oplus^D A]$, and the map ϕ is well defined. It is clearly an injective map. It is surjective by Theorem 2.8 and the claim of the theorem follows. \square

Theorem 2.10 shows that there is a structure of left and right $C(A)$ -module on the set of equivalence classes of extensions $\text{Exan}_k(A, I)$. The structure as left $C(A)$ -module agrees with the one defined in [3].

Let $\phi \in \text{Hom}_k(A, I)$. Let $C^\phi \in \text{Hom}_k(A \otimes_k A, I)$ be defined by

$$C^\phi(x, y) = x\phi(y) - \phi(xy) + \phi(x)y.$$

One checks that $C^\phi \in \text{exan}_k(A, I)$ for all $\phi \in \text{Hom}_k(A, I)$.

Definition 2.11. Let $\text{exan}_k^{\text{inn}}(A, I)$ be the subset of $\text{exan}_k(A, I)$ of maps C^ϕ for $\phi \in \text{Hom}_k(A, I)$.

Lemma 2.12. *The set $\text{exan}_k^{\text{inn}}(A, I) \subseteq \text{exan}_k(A, I)$ is a left and right sub $C(A)$ -module.*

Proof. The proof is left to the reader as an exercise. \square

Definition 2.13. Let $\text{Exan}_k^{\text{inn}}(A, I) \subseteq \text{Exan}_k(A, I)$ be the image of $\text{exan}_k^{\text{inn}}(A, I)$ under the bijection $\text{exan}_k(A, I) \cong \text{Exan}_k(A, I)$.

It follows that $\text{Exan}_k^{\text{inn}}(A, I) \subseteq \text{Exan}_k(A, I)$ is a left and right sub $C(A)$ -module.

Recall the definition of the *Hochschild complex* as follows.

Definition 2.14. Let A be an associative k -algebra, and let I be a left and right A -module. Let $C^p(A, I) = \text{Hom}_k(A^{\otimes p}, I)$. Let $d^p : C^p(A, I) \rightarrow C^{p+1}(A, I)$ be defined as follows:

$$\begin{aligned} d^p(\phi)(a_1 \otimes \cdots \otimes a_{p+1}) &= a_1\phi(a_2 \otimes \cdots \otimes a_{p+1}) \\ &\quad + \sum_{1 \leq i \leq p} (-1)^i \phi(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) \\ &\quad + (-1)^{p+1} \phi(a_1 \otimes \cdots \otimes a_p) a_{p+1}. \end{aligned}$$

We let $\text{HH}^i(A, I)$ denote the i 'th cohomology of this complex. It is the i 'th *Hochschild cohomology* of A with values in I .

Proposition 2.15. *There is an exact sequence:*

$$0 \longrightarrow \text{Exan}_k^{\text{inn}}(A, I) \longrightarrow \text{Exan}_k(A, I) \longrightarrow \text{HH}^2(A, I) \longrightarrow 0$$

of left and right $C(A)$ -modules.

Proof. The proof is left to the reader as an exercise. \square

Example 2.16. Characteristic classes of L -connections.

Let A be a commutative k -algebra and let $\alpha : L \rightarrow \text{Der}_k(A)$ be a Lie-Rinehart algebra. Let W be a left A -module with an L -connection $\nabla : L \rightarrow \text{End}_k(W)$. In [6], we define a characteristic class $c_1(E) \in H^2(L|_U, \mathcal{O}_U)$ when W is of finite presentation, $U \subseteq \text{Spec}(A)$ is the open set, where W is locally free, and $H^2(L|_U, \mathcal{O}_U)$ is the Lie-Rinehart cohomology of $L|_U$ with values in \mathcal{O}_U . If L is locally free, it follows that $H^2(L, A) \cong \text{Ext}_{U(L)}^2(A, A)$, where $U(L)$ is the generalized universal enveloping algebra of L . There is an obvious structure of left and right $U(L)$ -module on $\text{End}_k(A)$ and an isomorphism:

$$\text{HH}^2(U(L), \text{End}_k(A)) \cong \text{Ext}_{U(L)}^2(A, A)$$

of abelian groups. The exact sequence 2.15 gives a sequence:

$$\begin{aligned} 0 &\longrightarrow \text{Exan}_k^{\text{inn}}(U(L), \text{End}_k(A)) \longrightarrow \text{Exan}_k(U(L), \text{End}_k(A)) \\ &\longrightarrow \text{Ext}_{U(L)}^2(A, A) \longrightarrow 0 \end{aligned}$$

with $A = U(L)$ and $I = \text{End}_k(A)$. If we can construct a lifting:

$$\tilde{c}_1(W) \in \text{Exan}_k(U(L), \text{End}_k(A))$$

of the class:

$$c_1(W) \in \text{Ext}_{U(L)}^2(A, A) = \text{HH}^2(U(L), \text{End}_k(A)),$$

we get a generalization of the characteristic class from [6] to arbitrary Lie-Rinehart algebras L . This problem will be studied in a future paper on the subject (see [7]).

Example 2.17. Non-commutative Kodaira-Spencer maps.

Let A be an associative k -algebra, and let M be a left A -module. Let $D^1(A) \subseteq \text{End}_k(A)$ be the *module of first-order differential operators* on A . It is defined as follows: an element $\partial \in \text{End}_k(A)$ is in $D^1(A)$ if and only if $[\partial, a] \in D^0(A) = A \subseteq \text{End}_k(A)$ for all $a \in A$. Define the following map:

$$f : D^1(A) \longrightarrow \text{Hom}_k(A, \text{End}_k(M))$$

by

$$f(\partial)(a, m) = [\partial, a]m = (\partial(a) - a\partial(1))m.$$

Here, $\partial \in D^1(A)$, $a \in A$, and $m \in M$. Since $[\partial, a] \in A$, we get a well-defined map. Let for any $a \in A$ and $m \in M$ $\phi_a(m) = am$. It follows $\phi_a \in \text{End}_k(M)$ is an endomorphism of M . We get

$$\begin{aligned} f(\partial)(ab, m) &= (\partial(ab) - ab\partial(1))m = (\partial\phi_{ab} - \phi_{ab}\partial)(1)m \\ &= (\partial\phi_{ab} - \phi_a\partial\phi_b + \phi_a\partial\phi_b - \phi_{ab}\partial)(1)m \\ &= (\partial\phi_a - \phi_a\partial)\phi_b(1)m + \phi_a(\partial\phi_b - \phi_b\partial)(1)m \\ &= f(\partial)(a, bm) + af(\partial)(b, m). \end{aligned}$$

Hence,

$$f(\partial)(ab) = af(\partial)(b) + f(\partial)(a)b$$

for all $\partial \in D^1(A)$ and $a, b \in A$. The Hochschild complex gives a map:

$$d^1 : \text{Hom}_k(A, \text{End}_k(M)) \longrightarrow \text{Hom}_k(A \otimes A, \text{End}_k(M)),$$

and

$$\ker(d^1) = \text{Der}_k(A, \text{End}_k(M)).$$

It follows that we get a map:

$$f : D^1(A) \longrightarrow \text{Der}_k(A, \text{End}_k(M)).$$

We get an induced map:

$$f : D^1(A) \longrightarrow \text{HH}^1(A, \text{End}_k(M)) = \text{Ext}_A^1(M, M).$$

Lemma 2.18. *The following holds $f(D^0(A)) = f(A) = 0$*

Proof. The proof is left to the reader as an exercise. □

One checks that $D^1(A)/D^0(A) = D^1(A)/A \cong \text{Der}_k(A)$. It follows that we get an induced map:

$$g : \text{Der}_k(A) = D^1(A)/D^0(A) \longrightarrow \text{Ext}_A^1(M, M),$$

the *non-commutative Kodaira-Spencer map*.

Lemma 2.19. *Assume A is commutative. The following hold:*

$$\mathbb{V}_M = \ker(g) \subseteq \text{Der}_k(A) \text{ is a Lie-Rinehart algebra,} \tag{2.3}$$

$$g(\delta) = 0 \iff \exists \phi \in \text{End}_k(M), \phi(am) = a\phi(m) + \delta(a)m, \tag{2.4}$$

$$\exists \nabla \in \text{Hom}_k(\mathbb{V}_M, \text{End}_k(M)) \text{ with } \nabla(\delta)(am) = a\nabla(\delta)(m) + \delta(a)m, \tag{2.5}$$

$$\mathbb{V}_M \text{ is the maximal Lie-Rinehart algebra satisfying (2.5).} \tag{2.6}$$

Proof. We first prove (2.3): assume $g(\delta) = g(\eta) = 0$. By definition, this is if and only if there are maps $\phi, \psi \in \text{End}_k(M)$ such that the following hold:

$$d^0\phi = g(\delta), \tag{2.7}$$

$$d^0\psi = g(\eta). \tag{2.8}$$

One checks that conditions (2.7) and (2.8) hold if and only if the following hold:

$$\phi(am) = a\phi(m) + \delta(a)m,$$

and

$$\psi(am) = a\psi(m) + \eta(a)m.$$

We claim $d^0[\delta, \eta] = g([\delta, \eta])$. We get

$$\begin{aligned} [\phi, \psi](am) &= \phi\psi(am) - \psi\phi(am) = \phi(a\psi(m) + \eta(a)m) - \psi(a\phi(m) + \delta(a)m) \\ &= a\phi\psi(m) + \delta(a)\psi(m) + \eta(a)\phi(m) + \delta\eta(a)m - a\psi\phi(m) - \eta(a)\phi(m) \\ &\quad - \delta(a)\psi(m) - \eta\delta(a)m = a[\phi, \psi](m) + [\delta, \eta](a)m. \end{aligned}$$

Hence, $g([\delta, \eta]) = 0$ and $\mathbb{V}_M \subseteq \text{Der}_k(A)$ is a k -Lie algebra. It is an A -module since g is A -linear, hence it is a Lie-Rinehart algebra. Claim (2.3) is proved. Claim (2.4) and (2.5) follows from the proof of (2.3). Claim (2.6) is obvious and the lemma is proved. \square

The Lie-Rinehart algebra \mathbb{V}_M is the *linear Lie-Rinehart algebra* of M .

Let in the following E be a left and right A -module.

Definition 2.20. Let

$$\mathcal{J}_I^1(E) = I \otimes_A E \oplus E$$

be the *first-order I -jet bundle* of E .

Pick a derivation $d \in \text{Der}_k(A, I)$ of left and right modules. This means that

$$d(xy) = xd(y) + d(x)y$$

for all $x, y \in A$. Let $B^C = I \oplus^C A$ and define the following left B^C -action on $\mathcal{J}_I^1(E)$:

$$(u, x)(w \otimes e, f) = (u \otimes f + xw \otimes e + d(x) \otimes f, xf)$$

for any elements $(u, x) \in B^C$ and $(w \otimes e, f) \in \mathcal{J}_I^1(E)$.

Proposition 2.21. *The abelian group $\mathcal{J}_I^1(E)$ is a left B^C -module if and only if $C(y, x) \otimes f = 0$ for all $y, x \in A$ and $f \in E$.*

Proof. One easily checks that for any $a, b \in B^C$ and $i, j \in \mathcal{J}_I^1(E)$ the following hold:

$$\begin{aligned} (a + b)i &= ai + bi, \\ a(i + j) &= ai + aj. \end{aligned}$$

Moreover,

$$\mathbf{1}i = i.$$

It remains to check that $a(bi) = (ab)i$. Let $a = (v, y) \in B^C$ and $b = (u, x) \in B^C$. Let also $i = (w \otimes e, f) \in \mathcal{J}_I^1(E)$. We get

$$a(bi) = (v, y)((u, x)(w \otimes e, f)) = (vx \otimes f + yu \otimes f + yxw \otimes e + d(yx) \otimes f, yxf).$$

We also get

$$(ab)i = (vx \otimes f + yu \otimes f + yxw \otimes e + d(yx) \otimes f + C(y, x) \otimes f, yxf).$$

It follows that

$$(ab)i - a(bi) = 0,$$

if and only if

$$C(y, x) \otimes f = 0,$$

and the claim of the proposition follows. \square

Note the abelian group $\mathcal{J}_I^1(E)$ is always a left A -module and there is an exact sequence of left A -modules:

$$0 \longrightarrow I \otimes E \longrightarrow \mathcal{J}_I^1(E) \longrightarrow E \longrightarrow 0,$$

defining a characteristic class:

$$c_I(E) \in \text{Ext}_A^1(E, E \otimes I).$$

The class $c_I(E)$ has the property that $c_I(E) = 0$ if and only if E has an I -connection:

$$\nabla : E \longrightarrow I \otimes E$$

with

$$\nabla(xe) = x\nabla(e) + d(x) \otimes e.$$

Let $J \subseteq I \subseteq B^C$ be the smallest two-sided ideal containing $\text{Im}(C)$, where $C : A \otimes_k A \rightarrow I$ is the cocycle defining B^C . Let $D^C = B^C/J$ and $I^C = I/J$. We get a square zero extension:

$$0 \longrightarrow I^C \longrightarrow D^C \longrightarrow A \longrightarrow 0$$

of A by the square zero ideal I^C . It follows that $D^C = I^C \oplus A$ as abelian group. Since $\overline{C(x, y)} = 0$ in I^C , it follows that D^C has a well-defined associative multiplication defined by

$$(u, x)(v, y) = (uy + xv, xy).$$

Also D^C is the largest quotient of B^C such that the ring homomorphism $B^C \rightarrow D^C$ fits into a commutative diagram of square zero extensions:

$$\begin{array}{ccccccc} 0 & \longrightarrow & I & \longrightarrow & B^C & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow = \\ 0 & \longrightarrow & I^C & \longrightarrow & D^C & \longrightarrow & A \longrightarrow 0. \end{array}$$

Definition 2.22. Let

$$\mathcal{J}_{I^C}^1(E) = I^C \otimes E \oplus E$$

be the *first-order I^C -jet bundle of E* .

Example 2.23. First-order commutative jets.

Let $k \rightarrow A$ be a commutative k -algebra, and let $I \subseteq A \otimes_k A$ be the ideal of the diagonal. Let $\mathcal{J}_A^1 = A \otimes A/I^2$ and $\Omega_A^1 = I/I^2$. We get an exact sequence of left A -modules:

$$0 \longrightarrow \Omega_A^1 \longrightarrow \mathcal{J}_A^1 \longrightarrow A \longrightarrow 0. \quad (2.9)$$

It follows that $\mathcal{J}_A^1 \cong \Omega_A^1 \oplus A$ with the following product:

$$(\omega, a)(\eta, b) = (\omega a + b\eta, ab),$$

hence the sequence (2.9) splits. Let $\mathcal{J}_A^1(E) = \Omega_A^1 \otimes E \oplus E$ be the first-order Ω_A^1 -jet of E . We get an exact sequence of left A -modules:

$$0 \longrightarrow \Omega_A^1 \otimes E \longrightarrow \mathcal{J}_A^1(E) \longrightarrow E \longrightarrow 0.$$

Since the sequence (2.9) splits, it follows that $\mathcal{J}_A^1(E)$ is a lifting of E to the first-order jet \mathcal{J}_A^1 .

3 Atiyah classes and Kodaira-Spencer classes

In this section, we define and prove some properties of Atiyah classes and Kodaira-Spencer classes.

Let X be any scheme defined over an arbitrary basefield F and let $\text{Pic}(X)$ be the *Picard group* of X . Let $\mathcal{O}^* \subseteq \mathcal{O}_X$ be the following subsheaf of abelian groups: for any open set $U \subseteq X$, the group $\mathcal{O}(U)^*$ is the multiplicative group of units in $\mathcal{O}_X(U)$. Define for any open set $U \subseteq X$ the following morphism:

$$\text{dlog} : \mathcal{O}(U)^* \longrightarrow \Omega_X^1(U),$$

defined by

$$\text{dlog}(x) = d(x)/x,$$

where d is the universal derivation and $x \in \mathcal{O}(U)^*$.

Lemma 3.1. *The following holds:*

$$\text{dlog}(xy) = \text{dlog}(x) + \text{dlog}(y)$$

for $x, y \in \mathcal{O}(U)^*$

Proof. The proof is left to the reader as an exercise. □

Hence, $\text{dlog} : \mathcal{O}^* \rightarrow \Omega_X^1$ defines a map of sheaves of abelian groups. The map dlog induces a map on cohomology

$$\text{dlog} : \text{Pic}(X) = H^1(X, \mathcal{O}^*) \longrightarrow H^1(X, \Omega_X^1),$$

and by definition

$$\text{dlog}(\mathcal{L}) = c_1(\mathcal{L}) \in H^1(X, \Omega_X^1).$$

Let $\mathcal{I} \subseteq \Omega_X^1$ be any sub \mathcal{O}_X -module, and let $\mathcal{F} = \Omega_X^1/\mathcal{I}$ be the quotient sheaf. We get a derivation:

$$d : \mathcal{O}_X \longrightarrow \mathcal{F}$$

by composing with the universal derivation. We get a canonical map:

$$H^1(X, \Omega_X^1) \longrightarrow H^1(X, \mathcal{F}),$$

and we let

$$\bar{c}_1(\mathcal{L}) \in H^1(X, \mathcal{F})$$

be the image of $c_1(\mathcal{L})$ under this map.

Definition 3.2. The class $c_1(\mathcal{L}) \in H^1(X, \Omega_X^1)$ is the *first Chern class* of the line bundle $\mathcal{L} \in \text{Pic}(X)$. The class $\bar{c}_1(\mathcal{L}) \in H^1(X, \mathcal{F})$ is the *generalized first Chern class* of \mathcal{L} .

Let \mathcal{E} be any \mathcal{O}_X -module and consider the following sequence of sheaves of abelian groups:

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathcal{J}_{\mathcal{F}}^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0,$$

where

$$\mathcal{J}_{\mathcal{F}}^1(\mathcal{E}) = \mathcal{F} \otimes \mathcal{E} \oplus \mathcal{E}$$

as sheaf of abelian groups. Let s be a local section of \mathcal{O}_X , and let $(x \otimes e, f)$ be a local section of $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E})$ over some open set U . Make the following definition:

$$s(x \otimes e, f) = (sx \otimes e + ds \otimes f, sf).$$

It follows that the sequence

$$0 \longrightarrow \mathcal{F} \otimes \mathcal{E} \longrightarrow \mathcal{J}_{\mathcal{F}}^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0$$

is a short exact sequence of sheaves of abelian groups. It is called the *Atiyah-Karoubi sequence*.

Definition 3.3. An \mathcal{F} -connection ∇ is a map:

$$\nabla : \mathcal{E} \longrightarrow \mathcal{F} \otimes \mathcal{E}$$

of sheaves of abelian groups with

$$\nabla(se) = s\nabla(e) + d(s) \otimes e.$$

Proposition 3.4. *The Atiyah-Karoubi sequence is an exact sequence of left \mathcal{O}_X -modules. It is left split by an \mathcal{F} -connection.*

Proof. We first show that it is an exact sequence of left \mathcal{O}_X -modules. The \mathcal{O}_X -module structure is twisted by the derivation d , hence we must verify that this gives a well-defined left \mathcal{O}_X -structure on $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E})$. Let $\omega = (x \otimes e, f)$ be a local section of $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E})$, and let s, t be local sections of \mathcal{O}_X . We get the following calculation:

$$\begin{aligned} (st)\omega &= (st)(x \otimes e, f) = ((st)x \otimes e + d(st) \otimes f, (st)f) \\ &= (stx \otimes e + sdt \otimes f + (ds)t \otimes f, stf) = (s(tx \otimes e + dt \otimes f) + ds \otimes tf, s(tf)) \\ &= s(tx \otimes e + dt \otimes f, tf) = s(t(x \otimes e, f)) = s(t\omega). \end{aligned}$$

It follows that $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E})$ is a left \mathcal{O}_X -module and the sequence is left exact. Assume that

$$s : \mathcal{E} \longrightarrow \mathcal{J}_{\mathcal{F}}^1(\mathcal{E}) = \mathcal{F} \otimes \mathcal{E} \oplus \mathcal{E}$$

is a left splitting. It follows that $s(e) = (\nabla(e), e)$ for e a local section of \mathcal{E} . It follows that ∇ is a generalized connection and the theorem is proved. \square

Note: If $\mathcal{I} = 0$, we get that $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E}) = \mathcal{J}_X^1(\mathcal{E})$ is the first-order jet bundle of \mathcal{E} and the exact sequence above specializes to the well-known *Atiyah sequence*:

$$0 \longrightarrow \Omega_X^1 \otimes \mathcal{E} \longrightarrow \mathcal{J}_X^1(\mathcal{E}) \longrightarrow \mathcal{E} \longrightarrow 0.$$

The Atiyah sequence is left split by a connection:

$$\nabla : \mathcal{E} \longrightarrow \Omega_X^1 \otimes \mathcal{E}.$$

The \mathcal{O}_X -module $\mathcal{J}_{\mathcal{F}}^1(\mathcal{E})$ is the *generalized first-order jet bundle* of \mathcal{E} .

Definition 3.5. The characteristic class:

$$\text{AT}(\mathcal{E}) \in \text{Ext}_{\mathcal{O}_X}^1(\mathcal{E}, \mathcal{F} \otimes \mathcal{E})$$

is called the *Atiyah class* of \mathcal{E} .

The class $\text{AT}(\mathcal{E})$ is defined for an arbitrary \mathcal{O}_X -module \mathcal{E} and an arbitrary sub module $\mathcal{I} \subseteq \Omega_X^1$.

Assume that $\mathcal{E} = \mathcal{L} \in \text{Pic}(X)$ is a line bundle on X . We get isomorphisms:

$$\begin{aligned} \text{Ext}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L} \otimes \mathcal{F}) &\cong \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \mathcal{L}^* \otimes \mathcal{L} \otimes \mathcal{F}) \\ &\cong \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \mathcal{F}) \longrightarrow \text{H}^1(X, \mathcal{F}). \end{aligned}$$

We get a morphism:

$$\phi : \text{Ext}_{\mathcal{O}_X}^1(\mathcal{L}, \mathcal{L} \otimes \mathcal{F}) \longrightarrow \text{H}^1(X, \mathcal{F}).$$

Proposition 3.6. *The following holds:*

$$\phi(\text{AT}(\mathcal{L})) = \bar{c}_1(\mathcal{L}).$$

Hence, the Atiyah class calculates the generalized first Chern class of a line bundle.

Proof. Let $\mathcal{I} = 0$. It is well known that $\text{AT}(\mathcal{L})$ calculates the first Chern class $c_1(\mathcal{L})$. From this, the claim of the proposition follows. \square

Let T_X be the tangent sheaf of X . It has the property that for any open affine set $U = \text{Spec}(A) \subseteq X$ the local sections $T_X(U)$ equal the module $\text{Der}_F(A)$ of derivations of A . Let $\mathbb{V}_{\mathcal{E}} \subseteq T_X$ be the subsheaf of local sections ∂ of T_X with the following property: the section $\partial \in T_X(U)$ lifts to a local section $\nabla(\partial)$ of $\text{End}_F(\mathcal{E}|_U)$ with the following property:

$$\nabla(\partial) : \mathcal{E}|_U \longrightarrow \mathcal{E}|_U$$

which satisfies

$$\nabla(\partial)(se) = s\nabla(\partial)(e) + \partial(s)e.$$

It follows that $\mathbb{V}_{\mathcal{E}} \subseteq T_X$ is a subsheaf of Lie algebras – the *Kodaira-Spencer sheaf* of \mathcal{E} .

Define for any local sections a, b of \mathcal{O}_X , ∂ of $\mathbb{V}_{\mathcal{E}}$ and e of \mathcal{E} the following:

$$L(a, \partial)(e) = a\nabla(\partial)(e) - \nabla(a\partial)(e).$$

Lemma 3.7. *It follows that $L(a, \partial) \in \text{End}_{\mathcal{O}_U}(\mathcal{E}|_U)$.*

Proof. The following holds:

$$\begin{aligned} L(a, \partial)(be) &= a\nabla(\partial)(be) - \nabla(a\partial)(be) \\ &= a(b\nabla(\partial)(e) + \partial(b)e) - b\nabla(a\partial)(e) - a\partial(b)e \\ &= ab\nabla(\partial)(e) + a\partial(b)e - b\nabla(a\partial)(e) - a\partial(b)e \\ &= b(a\nabla(\partial)(e) - \nabla(a\partial)(e)) = b(a\nabla(\partial) - \nabla(a\partial))(e) \\ &= bL(a, \partial)(e), \end{aligned}$$

and the lemma is proved. \square

Lemma 3.8. *The following formula holds:*

$$L(ab, \partial) = aL(b, \partial) + L(a, b\partial)$$

for all local sections a , b , and ∂ .

Proof. We get

$$\begin{aligned} L(ab, \partial) &= ab\nabla(\partial) - \nabla(ab\partial) \\ &= ab\nabla(\partial) - a\nabla(b\partial) + a\nabla(b\partial) - \nabla(ab\partial) \\ &= a(b\nabla(\partial) - \nabla(b\partial)) + (a\nabla - \nabla a)(b\partial) \\ &= aL(b, \partial) + L(a, b\partial), \end{aligned}$$

and the lemma is proved. \square

Let $\text{LR}(\mathbb{V}_{\mathcal{E}}) = \text{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}}$ be the *linear Lie-Rinehart algebra* of \mathcal{E} . Let $\text{LR}(\mathbb{V}_{\mathcal{E}})$ have the following left \mathcal{O}_X -module structure:

$$a(\phi, \partial) = (a\phi + L(a, \partial), a\partial).$$

Here, a , ϕ , and ∂ are local sections of \mathcal{O}_X , $\text{End}_{\mathcal{O}_X}(\mathcal{E})$, and $\mathbb{V}_{\mathcal{E}}$. We twist the trivial \mathcal{O}_X structure on $\text{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}}$ with the element L . We get a sequence of sheaves of abelian groups:

$$0 \longrightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E}) \xrightarrow{i} \text{LR}(\mathbb{V}_{\mathcal{E}}) \xrightarrow{p} \mathbb{V}_{\mathcal{E}} \longrightarrow 0,$$

where i and p are the canonical maps. An \mathcal{O}_X -linear map:

$$\nabla : \mathbb{V}_{\mathcal{E}} \longrightarrow \text{End}_F(\mathcal{E}),$$

satisfying

$$\nabla(\partial)(ae) = a\nabla(\partial)(e) + \partial(a)e$$

is a $\mathbb{V}_{\mathcal{E}}$ -connection on \mathcal{E} .

Proposition 3.9. *The sequence defined above is an exact sequence of left \mathcal{O}_X -modules. It is left split by a $\mathbb{V}_{\mathcal{E}}$ -connection ∇ .*

Proof. We need to check that $\text{LR}(\mathbb{V}_{\mathcal{E}})$ has a well-defined left \mathcal{O}_X -module structure. By definition,

$$a(\phi, \partial) = (a\phi + L(a, \partial), a\partial).$$

We get

$$\begin{aligned} (ab)x &= (ab)(\phi, \partial) = ((ab)\phi + L(ab, \partial), (ab)\partial) \\ &= (ab\phi + aL(b, \partial) + L(a, b\partial), ab\partial) \\ &= a(b\phi + L(b, \partial), b\partial) = a(b(\phi, \partial)) = a(bx), \end{aligned}$$

and it follows that the sequence is a left exact sequence of \mathcal{O}_X -modules. If

$$s : \mathbb{V}_{\mathcal{E}} \longrightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E}) \oplus \mathbb{V}_{\mathcal{E}} = \text{LR}(\mathbb{V}_{\mathcal{E}})$$

is a section, it follows that $s(e) = (\nabla(e), e)$. One checks that ∇ is a $\mathbb{V}_{\mathcal{E}}$ -connection, and the theorem is proved. \square

Definition 3.10. We get a characteristic class:

$$\text{KS}(\mathcal{E}) \in \text{Ext}_{\mathcal{O}_X}^1(\mathbb{V}_{\mathcal{E}}, \text{End}_{\mathcal{O}_X}(\mathcal{E})),$$

the *Kodaira-Spencer class* of \mathcal{E} .

Assume that $\mathbb{V}_{\mathcal{E}}$ is locally free and $\mathcal{E} = \mathcal{L} \in \text{Pic}(X)$ is a line bundle on X . Assume also that $\mathbb{V}_{\mathcal{E}}^* = \mathcal{F} = \Omega_X^1/\mathcal{I}$ for some submodule \mathcal{I} . We get the following calculation:

$$\begin{aligned} \text{Ext}_{\mathcal{O}_X}^1(\mathbb{V}_{\mathcal{E}}, \text{End}_{\mathcal{O}_X}(\mathcal{L})) &\cong \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \text{End}_{\mathcal{O}_X}(\mathcal{L}) \otimes \mathbb{V}_{\mathcal{E}}^*) \\ &\cong \text{Ext}_{\mathcal{O}_X}^1(\mathcal{O}_X, \text{End}_{\mathcal{O}_X}(\mathcal{L}) \otimes \mathcal{F}) \longrightarrow \text{H}^1(X, \mathcal{F}). \end{aligned}$$

We get a map:

$$\psi : \text{Ext}_{\mathcal{O}_X}^1(\mathbb{V}_{\mathcal{E}}, \text{End}_{\mathcal{O}_X}(\mathcal{L})) \longrightarrow \text{H}^1(X, \mathcal{F})$$

of sheaves.

Proposition 3.11. *The following holds: there is an equality:*

$$\psi(\text{KS}(\mathcal{L})) = \bar{c}_1(\mathcal{L})$$

in $\text{H}^1(X, \mathcal{F})$. Hence the Kodaira-Spencer class calculates the class $\bar{c}_1(\mathcal{L})$.

Proof. The proof is left to the reader as an exercise. □

We get the following diagram expressing the relationship between the characteristic classes defined above:

$$\begin{array}{ccc} \text{Ext}_{\mathcal{O}_X}^1(\mathbb{V}_{\mathcal{L}}, \text{End}_{\mathcal{O}_X}(\mathcal{L})) & & \\ & \searrow \psi & \\ & & \text{H}^1(X, \mathcal{F}) \xleftarrow{\bar{c}_1(-)} \text{Pic}(X) . \\ & \nearrow \phi & \\ \text{Ext}_{\mathcal{O}_X}^1(\mathcal{L}, \mathcal{F} \otimes \mathcal{L}) & & \end{array}$$

The following equation holds in $\text{H}^1(X, \mathcal{F})$:

$$\phi(\text{AT}(\mathcal{L})) = \psi(\text{KS}(\mathcal{L})) = \bar{c}_1(\mathcal{L}).$$

References

- [1] M. André. *Homologie des algèbres commutatives*. Die Grundlehren der mathematischen Wissenschaften, **206**, Springer-Verlag, Berlin, 1974.
- [2] M. Atiyah. Complex analytic connections in fibre bundles. *Trans. Amer. Math. Soc.*, **85** (1957), 181–207.
- [3] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale de schémas et des morphismes de schémas. *Inst. Hautes Études Sci. Publ. Math.*, **20** (1964), 101–355.
- [4] M. Karoubi. Homologie cyclique et K -theorie. *Astérisque*, no. 149 (1987), 147.
- [5] H. Maakestad. A note on principal parts on projective space and linear representations. *Proc. Amer. Math. Soc.*, **133** (2005), 349–355.

-
- [6] H. Maakestad. Chern classes and Lie-Rinehart algebras. *Indag. Math. (N.S.)*, **18** (2007), 589–599.
 - [7] H. Maakestad. Chern classes and Exan functors. Preprint 2009.
 - [8] H. Maakestad. Principal parts on the projective line over arbitrary rings. *Manuscripta Math.*, **126** (2008), 443–464.

Received November 09, 2009

Revised July 19, 2010