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On Remarkable Relations and the Passage to the Limit in the Theory of Infinite Systems

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Abstract

The present paper is about the problem of the passage to the limit from finite truncated systems to infinite system of linear algebraic equations. We consider the four important relations that arise in dealing with finite truncated Gaussian systems. These remarkable relations in fact give the opportunity to make transition from the solutions of finite systems to the solution of infinite system.

Keywords: Infinite system; Homogeneous; Inhomogeneous; Narrow sense; Broad sense; Strictly particular solution; Cramer's rule; Nontrivial solution

Introduction

Recently, we have discovered and described in detail a new class of infinite systems, called periodic class of infinite systems [1]. Namely, the elaboration of the theory of this class of systems enabled us to study the infinite systems with common positions and has recently allowed to move on from the critical point. In the author's review monograph [2] classes of infinite systems had been systematized and studied since their emergence as independent theory. This article focuses on the main and at the same time the most difficult issue, namely, the problem of passage to the limit from finite systems solution to infinite systems solution. Basic information, concepts and definitions of infinite systems, matrices and determinants can be studied in the articles [1-5].

There is an infinite system of linear algebraic equations with an infinite number of the unknown

$$\left.\begin{array}{l} a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + \ldots = b_1, \\ a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + \ldots = b_2, \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ a_{n,1}x_1 + a_{n,2}x_2 + \ldots + a_{n,n}x_n + \ldots = b_n, \\ \ldots\ldots\ldots\ldots\ldots\ldots \end{array}\right\} \qquad (1)$$

where $a_{j,i}$ – are known coefficients, b_j – are constant terms, x_i – are unknown quantities in a field F.

A set of numerical values $\bar{x}_1, \bar{x}_2$ ... is called a solution of system (1), if, after substituting these values in the left-hand side of (1) we obtain convergent series, and all of these equations will be satisfied, otherwise the $\{\bar{x}_1\}$ numbers will not be considered as solutions.

In the case of the solvability, the infinite system is called consistent, otherwise – inconsistent.

Under the infinite matrix we consider the table of coefficients of an infinite system (1):

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & \ldots & a_{1,n} & \ldots \\ a_{2,1} & a_{2,2} & \ldots & a_{2,n} & \ldots \\ . & . & \ldots & . & \ldots \\ a_{n,1} & a_{n,2} & \ldots & a_{n,n} & \ldots \\ . & . & \ldots & . & \ldots \end{pmatrix}, \qquad (2)$$

which is called the coefficient matrix of the system (1), and matrix

$$\bar{A} = \begin{pmatrix} b_1 & a_{1,1} & a_{1,2} & \ldots & a_{1,n} & \ldots \\ b_2 & a_{2,1} & a_{2,2} & \ldots & a_{2,n} & \ldots \\ . & . & . & \ldots & . & \ldots \\ b_n & a_{n,1} & a_{n,2} & \ldots & a_{n,n} & \ldots \\ . & . & . & \ldots & . & \ldots \end{pmatrix}, \qquad (3)$$

– the augmented matrix of the system (1).

To develop a general theory of infinite systems (1) we propose to use the reduction method as the major method for solving them [1,2], but not in the classical sense. So far, the reduction method is used only for solving a system for general form (1) [4,6]. In this case, an exact solution (without the use of the theory of determinants) of the finite truncated nth order system of (1) is impossible to obtain for any n. So, to solve the truncated system only approximate methods should be used, most often – the method of successive approximations. Therefore, in dealing with the general system (1) two approximate methods are simultaneously applied: reduction method and the method of successive approximations. Thus, in the case where it is impossible to obtain an exact solution of (1), it is difficult to say which one of these methods does not converge, and in finally, whether the system (1) is consistent or not? To answer this question, we introduced the concept of strictly particular solution of the infinite system (1) [7-9]. This strictly partial solution we obtain by the reduction method in the narrow sense, i.e. by a simple reduction method (see definition 1). To do this, it is necessary to find the exact solution of finite system of any order n by one algorithm. And this is possible only when we use the Gaussian elimination [10], which is always possible if an infinite determinant is nonzero. Therefore, here we assume that the infinite determinant of the system (1) is not zero.

Reduction method in the classical sense (simple reduction

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method) is not used for determining the nontrivial solutions (if it exists) of the corresponding homogeneous (reduced) system (1).

Therefore, we have introduced a different interpretation of the reduction method [1,2].

Definition 1: If in the reduction method for solving infinite systems of algebraic equations the number of unknowns and the number of equations remain the same in the truncated system, then we can say that reduction method is understood in the narrow sense (simple reduction), and if the number of unknowns is greater than the number of equations, then we say that the method of reduction is understood in a broad sense.

Definition 2: If the elements $a_{i,j}$ of the infinite matrix (2) is equal to zero for all $i > j$ and $a_{j,j} \neq 0$, then infinite matrix (2) is called *a Gaussian infinite matrix*, and its associated infinite system of linear algebraic equations is called an infinite Gaussian system.

Naturally, the reduction method in its different understanding can give different solutions to the same infinite system. Details on this will be reviewed in the next section. Here we note that the method of reduction in the narrow sense we use to obtain a strictly particular solution of the inhomogeneous infinite Gaussian system, and the method of reduction in the broad sense for solving a non-trivial solution of the homogeneous an infinite Gaussian system if it exists.

In this paper we will focus on some remarkable relations that arise in dealing with finite truncated Gaussian systems. These relations allow us to make transition from the solution of the truncated system to the solution of the corresponding infinite system. Most of the results were described in many of our earlier works, for example, in [7,9,11,12], but these results are shown there in order to solve specific problems of these papers. In the present paper these results are collected for one purpose: to answer the question: how to make the passage to the limit from the truncated Gaussian system solution to the solution of the general infinite system? Therefore, to maintain the integrity of the work here we repeat and clarify proofs of some theorems.

So, the infinite determinant $|A|$ is nonzero. Therefore, Gaussian elimination is possible [10], so instead of general infinite system (1), we solve an infinite Gaussian system ($a_{j,j} \neq 0$ for any j):

$$\sum_{p=0}^{\infty} a_{j,j+p} x_{j+p} = b_j, \quad j = 1,2,3,..., \tag{4}$$

with the following matrices, respectively the coefficient matrix A and the augmented matrix $\overline{A}$:

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & . & a_{1,n} & . \\ 0 & a_{2,2} & . & a_{2,n} & . \\ . & . & . & . & . \\ 0 & 0 & . & a_{n,n} & . \\ . & . & . & . & . \end{pmatrix}, \overline{A} = \begin{pmatrix} b_1 & a_{1,1} & . & a_{1,n} & . \\ b_2 & 0 & . & a_{2,n} & . \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ b_n & 0 & . & a_{n,n} & . \\ . & . & . & . & . \end{pmatrix}, \tag{5}$$

The Solution of the Finite Truncated Systems

Thus, only after changing the general infinite system (1) into infinite Gaussian system (4) we can apply the reduction method, namely in two of its aspects. First, system (4) will be solved by the method of reduction in the narrow sense, i.e. by simple reduction.

Theorem 1: Let the system (4) is truncated by the reduction method in the narrow sense into the finite Gaussian system of the form

$$\sum_{p=0}^{n-j} a_{j,j+p} \overset{n}{x}_{j+p} = b_j, \qquad a_{j,j} \neq 0, \qquad j = \overline{1,n}. \tag{6}$$

Then the solution of a finite system (6) is an expression:

$$\overset{n}{x}_j = B_{n-j}, \quad j = 1,2,...,n, \tag{7}$$

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=0}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p \quad B_0 = \frac{b_n}{a_{n,n}}, \ j = \overline{1, n-1}, \tag{8}$$

The proof is given in the work [9]. Here we only note that it is carried out by building recursive process (8), which is required in the transition to an infinite system. It is clear that this process demonstrates the meaning of reduction, because if it does not converge, the reduction method will not converge either.

Let us consider the homogenous infinite Gaussian system ($b_j \equiv 0$ for all j) (4). As shown in the examples [1,2], there exist nontrivial solutions of the homogeneous infinite systems. Moreover, the subspace of such solutions can be infinite-dimensional. But if we try to solve the homogeneous infinite Gaussian system (4) by the reduction in the narrow sense, i.e. with the use of Theorem 1, it is difficult to expect to obtain nontrivial solution. From the Theorem 1 it is pointed out that for each n we obtain the trivial solution, and it is likely that if n goes to infinity we will get only the trivial solution of the homogeneous infinite Gaussian system (4). Therefore we will solve the homogeneous infinite Gaussian system (4) by the method of reduction in the broad sense. It means that the finite truncated system for any n has at least one unknown with an arbitrary value. It is convenient to assume such an unknown to be, for example, x_1.

Theorem 2: Let the system (4) is truncated by the reduction method in the broad sense into the finite Gaussian system of the form

$$\sum_{p=0}^{n-j} a_{j,j+p} \overset{n}{x}_{j+p} = 0, \ a_{jj} \neq 0, \ j = \overline{1, n-1}. \tag{9}$$

Then a solution of (9) is the expression

$$\overset{n}{x}_j = \frac{(-1)^j x_1}{\prod_{k=1}^{j} S_{n-j+k}}, \quad j = \overline{2, n-1}, \tag{10}$$

where,

$$S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j} \frac{(-1)^{p+1} a_{j,j+p}}{a_{j,j} \prod_{k=1}^{p-1} S_{n-j-k}}, S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \ j = \overline{1, n-1}, \tag{11}$$

and x_1 is an arbitrary real number.

Proof: Although the proof is given [2], here we repeat it with some clarification. At the same time, we should act in the same way as with the proof of Theorem 1. To do this, in the equations of the system (9), transferring members, containing the unknowns $\overset{n}{x}_n$ to the right-hand side, we obtain

$$\sum_{p=0}^{n-j-1} a_{j,j+p} \overset{n}{x}_{j+p} = -a_{j,n} \overset{n}{x}_n, \qquad j = \overline{1, n-1}. \tag{12}$$

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To solve the finite system (12), we will firstly build a recursive process (11), similar to the process (8). Then the last equation in (12) (i.e. when j=n-1) is given by: $a_{n-1,n-1}\overset{n}{x}_{n-1} = -a_{n-1,n}\overset{n}{x}_{n}$. Hence, by introducing symbol $S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}$, we obtain $\overset{n}{x}_{n-1} = -S_1\overset{n}{x}_{n}$. Doing similarly with the penultimate equation of system (12), we have $\overset{n}{x}_{n-2} = -S_2\overset{n}{x}_{n-1}$,where $S_2 = \frac{a_{n-2,n-1}}{a_{n-2,n-2}} - \frac{a_{n-2,n}}{a_{n-2,n-2}S_1}$ Inductively continuing this, we obtain the relation (11), wherein it is valid that

$$\overset{n}{x}_{j} = -S_{n-j}\overset{n}{x}_{j+1},\ j=\overline{1,n-1}. \tag{13}$$

Solving the recurrence equation (13), we obtain (10).

Now let us solve the inhomogeneous infinite Gaussian system (4) by the reduction method in a broad sense

Theorem 3: Let the inhomogeneous infinite Gaussian system (4) is truncated by the reduction method in the broad sense into the inhomogeneous finite Gaussian system of the form

$$\sum_{p=0}^{n-j} a_{j,j+p}\overset{n}{x}_{j+p} = b_j, \qquad a_{jj}\neq 0, \qquad j=\overline{1,n-1}. \tag{14}$$

Then the solution of (14) is the expression

$$x \ = B_{n\ \ j} + \frac{(\ \ 1)}{\prod_{k=1}^{j} S_{n-j+k}} + \frac{(\ \ 1)}{\prod_{k=1}^{j} S_{n-j+k}}, \quad j=\overline{1,n-1} \tag{15}$$

where

$$B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=1}^{n-j-1}\frac{a_{j,n-p}}{a_{j,j}}B_p, \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}, \qquad j=\overline{1,n-1}, \tag{16}$$

$$S_{n-j} = \frac{a_{j,j+1}}{a_{j,j}} + \sum_{p=2}^{n-j}\frac{(-1)^{p+1}a_{j,j+p}}{a_{j,j}\prod_{k=1}^{p-1}S_{n-j-k}}, S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \quad j=\overline{1,n-1}, \tag{17}$$

x_1 is an arbitrary real number.

Proof: We proceed in the same way as in the proof of Theorems 2 and 1. According to it in the equations of system (14) members containing the unknowns $\overset{n}{x}_{n}$, we transfer to the right-hand side of the equations, we obtain

$$\sum_{p=0}^{n-j-1} a_{j,j+p}\overset{n}{x}_{j+p} = -a_{j,n}\overset{n}{x}_{n} + b_j, \qquad j=\overline{1,n-1}. \tag{18}$$

To solve the finite system (18) we offer to enter two recursive processes, similar to the previous processes (8) and (11). From the last equation of (18) we obtain:

$$\overset{n}{x}_{n-1} = -\frac{a_{n-1,n}}{a_{n-1,n-1}}\overset{n}{x}_{n} + \frac{b_{n-1}}{a_{n-1,n-1}}.$$

Introducing the notation $S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}$, we have $\overset{n}{x}_{n-1} = -S_1\overset{n}{x}_{n} + B_1$ or $\overset{n}{x}_{n} = \frac{B_1}{S_1} - \frac{\overset{n}{x}_{n-1}}{S_1}$.

Taking into account the last relation, the penultimate equation in (18) gives

$$\overset{n}{x}_{n-2} + \frac{a_{n-2,n-1}}{a_{n-2,n-2}}\overset{n}{x}_{n-1} = -\frac{a_{n-2,n}}{a_{n-2,n-2}}\left(\frac{B_1}{S_1} - \frac{\overset{n}{x}_{n-1}}{S_1}\right) + \frac{b_{n-2}}{a_{n-2,n-2}}.$$

Hence, producing a transformation in order to obtain the expression (16) (for example, by adding and substracting the member $\frac{a_{n-2,n-1}}{a_{n-2,n-2}}B_1$) and introducing the following notations

$$S_2 = \frac{a_{n-2,n-1}}{a_{n-2,n-2}} - \frac{a_{n-2,n}}{a_{n-2,n-2}S_1}; \quad B_2 = \frac{b_{n-2}}{a_{n-2,n-2}} - \frac{a_{n-2,n-1}}{a_{n-2,n-2}}B_1,$$

We obtain

$$\overset{n}{x}_{n-2} = -S_2\overset{n}{x}_{n-1} + S_2B_1 + B_2.$$

Continuing in this way, we inductively conclude that

$$\overset{n}{x}_{n-j} = B_j + S_jB_{j-1} - S_j\overset{n}{x}_{n-j+1}, \tag{19}$$

where

$$B_j = \frac{b_{n-j}}{a_{n-j,n-j}} - \sum_{p=1}^{j-1}\frac{a_{n-j,n-p}}{a_{n-j,n-j}}B_p \quad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}, \quad j=\overline{2,n-1}, \tag{20}$$

$$S_j = \frac{a_{n-j,n-j+1}}{a_{n-j,n-j}} + \sum_{p=2}^{j}\frac{(-1)^{p+1}a_{n-j,n-j+p}}{a_{n-j,n-j}\prod_{k=1}^{p-1}S_{j-k}}, S_1 = \frac{a_{n-1,n}}{a_{n-1,n-1}}, \qquad j=\overline{2,n}. \tag{21}$$

Obviously, the relations (20) and (21) respectively coincide with by expressions (16) and (17).

For the formula (19) to take place for j=1 we formally consider that B_0=0.

Replacing in (19), the index n-j+1 to j, and solving it for the unknown $\overset{n}{x}_{j}$, we obtain

$$\overset{n}{x}_{j} = B_{n-j} + \frac{B_{n-j+1}}{S_{n-j+1}} - \frac{\overset{n}{x}_{j-1}}{S_{n-j+1}}, \tag{22}$$

where B_{n-j}, S_{n-j} are defined by the formulas (20) and (21) respectively.

We solve the recurrence equation (22) lowering the index j of the unknown $\overset{n}{x}_{j-1}$ and repeating the formula (22). For example lowering once, we can get

$$\begin{aligned}\overset{n}{x}_{j} &= B_{n-j} + \frac{B_{n-j+1}}{S_{n-j+1}} - \frac{1}{S_{n-j+1}}\left(B_{n-j+1} + \frac{B_{n-j+2}}{S_{n-j+2}} - \frac{\overset{n}{x}_{j-2}}{S_{n-j+2}}\right) = \\ &= B_{n-j} - \frac{B_{n-j+2}}{S_{n-j+1}S_{n-j+2}} + \frac{\overset{n}{x}_{j-2}}{S_{n-j+1}S_{n-j+2}}.\end{aligned} \tag{23}$$

Continuing in this way, we obviously obtain (15). We can show that expression (15) is indeed a solution of the finite system (14). Substituting (15) into (14) we obtain

$$\sum_{p=0}^{n-j} a_{j,j+p}B_{n-j-p} + \sum_{p=0}^{n-j} a_{j,j+p}\frac{(-1)^{j+p}x_0}{\prod_{k=1}^{j+p}S_{n-j-p+k}} + $$
$$+\sum_{p=0}^{n-j} a_{j,j+p}\frac{(-1)^{j+p-1}B_n}{\prod_{k=1}^{j+p}S_{n-j-p+k}} = J_1 + J_2 + J_3 = J.$$

First, we calculate J_2:

$$J_2 = \frac{(-1)^j a_{j,j}x_0}{\prod_{k=1}^{j}S_{n-j+k}} - \frac{(-1)^j a_{j,j+1}x_0}{\prod_{k=1}^{j+1}S_{n-j-1+k}} + \sum_{p=2}^{n-j}\frac{(-1)^{j+p}a_{j,j+p}x_0}{\prod_{k=0}^{j}S_{n-j+k}\prod_{k=1}^{p-1}S_{n-j-k}}.$$

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Further, considering the relation (17) we find

$$J_2 = \frac{(-1)^j a_{j,j} x_0}{\prod_{k=1}^{j} S_{n-j+k}} - \frac{(-1)^j a_{j,j+1} x_0}{\prod_{k=1}^{j+1} S_{n-j-1+k}} + \frac{(-1)^{j+1} x_0 a_{j,j}}{\prod_{k=0}^{j} S_{n-j+k}} \left(S_{n-j} - \frac{a_{j,j+1}}{a_{j,j}} \right).$$

Hence we conclude that J_2=0. Similarly, we see that J_3=0. Then, we can calculate J_1:

$$J_1 = a_{j,j} B_{n-j} + \sum_{p=1}^{n-j} a_{j,j+p} B_{n-j-p}.$$

Replacing the summation index *n-j-p* by *p* and taking into account the expression (16), obtain

$$J_1 = a_{j,j} B_{n-j} + a_{j,n} B_0 + \sum_{p=1}^{n-j-1} a_{j,n-p} B_p = \sum_{p=1}^{n-j} a_{j,n-p} B_p = b_j,$$

on the assumption that B_0=0. Therefore, $J=J_1+J_2+J_3=b_j$.

Thus, the expressions (15) satisfy all of the equations of the system (14), as required.

Corollary 1: Between neighboring unknowns of inhomogeneous finite system (14) there is the following relation:

$$\overset{n}{x}_j = B_{n-j} + S_{n-j} B_{n-j-1} - S_{n-j} \overset{n}{x}_{j+1}, \quad j = \overline{1, n-1}, \tag{24}$$

where B_{n-j} and S_{n-j} are recursively defined by formulas (16) and (17), and for the unification of notations above we agreed to consider that B_0=0.

For the homogeneous finite system (14) from the (24) obviously the relation (13) follows.

Remark 1: Clearly, the expression (8) and (16) are the same, they differ only in initial values, respectively, B_0 for (8), and B_1 for (16). By recalling, however, that these expressions reflect the solutions of finite systems of different orders of *n* and *n*-1 respectively, we can properly denote them and thus achieve their total coincidence. Besides, it is obvious, that $\lim_{n\to\infty} \frac{b_{n-1}}{a_{n-1,n-1}} = \lim_{n\to\infty} \frac{b_n}{a_{n,n}}$ if these limits exist. Therefore, in the future, without compromising generality, we can consider only the expression (16), considering that in the formulas (7) and (8), *n-1* is taken for the number *n*, i.e., the finite system of order *n-1* is considered.

Remarkable Relations for the Numbers B_{n-j}

We now turn to important relations for the numbers B_{n-j}, which in fact give the opportunity to make transition from the solutions of finite systems to the solution of corresponding infinite system, i.e. these relations allow us to make the passage to the limit from the finite systems to infinite system.

Theorem 4: For numbers B_{n-j} we have the following relations:

$$I. \quad B_{n-j} = \frac{b_j}{a_{j,j}} - \sum_{p=1}^{n-j-1} \frac{a_{j,n-p}}{a_{j,j}} B_p \qquad B_1 = \frac{b_{n-1}}{a_{n-1,n-1}};$$

$$II. \quad B_{n-j} = \frac{|A_{n-1}^{(j)}|}{|A_{n-1}|},$$

where $|A_{n-1}|$ is determinant of the finite system of type (6) of order *n*-1, $|A^{(j)}_{n-1}|$ – Cramer determinant of the same system (determinant obtained by replacing the *j* column of $|A_{n-1}|$ with the right-hand side of system of type (6);

$$III. \quad B_{n-j} = \begin{vmatrix} \frac{b_j}{a_{j,j}} & \frac{b_{j+1}}{a_{j+1,j+1}} & \frac{b_{j+2}}{a_{j+2,j+2}} & \cdots & \frac{b_{n-1}}{a_{n-1,n-1}} \\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \cdots & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \cdots & 0 \\ . & . & . & \cdots & . \\ \frac{a_{j,j+j}}{a_{j,j}} & \frac{a_{j+1,j+j}}{a_{j+1,j+1}} & \frac{a_{j+2,j+j}}{a_{j+2,j+2}} & \cdots & 0 \\ . & . & . & \cdots & \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & \cdots & 0 \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & \cdots & 1 \end{vmatrix}$$

$$IV. \quad B_{n-j} = \sum_{p=0}^{n-j-1} (-1)^p A_p(j) \frac{b_{j+p}}{a_{j+p,j+p}}, \quad \text{where}$$

$$A_p(j) = \sum_{k=0}^{p-1} \frac{(-1)^{p-1-k} a_{j+k,j+p}}{a_{j+k,j+k}} A_k(j), A_0(j) = 1 \, where \, \forall j, \text{ and } \; j = \overline{1, n-1}$$

for all relations.

Proof: Obviously, the first relation is the result of the Theorem 3 – more precisely, the expression (16), but if in the formulas (7) and (8), *n-1* is taken for *n*, it will be the result of Theorem 1. The second relation follows directly from the Cramer's rule for finite systems of order *n-1*. But it is possible to obtain it directly from the relation I. First, we will prove relation III, but actually it was obtained [12]. Here we will only recall highlights of the proof. To do this, we will calculate the determinant of *n-j*[th] order on the right-hand of the expression (25), denoting it with Δ_{n-j}. If *n-j*=1, i.e. *j*=*n-1*, then (25) implies that $\Delta_1 = \frac{b_{n-1}}{a_{n-1,n-1}}$ i. e. $B_1=\Delta_1$. Proof can be fulfilled by expansion of the determinant (25) on the cofactors $A_{i,1}$ of the first column, i.e

$$\Delta_{n-j} = \frac{b_j}{a_{j,j}} A_{1,1} + \sum_{i=2}^{n-j} \frac{a_{j,j+i-1}}{a_{j,j}} A_{i,1}, \tag{27}$$

where $A_{i,1} = (-1)^{i+1} M_{i,1}, M_{i,1}$ – complementary minor of the *i*[th] row of the first column of the determinant (25). The calculation of these minors is actually given in [12], following on it, we see that $\Delta_{n-j} \equiv B_{n-j}$. Now let us return to the proof of the relation II directly from the expressions I. For this, we consider the following Cramer determinant of the order *n-1*

$$\Delta_{n-1}^{(j)} = \begin{vmatrix} 1 & \frac{a_{1,2}}{a_{1,1}} & \frac{a_{1,3}}{a_{1,1}} & \cdots & \frac{a_{1,j-1}}{a_{1,1}} & \overset{(j)}{\frac{b_1}{a_{1,1}}} & \frac{a_{1,j+1}}{a_{1,1}} & \cdots & \frac{a_{1,n-1}}{a_{1,1}} \\ 0 & 1 & \frac{a_{2,3}}{a_{2,2}} & \cdots & \frac{a_{2,j-1}}{a_{2,2}} & \frac{b_2}{a_{2,2}} & \frac{a_{2,j+1}}{a_{2,2}} & \cdots & \frac{a_{2,n-1}}{a_{2,2}} \\ . & . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 1 & \frac{b_{j-1}}{a_{j-1,j-1}} & \frac{a_{j-1,j+1}}{a_{j-1,j-1}} & \cdots & \frac{a_{j-1,n-1}}{a_{j-1,j-1}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{b_j}{a_{j,j}} & \frac{a_{j,j+1}}{a_{j,j}} & \cdots & \frac{a_{j,n-1}}{a_{j,j}} \\ 0 & 0 & 0 & \cdots & 0 & \frac{b_{j+1}}{a_{j+1,j+1}} & 1 & \cdots & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} \\ . & . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & 0 & \cdots & 0 & \frac{b_{n-1}}{a_{n-1,n-1}} & 0 & \cdots & 1 \end{vmatrix}. \tag{28}$$

On the one hand, factoring out the element $\frac{1}{a_{k,k}}$ of each k_{th} row,

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in determinant (28), we get:

$$\Delta_{n-1}^{(j)} = \frac{1}{\prod_{k=1}^{n-1} a_{k,k}} \, |A_{n-1}^{(j)}| = \frac{|A_{n-1}^{(j)}|}{|A_{n-1}|},$$

where $|A_{n-1}|$ is the determinant of the truncated system (6) of order $n-1$, $|A^{(j)}_{n-1}|$ is Cramer determinant of the same system.

On the other hand, we expand the determinant (28) along the first column, and then expand the obtained determinant along the its first column, and then we continue to do so $\boldsymbol{j}$ times. Thus we obtain the determinant of n-$\boldsymbol{j}$ order, taking the transpose of this determinant we will get $\Delta_{n-1}^{(j)}$

$$\Delta_{n-1}^{(j)} = \begin{vmatrix} \frac{b_j}{a_{j,j}} & \frac{a_{j,j+1}}{a_{j,j}} & \dots & \frac{a_{j,n-1}}{a_{j,j}} \\ \frac{b_{j+1}}{a_{j+1,j+1}} & 1 & \dots & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} \\ . & . & \dots & . \\ \frac{b_{n-1}}{a_{n-1,n-1}} & 0 & \dots & 1 \end{vmatrix} = \begin{vmatrix} \frac{b_j}{a_{j,j}} & \frac{b_{j+1}}{a_{j+1,j+1}} & \dots & \frac{b_{n-1}}{a_{n-1,n-1}} \\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & \dots & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & \dots & 0 \\ . & . & \dots & \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \dots & 1 \end{vmatrix} = B_{n-j}.$$

Thus, the relation II is obtained. Therefore, the determinant B_{n-j} (25) can be called generalized Cramer determinant.

Let us consider the proof of the relation IV in more detail, since it plays a key role in the transition to infinite systems. It can be straightaway noted that in the right-hand side of IV, the sum does not contain numbers with the index n, in contrast to relation I. Before proving let us pay attention to a very important moment. The index $\boldsymbol{j}$ in the determinant B_{n-j} is the number of column of the determinant $|A_{n-1}|$, which is replaced by the constant terms of system of type (6). That can be seen from II, and also from the transpose of (28). It is clear that $\boldsymbol{j}$ does not depend on n, to be more precise, on the order $n-1$ of the truncated system (14), and in arbitrary manner varies from 1 to $n-1$. As it was mentioned before, the index n describes order of the truncated system (14), and the index n-j is the order of the determinant B_{n-j} which varies with changes in the number of j. For example, if $\boldsymbol{j}=n-1$, i.e. when in $|A_{n-1}|$ the last column is replaced by the constant terms of system, we can obtain $n - j=1$ and $B_1 = \frac{b_{n-1}}{a_{n-1,n-1}}$.

And if $j=1$, the determinant B_{n-j} has order $n-1$ and coincides with transpose of (28) for $\boldsymbol{j}=1$. Thus, in order to emphasize this dependence, we can assume that the determinant in the III is a function of the index j, i.e $B_{n-j} \equiv |B_{n-j}(j)|$. For convenience, we will omit the symbol of determinant.

Now we will proceed with the proof of the relation IV. Having deleted the first row from the determinant (25) and then adding appropriate last row, we get the determinant $|A(j)|$ of n-j order, i.e

$$|A_{n-j}(j)| = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \dots & 0 & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \dots & 0 & 0 \\ \frac{a_{j,j+3}}{a_{j,j}} & \frac{a_{j+1,j+3}}{a_{j+1,j+1}} & \frac{a_{j+2,j+3}}{a_{j+2,j+2}} & \dots & 0 & 0 \\ . & . & . & \dots & . & . \\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & \dots & 1 & 0 \\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & \dots & \frac{a_{n-2,n-1}}{a_{n-2,n-2}} & 1 \\ \frac{a_{j,n}}{a_{j,j}} & \frac{a_{j+1,n}}{a_{j+1,j+1}} & \frac{a_{j+2,n}}{a_{j+2,j+2}} & \dots & \frac{a_{n-2,n}}{a_{n-2,n-2}} & \frac{a_{n-1,n}}{a_{n-1,n-1}} \end{vmatrix}. \qquad (29)$$

Here and below the symbol of determinant $|A(j)|$ is also omitted. We construct a sequence of determinants $A_p(j)$ $0 \le p \le n$-j, assuming that $A_0(j)=1$ for all j, and for other p values we take principal minors of the determinant (29), i.e.

$$A_1(j) = \frac{a_{j,j+1}}{a_{j,j}}, \quad A_2(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} \end{vmatrix}, \dots,$$

$$A_p(j) = \begin{vmatrix} \frac{a_{j,j+1}}{a_{j,j}} & 1 & \dots & 0 & 0 \\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & \dots & 0 & 0 \\ . & . & . & . & . \\ \frac{a_{j,j+p-2}}{a_{j,j}} & \frac{a_{j+1,j+p-2}}{a_{j+1,j+1}} & \dots & 1 & 0 \\ \frac{a_{j,j+p-1}}{a_{j,j}} & \frac{a_{j+1,j+p-1}}{a_{j+1,j+1}} & \dots & \frac{a_{j+p-2,j+p-1}}{a_{j+p-2,j+p-2}} & 1 \\ \frac{a_{j,j+p}}{a_{j,j}} & \frac{a_{j+1,j+p}}{a_{j+1,j+1}} & \dots & \frac{a_{j+p-2,j+p}}{a_{j+p-2,j+p-2}} & \frac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}} \end{vmatrix}. \qquad (30)$$

Using the sequence (30), recurrence relations (26) can easily be proved by induction. The only thing we can note that when expanding the determinant of $A_p(j)$ of p order along the last column, we get:

$$A_p(j) = \frac{a_{j+p-1,j+p}}{a_{j+p-1,j+p-1}} A_{p-1}(j) - A'_{p-1}(j),$$

where A'_{p-1} is the determinant $A_{p-1}(j)$, where the last row is replaced by the last row of the determinant $A_p(j)$ without the last element. The inductive assumption can be induced afterwards.

Further, in the similar way, by induction on the order of the determinant (25), we can prove the validity of IV (the determinant (25) is expanded along the last column).

The Transition from the Finite System Solutions to the Solution of the Infinite System

Although the main result of this section was published [9], we repeat here the main points concerning the relations I–IV. We will describe more in detail the role played by each of the relations I-IV in the transition from finite systems to infinite systems. Let us start with the relation I. We assume that the following two conditions hold:

1) Suppose that the limit $\lim_{n\to\infty} B_{n-j}(j) = B(j)$ exists. This condition guarantees, as it can be seen from the expression (7), that the method of reduction in the narrow sense converges;

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2) Suppose that in (8) (i.e. in the relation I) it is possible to pass term-by-term to the limit in the sense of formula

$$\lim_{n\to\infty}\sum_{p=j+1}^{n}\frac{a_{j,p}}{a_{j,j}}B_{n-p}=\sum_{p=j+1}^{\infty}\frac{a_{j,p}}{a_{j,j}}\lim_{n\to\infty}B_{n-p}. \tag{31}$$

As it will be seen below, the condition 2) is a sufficient condition for numbers *B(j)* to be a particular solution of the original system (4). Thus, the performance of only one condition 1) is not sufficient for numbers *B(j)* to satisfy the infinite system (4), i.e. the convergence of the method of reduction does not guarantee the existence of solution of the original infinite system.

Theorem 5: Let the conditions 1) and 2) hold, then the limit value $\lim_{n\to\infty}B_{n-j}=B(j)$ particular solution of inhomogeneous infinite Gaussian system (4).

Proof: The passage to the limit in relation I and theorem assumptions allow us to obtain the following equality for each *j*:

$$B(j)=\frac{b_j}{a_{j,j}}-\sum_{p=j+1}^{\infty}\frac{a_{j,p}}{a_{j,j}}B(p). \tag{32}$$

Hence, since $a_{j,j}\neq 0$, obviously we have:

$$\sum_{p=j}^{\infty}a_{j,p}B(p)=b_j.$$

Comparing the last expression with (4), we see that the numbers *B*(*j*) form a particular solution of Gaussian system (4).

Definition 3: The particular solution $x_j=B(j)$ of inhomogeneous infinite Gaussian system (4) is called a strictly particular solution of the system (4).

Thus, the solution of Gaussian system (4) obtained by a simple reduction (reduction in the narrow sense) is a strictly particular solution.

As shown in [9], we can easily obtain the following theorem.

Theorem 6. Let the condition 1) hold. The passage to the limit in relation I is possible if and only if the set of numbers *B(j) j*=1, 2, ... is a strictly particular solution of infinite Gaussian system (4).

Proof: Necessity. Suppose that passage to the limit in relation I is possible. Then by Theorem 5 we conclude that the set of numbers *B*(*j*) *j*=1, 2, ... is a strictly particular solution of Gaussian system (4).

Sufficiency: Suppose that the set of numbers *B(j) j*=1, 2, ... is a strictly particular solution of Gaussian system (4), i.e. the equalities are valid (32). We can prove the expression (31).

Changing the summation index in the relation I and passing to the limit in it in view of the conditions 1), we obtain

$$\lim_{n\to\infty}B_{n-j}=B(j)=\frac{b_j}{a_{j,j}}-\lim_{n\to\infty}\sum_{p=j+1}^{n}\frac{a_{j,p}}{a_{j,j}}B_{n-p}. \tag{33}$$

Subtracting equality (32) from equality (33) we can obtain:

$$\lim_{n\to\infty}\sum_{p=j+1}^{n}\frac{a_{j,p}}{a_{j,j}}B_{n-p}=\sum_{p=j+1}^{\infty}\frac{a_{j,p}}{a_{j,j}}B(p)=\sum_{p=j+1}^{\infty}\frac{a_{j,p}}{a_{j,j}}\lim_{n\to\infty}B_{n-p},$$

which was to be proved.

Thus, under the conditions 1) and 2) the reduction method in a narrow sense (i.e. a simple reduction) converges to particular solution of Gaussian system (4), which we have called a strictly particular solution. We will explain later why it is called so. Unfortunately, it is impossible to define the limits themselves, i.e. the numbers *B*(*j*), from the equalities (32), because solving (32) is equivalent to finding the solution of the original Gaussian system (4). Therefore, to calculate these limits we will use other relations.

From the relation II it can be concluded that if the limit $\lim_{n\to\infty}B_{n-j}$ exists, it will be equal to the Cramer's formula that i.e. $B(j)=\frac{|A^{(j)}|}{|A|}$. Indeed, passing to the limit in the relation II and using the definition of infinite determinants, we can obtain

$$\lim_{n\to\infty}B_{n-j}=B(j)=\lim_{n\to\infty}\frac{|A_{n-1}^{(j)}|}{|A_{n-1}|}=\frac{\lim_{n\to\infty}|A_{n-1}^{(j)}|}{\lim_{n\to\infty}|A_{n-1}|}=\frac{|A^{(j)}|}{|A|}.$$

Therefore, calculating infinite determinants $|A^{(j)}|$ and $|A|$, we can find the numbers *B*(*j*). Thus, in this case, the transition from finite systems solutions to infinite system solution is based on the definition of infinite determinants. More specifically, this idea is realized by means of generalized Cramer determinant (25) and the relation IV. Passing to the limit in relation IV, we will obviously obtain

$$\lim_{n\to\infty}B_{n-j}=B(j)=\sum_{p=0}^{\infty}(-1)^p A_p(j)\frac{b_{j+p}}{a_{j+p,j+p}}, \tag{34}$$

where Ap is defined by the recurrent relation (26).

Thus, formally it is valid that:

$$\lim_{n\to\infty}B_{n-j}=B(j)=\lim_{n\to\infty}\begin{vmatrix}\frac{b_j}{a_{j,j}} & \frac{b_{j+1}}{a_{j+1,j+1}} & \cdots & \frac{b_{n-1}}{a_{n-1,n-1}}\\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & \cdots & 0\\ . & . & \cdots & .\\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \cdots & 1\end{vmatrix}=$$

$$=\begin{vmatrix}\frac{b_j}{a_{j,j}} & \frac{b_{j+1}}{a_{j+1,j+1}} & \frac{b_{j+2}}{a_{j+2,j+2}} & \cdots & \frac{b_{n-1}}{a_{n-1,n-1}} & .\\ \frac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \cdots & 0 & .\\ \frac{a_{j,j+2}}{a_{j,j}} & \frac{a_{j+1,j+2}}{a_{j+1,j+1}} & & & & \\ . & . & . & \cdots & . & .\\ \frac{a_{j,j+j}}{a_{j,j}} & \frac{a_{j+1,j+j}}{a_{j+1,j+1}} & \frac{a_{j+2,j+j}}{a_{j+2,j+2}} & \cdots & 0 & .\\ . & . & . & \cdots & . & .\\ \frac{a_{j,n-2}}{a_{j,j}} & \frac{a_{j+1,n-2}}{a_{j+1,j+1}} & \frac{a_{j+2,n-2}}{a_{j+2,j+2}} & \cdots & 0 & .\\ \frac{a_{j,n-1}}{a_{j,j}} & \frac{a_{j+1,n-1}}{a_{j+1,j+1}} & \frac{a_{j+2,n-1}}{a_{j+2,j+2}} & \cdots & 1 & .\\ . & . & . & \cdots & . & .\end{vmatrix}=\Delta(j) \tag{35}$$

But if the series in (34) converge, than $\lim_{n\to\infty}B_{n-j}$ definitely equals to the infinite determinant Δ(*j*), i.e. the passage to the limit is done.

Consistency of inhomogeneous infinite systems

Theorem 7: If inhomogeneous Gaussian system (4) has a unique solution, then this solution will certainly be its strictly particular solution, and this solution is given by Cramer's formula.

Proof: Let $\{y_i\}_1^{\infty}$ be the unique solution of the system (4), i.e. system (4) is satisfied for these numbers:

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + \ldots + a_{1,N}y_N + \ldots &= b_1 \\ a_{2,2}y_2 + a_{2,3}y_3 + a_{2,4}y_4 + \ldots + a_{2,N}y_N + \ldots &= b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N + \ldots &= b_{N-1} \\ a_{N,N}y_N + \ldots &= b_N \\ \ldots\ldots\ldots\ldots & \end{aligned} \tag{36}$$

Then we will use Theorem 1 and for this we rewrite (36) in this way:

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + \ldots + a_{1,N}y_N &= b_1 - b_1^N = \overline{b}_1^N \\ a_{2,2}y_1 + a_{2,3}y_3 + a_{2,4}y_4 + \ldots + a_{2,N}y_N &= b_2 - b_2^N = \overline{b}_2^N \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N &= b_{N-1} - b_{N-1}^N = \overline{b}_{N-1}^N \\ a_{N,N}y_N &= b_N - b_N^N = \overline{b}_N^N \\ \ldots\ldots\ldots\ldots, & \end{aligned} \tag{37}$$

where $b_j^N = \sum_{p=N+1}^{\infty} a_{j,p}y_p,\ j = \overline{1,N}$.

It should be noted, that since y_i satisfies every equation of system (4), then $\lim_{N\to\infty} b_j^N = 0$ independently from fixed *j*.

As with the reduction method in a narrow sense, the system (37) will be truncated, leaving *N* equations with *N* unknowns x_j, but in this case, the x_j are known: $x_j = y_j$, where *j* varies from 1 to *N*.

On the other hand, according to Theorem 1, y_j are expressed by the formula (7), i.e. the following equality is valid:

$$y_j = \overline{B}_{N-j}, \quad j = 1,2,\ldots,N, \tag{38}$$

where $\overline{B}_{N-j}$ is determined from relations I–IV, where $(b_j - b_j^N)$ are taken for the b_j and $b_j^N = \sum_{p=N+1}^{\infty} a_{j,p}y_p$.

It is necessary to give some explanation. Components y_j of $\{y_i\}_1^\infty$ in (38) depend on *N* only in the sense that with increasing *N* a new components y_{j0} appear where j_0 is greater than the initial value of *N*.

Passing to the limit as $N \to \infty$ in (38), we have

$$y_j = \lim_{N\to\infty} \overline{B}_{N-j}, \quad j = 1,2,\ldots,\infty. \tag{39}$$

The equality (39) shows that, firstly, a simple reduction method converges since the limit $\lim_{N\to\infty} \overline{B}_{N-j}$ exists, and secondly, this limit is the solution of the system (4). We must show that $\lim_{N\to\infty} \overline{B}_{N-j} = \lim_{N\to\infty} B_{N-j} = B(j)$. This equation can be obtained in two ways. Firstly – on the basis of the equalities (35). Indeed, the limit $\lim_{N\to\infty} \overline{B}_{N-j}$ equals to the infinite determinant Δ(*j*) (35), where $(b_j - \lim_{N\to\infty} b_j^N)$ are taken for the b_j, but $\lim_{N\to\infty} b_j^N = 0$ by the its construction, therefore $\lim_{N\to\infty} \overline{B}_{N-j} = \Delta(j) = B(j)$. Secondly, we can make sure about this equality in the following manner. Directly passing to the limit as $N \to \infty$ in the finite system (37), we will obviously get the original infinite system (36). As a result we see the equality of these infinite determinants. Thus, from (35) and (39), we obtain $\lim_{N\to\infty} \overline{B}_{N-j} = \Delta(j) = B(j) = y_j$. It means that y_j is a strictly particular solution of the system (4) according to definition 3. The second part of the theorem has been proved above. Here we passed to the limit using the concept of an infinite determinant. We got that which was to be proved.

Note 2: The strictly particular solution of (4) is unique, if it exists, and is expressed by Cramer's formula. This statement follows from the uniqueness of infinite Cramer determinants |*A*(*j*)| for each *j* and uniqueness of determinant |*A*| of (4), in case if $|A| \neq 0$.

The existence of nontrivial solutions of the homogeneous infinite systems

Let us consider the conditions of existence of nontrivial solutions of homogeneous infinite Gaussian systems (4).

Theorem 8: For any nontrivial solution of the homogeneous infinite Gaussian system (4) there is a characteristic numbers *S(j)*, i.e. there is a limit $\lim_{n\to\infty} S_{n-j} = S(j)$, where numbers S_{n-j} are determined by (11).

Proof: Let $\{y_i\}_1^\infty$ be a nontrivial solution of the homogeneous system (4), i.e., the system (36) is valid, here b_j=0 for all *j*. Further we rewrite it in the form (37) (b_j=0), then truncate it leaving *N-1* equations with *N* unknowns (the method of reduction in a broad sense), and we will obtain:

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + \ldots + a_{1,N-1}y_{N-1} + a_{1,N}y_N &= b_1^N \\ a_{2,2}y_1 + a_{2,3}y_3 + a_{2,4}y_4 + \ldots + a_{2,N-1}y_{N-1} + a_{2,N}y_N &= b_2^N \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{N-2,N-2}y_{N-2} + a_{N-2,N-1}y_{N-1} + a_{N-2,N}y_N &= b_{N-2}^N \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N &= b_{N-1}^N, \end{aligned} \tag{40}$$

where $b_j^N = -\sum_{p=N+1}^{\infty} a_{j,p}y_p$ in which the upper index *N* emphasizes, that every constant term of the system (40) tends to zero with increasing *N*. Let us clarify this.

Firstly, since y_i satisfies each equation of the system (4), then $\lim_{N\to\infty} b_j^N = 0$ independently on *j*.

Secondly, by increasing *N*, for example by unity, a new component y_{N+1} will appear in the left side of finite system (40) and one new equation will be added, i.e., instead of (40) we will have

$$\begin{aligned} a_{1,1}y_1 + a_{1,2}y_2 + a_{1,3}y_3 + a_{1,4}y_4 + \ldots + a_{1,N}y_N + a_{1,N+1}y_{N+1} &= b_1^{N+1} \\ a_{2,2}y_1 + a_{2,3}y_3 + a_{2,4}y_4 + \ldots + a_{2,N}y_N + a_{2,N+1}y_{N+1} &= b_2^{N+1} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ a_{N-1,N-1}y_{N-1} + a_{N-1,N}y_N + a_{N-1,N+1}y_{N+1} &= b_{N-1}^{N+1} \\ a_{N,N}y_N + a_{N,N+1}y_{N+1} &= b_N^{N+1}, \end{aligned} \tag{41}$$

where $b_j^{N+1} = -\sum_{p=N+2}^{\infty} a_{j,p}y_p$. If number *N* is big enough, it is clear that b_j^N decreases with *N*: i.e., as *N* increases, the b_j^N will get smaller.

Let us use Corollary 1, then the formula (24), in our case will look as follows:

$$y_j = \overline{B}_{N-j} + S_{n-j}\overline{B}_{N-j-1} - S_{N-j}y_{j+1}, \quad j = \overline{1, N-1}, \tag{42}$$

where $\overline{B}_{N-j}$ and S_{N-j} are recursively defined by formulas (16) and (17), but in (16) instead of b_j will be b^{N}_{j}, and y_j will be known solutions of the homogeneous system (4).

On the basis of Theorem 4, that is more precisely, on the basis of relation III, the $\overline{B}_{N-j}$ equals to the determinant (25) where the b^{N}_{j} is taken for the b_j. Then, on the basis of the (35) and reasoning about the system (41), it is formally true that:

$$\lim_{N\to\infty}\overline{B}_{N-j} = \begin{vmatrix} \dfrac{\lim\limits_{N\to\infty} b_j^N}{a_{j,j}} & \dfrac{\lim\limits_{N\to\infty} b_{j+1}^N}{a_{j+1,j+1}} & \dfrac{\lim\limits_{N\to\infty} b_{j+2}^N}{a_{j+2,j+2}} & \cdots & \dfrac{\lim\limits_{N\to\infty} b_{N-1}^N}{a_{N-1,N-1}} & . \\ \dfrac{a_{j,j+1}}{a_{j,j}} & 1 & 0 & \cdots & 0 & . \\ \dfrac{a_{j,j+2}}{a_{j,j}} & \dfrac{a_{j+1,j+2}}{a_{j+1,j+1}} & 1 & \cdots & 0 & . \\ . & . & . & \cdots & . & . \\ \dfrac{a_{j,j+j}}{a_{j,j}} & \dfrac{a_{j+1,j+j}}{a_{j+1,j+1}} & \dfrac{a_{j+2,j+j}}{a_{j+2,j+2}} & \cdots & 0 & . \\ . & . & . & \cdots & . & . \\ \dfrac{a_{j,N-2}}{a_{j,j}} & \dfrac{a_{j+1,N-2}}{a_{j+1,j+1}} & \dfrac{a_{j+2,N-2}}{a_{j+2,j+2}} & \cdots & 0 & . \\ \dfrac{a_{j,N-1}}{a_{j,j}} & \dfrac{a_{j+1,N-1}}{a_{j+1,j+1}} & \dfrac{a_{j+2,N-1}}{a_{j+2,j+2}} & \cdots & 1 & . \\ . & . & . & \cdots & . & . \end{vmatrix}. \tag{43}$$

But by assumption, $\lim\limits_{N\to\infty} b_j^N = 0$ independently on j and therefore the infinite determinant in (43) contains a top zero row, hence infinite determinant in (43) exists and is equal to zero [2,3], i.e. $\lim\limits_{N\to\infty}\overline{B}_{N-j} = 0$. Rewriting (42) in the form

$$y_j - \overline{B}_{N-j} = S_{N-j}(\overline{B}_{N-j-1} - y_{j+1}), \quad j = \overline{1, N-1},$$

and passing in it to the limit, we have

$$\lim_{N\to\infty}(y_j - \overline{B}_{N-j}) = \lim_{N\to\infty}[S_{N-j}(\overline{B}_{N-j-1} - y_{j+1})], \quad j = \overline{1,\infty}. \tag{44}$$

But considering that $\lim\limits_{N\to\infty}\overline{B}_{N-j} = 0$, we can obtain $\lim\limits_{N\to\infty}(y_j - \overline{B}_{N-j}) = y_j$ Then, on the basis of (44) we can conclude that numbers $\lim\limits_{N\to\infty} S_{N-j} = S(j)$ exist, i.e. the equality $y_j = -S(j)y_{j+1}$ is valid, as was to be proved.

Corollary 2: If the nontrivial solution $\{x_i\}_1^\infty$ of Gaussian homogeneous system (4) exists, it is given by

$$x_i = \frac{(-1)^i x_1}{\prod\limits_{k=1}^{i-1} S(k)}, \; i = \overline{2,\infty} \tag{45}$$

where x_1 is arbitrary real number, *S(k)* are characteristic numbers.

Corollary 3: The necessary condition for the existence of nontrivial solution of the homogeneous Gaussian system (4) is the convergence of (11).

Theorem 9: The necessary and sufficient condition for the existence of a nontrivial solution $\{x_i\}_1^\infty$ of homogeneous Gaussian system (4) is the characteristic numbers *S(i)* of this solution satisfy the following conditions for each *j*:

$$\sum_{p=0}^{\infty} \frac{(-1)^p a_{j,j+p}}{a_{j,j}\prod\limits_{k=0}^{p-1} S(j+k)} = 0, \quad j = 1, 2, \ldots. \tag{46}$$

Here to simplify the notation adopted $\prod\limits_{k=0}^{-1} S(j+k) = 1$ for all *j*.

Proof: Necessity: Let x_i be an arbitrary nontrivial solution of the homogeneous system (4), then, according to Corollary 2, this solution is given by (45). Taking into account the ratios

$$\prod_{k=1}^{j+p-1} S(k) = \prod_{k=1}^{j-1} S(k) \prod_{k=j}^{j+p-1} S(k) = \prod_{k=1}^{j-1} S(k) \prod_{k=0}^{p-1} S(j+k),$$

we substitute (45) into the homogeneous system (4) and obtain the condition (46) for each j. Thus the necessity is proved.

Sufficiency: Let the numbers *S(j)* be a solution of (46) for each *j*. Then we form numbers x_j like (45):

$$x_i = \frac{(-1)^i x_1}{\prod\limits_{k=0}^{i-1} S(k)}, \quad i = 2, 3, \ldots.$$

Substituting these values into homogeneous system (4), we see that all equations of the system (4) are satisfied, since the conditions (46) are done. The sufficiency is proved.

Corollary 4: *Let the limit* $\lim\limits_{n\to\infty} S_{n-j} = S(j)$ *exists. Then the necessary and sufficient condition for the existence of a nontrivial solution of the homogeneous Gaussian system (4) is possibility of the passage term-by-term to the limit in the expression (11).*

Proof: Sufficiency: Let the passage term-by-term to the limit is possible, then passing to the limit as $n \to \infty$ in (11), we obtain the relation (46) and on the basis of Theorem 9, we verify the existence of a nontrivial solution.

Necessity: Let the homogeneous Gaussian system (4) have a nontrivial solution, then by Theorem 9 the (46) is done. Writing the expression (46) and passing to the limit in the (11), we will proceed similarly to the proof of Theorem 6. As a result, we will obtain

$$\lim_{n\to\infty}\sum_{p=2}^{n-j} \frac{(-1)^p a_{j,j+p}}{a_{j,j}\prod\limits_{k=1}^{p-1} S_{n-j-k}} = \sum_{p=2}^{\infty} \frac{(-1)^p a_{j,j+p}}{a_{j,j}\prod\limits_{k=1}^{p-1} \lim\limits_{n\to\infty} S(j+k)} \qquad j = 1, 2, \ldots,$$

and we got that which was to be proved.

Note 3: From Corollary 2 it follows that if there is any nontrivial solution of homogeneous Gaussian system (4), then we have an infinite number of such solutions. Therefore, on the basis of Note 2, we conclude that strictly particular solution does not contain as a summand the nontrivial solution of the corresponding homogeneous Gaussian system (4). That is why this particular solution was called a strictly particular solution.

Existence of a strictly particular solution of inhomogeneous Gaussian systems

Theorem 10: If inhomogeneous Gaussian system (4) is consistent, then its strictly particular solution exists.

Proof: Let $\{y_i\}_1^\infty$ be some particular solution of the inhomogeneous Gaussian system (4). If it is a unique solution of (4), then the solution $\{y_i\}_1^\infty$ will be a strictly particular solution by Theorem 7. Hence, in general case, this solution can be represented as the sum of some particular solution of inhomogeneous system (4) and some nontrivial solution of the corresponding homogeneous system (4). Therefore, we will act in the same way, as when using the reduction method in a broad sense. We will use the approach proposed in the proof of Theorems 7 and 8, and as a result, we will obtain (36). Leaving therein *N-1* equations with *N* components, we can obtain a finite system like (14) and then, after connecting neighboring components, we will get the ratio of type (24):

$$y_j = \overline{B}_{n-j} + S_{n-j}\overline{B}_{n-j-1} - S_{n-j}y_{j+1}, \quad j=\overline{1,n-1}, \tag{47}$$

where S_{n-j} is defined by expression (17), $\overline{B}_{n-j}$ by (16), i.e. by the generalized Cramer determinant (25), in which $b_j - b_j^n = b_j - \sum_{p=n+1}^{\infty} a_{j,p}y_p$ are taken for b_j and $\lim_{n\to\infty} b_j^n = 0$ independently on *j*.

Passing to the limit in (47), we have:

$$y_j = \lim_{n\to\infty}\overline{B}_{n-j} + \lim_{n\to\infty}(S_{n-j}\overline{B}_{n-j-1}) - \lim_{n\to\infty}S_{n-j}y_{j+1}, \quad j=\overline{1,\infty}. \tag{48}$$

By Theorem 8, if a nontrivial solution of corresponding homogeneous system exists, then the characteristic numbers *S*(*j*) always exist, i.e. $\lim_{n\to\infty}S_{n-j} = S(j)$. From (35) we can formally find the limit $\lim_{n\to\infty}\overline{B}_{n-j}$ and absolutely in the same way as in the proof of Theorem 7, we will make sure that $\lim_{N\to\infty}\overline{B}_{N-j} = B(j)$

Here we have to note that the limit *B*(*j*) is written formally. Introducing the notation $\Delta(j) = y_j - B(j)$, we can obtain $\Delta(j) = -S(j)\Delta(j+1)$ from (48). Solving the last recurrent equation, we will get

$$\Delta(j) = \frac{(-1)^j\Delta(1)}{\prod_{k=1}^{j-1}S(k)}. \tag{49}$$

But by the Corollary 3, the expression (49) is a nontrivial solution of the corresponding homogeneous system (see. (45)). Thus, the solution y_j would look like this:

$$y_j = B(j) - \frac{(-1)^j\Delta(1)}{\prod_{k=1}^{j-1}S(k)}.$$

It follows that *B*(*j*) is indeed a solution of the original Gaussian system (4). Moreover, it was received by the reduction method, i.e., in fact it is a strictly particular solution. We got that which was to be proved.

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