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Research Article

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On the Fundamental Theorem in Arithmetic Progression of Primes

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Abstract

Using Jiang function we prove the fundamental theorem in arithmetic progression of primes. The primes contain only $k<P_g+1$ long arithmetic progression, but the primes have no $k>P_g+1$ long arithmetic progressions theorem.

Keywords: Arithmetic; Lie theory; Fundamental theorem; Progression; Asymptotic formula

Theorem

The fundamental theorem in arithmetic progression of primes.

We define the arithmetic progression of primes [1-3].

$$P_{i+1}=P_1+\omega_g i, i=0,1,2,\ldots,k-1, \tag{1}$$

Where $\omega_g=\prod_{2\le P\le P_g} P$ is called a common difference, P_g is called g-th prime.

We have Jiang function [1-3]:

$$\prod_{i=1}(q+\omega_g i)\equiv 0(\text{mod } P), \tag{2}$$

$X(P)$ denotes the number of solutions for the following congruence:

$$\prod_{i=1}^{k-1}(q+\omega_g i)\equiv 0(\text{mod } P), \tag{3}$$

Where $q=1,2,\ldots,P-1$.

If $P|\omega_g$, then $X(P)=0$;$X(P)=k-1$ otherwise. From eqn (3) we have:

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-k). \tag{4}$$

If $k=P_{g+1}$ then $J_2(P_{g+1})=0$, $J_2(\omega)=0$, there exist finite primes P_1 such that $P_2,\ldots,P_k$ are primes. If $k<P_{g+1}$ then $J_2()\neq 0$, there exist infinitely many primes P_1 such that $P_2,\ldots,P_k$ are primes. The primes contain only $k<P_{g+1}$ long arithmetic progression, but the primes have no $k>P_{g+1}$ long arithmetic progression geometry. We have the best asymptotic formula [1-3]:

$$\pi_k(N,2)=|\{P_1+\omega_g i=\text{prime}, 0\le i\le k-1, P_1\le N\}|$$

$$=\frac{J_2(\omega)\omega^{k-1}}{\varphi^k(\omega)}\frac{N}{\log^k N}(1+o(1)), \tag{5}$$

Where $\omega=\prod_{2\le P}P$, $\varphi(\omega)=\prod_{2\le P}(P-1)$, ω is called primorial, $\varphi(\omega)$ Euler function.

Suppose $k=P_{g+1}-1$. From eqn (1) we have:

$$P_{i+1}=P_1+\omega_g i, i=0,1,2,\ldots,P_{g+1}2. \tag{6}$$

From eqn (4) we have [1,2]:

$$J_2(\omega)=\prod_{3\le P\le P_g}(P-1)\prod_{P_{g+1}\le P}(P-P_{g+1}+1)\to\infty \text{ as } \omega\to\infty \tag{7}$$

We prove that there exist infinitely many primes P_1 such that $P_2,\cdots,P_{P_{g+1}-1}$ are primes for all P_{g+1}.

From eqn (5) we have:

$$\pi_{P_{g+1}-1}(N,2)=$$

$$\prod_{2\le P\le P_g}\left(\frac{P}{P-1}\right)^{P_{g+1}-2}\prod_{P_{g+1}\le P}=\frac{P^{P_{g+1}-2}(P-P_{g+1}+1)}{(P-1)^{P_{g+1}-1}}\frac{N}{(\log N)^{P_{g+1}-1}}(1+o(1)). \tag{8}$$

From eqn (8) we are able to find the smallest solutions $\pi_{P_{g+1}-1}(N,2)>1$ for large P_{g+1}.

Theorem is foundation for arithmetic progression of primes.

Example 1: Suppose $P_1=2$, $\omega_1=2$, $P_2=3$. From eqn (6) we have the twin primes theorem:

$$P_2=P_1+2. \tag{9}$$

From eqn (7) we have:

$$J_2(\omega)=\prod_{3\le P}(P-2)\to\infty \text{ as } \omega\to\infty, \tag{10}$$

We prove that there exist infinitely many primes P_1 such that P_2 are primes. From eqn (8) we have the best asymptotic formula [4-6]:

$$\pi_2(N,2)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N}{\log^2 N}(1+o(1)). \tag{11}$$

Twin prime theorem is the first theorem in arithmetic progression of primes.

Example 2: Suppose $P_2=3$, $\omega_2=6$, $P_3=5$. From eqn (6) we have:

$$P_{i+1}=P_1+6i, i=0,1,2,3. \tag{12}$$

From eqn (7) we have:

$$J_2(\omega)=2\prod_{5\le P}(P-4)\to\infty \text{ as } \omega\to\infty, \tag{13}$$

We prove that there exist infinite many primes P_1 such that P_2, P_3 and P_4 are primes. From eqn (8) we have the best generalized asymptotic formula:

$$\pi_4(N,2)=27\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N}{\log^4 N}(1+o(1)). \tag{14}$$

Example 3: Suppose $P_9=23$, $\omega_9=223092870$, $P_{10}=29$. From eqn (6) we have:

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$P_{i+1}=P_1+223092870i$, $i=0,1,2,\ldots,27$. (15)

From eqn (7) we have:

$$J_2(\omega)=36495360\prod_{29\le P}(P-28)\to\infty \text{ as } \omega\to\infty, \quad (16)$$

We prove that there exist infinitely many primes P_1 such that $P_2,\ldots,P_{28}$ are primes. From eqn (8) we have the best asymptotic formula [7]:

$$\pi_{28}(N,2)=\prod_{2\le P\le 23}\left(\frac{P}{P-1}\right)^{27}\prod_{29\le P}\frac{P^{27}(P-28)}{(P-1)^{28}}\frac{N}{\log^{28}N}(1+o(1)). \quad (17)$$

From eqn (17) we are able to find the smallest solutions $\pi_{28}(N_0,2)>1$.

On May 17, 2008, Wroblewski and Raanan Chermoni found the first known case of 25 primes:

$61711054912832631+366384\times\omega_{23}\times n$, for $n=0$ to 24.

Theorem can help in finding for 26, 27, 28,…, primes in arithmetic progressions of primes.

Corollary 1: Arithmetic progression with two prime variables.

Suppose $\omega_g=d$. From eqn (1) we have:

$P_1, P_2=P_1+d, P_3=P_1+2d,\ldots,P_k=P_1+(k-1)d$, $(P_1,d)=1$ (18)

From eqn (18) we obtain the arithmetic progression with two prime variables: P_1 and P_2,

$P_3=2P_2-P_1$, $P_j=(j-1)P_2-(j-2)P_1$, $3\le j\le k<P_{g+1}$. (19)

We have Jiang function [3]:

$$J_3(\omega)=\prod_{3\le P}[(P-1)^2-X(P)], \quad (20)$$

$X(P)$ denotes the number of solutions for the following congruence matrices:

$$\prod_{j=3}^{k}[(j-1)q_2-(j-2)q_1]\equiv 0(\text{mod } P), \quad (21)$$

where $q_1=1,2,\ldots,P-1$; $q2=1,2,\ldots,P-1$.

From eqn (21) we have:

$$J_3(\omega)=\prod_{3\le P\le k}(P-1)\prod_{k<P}(P-1)(P-k+1)\to\infty \text{ as } \omega\to\infty. \quad (22)$$

We prove that there exist infinitely many primes P_1 and P_2 such that $P_3,\ldots,P_k$ are primes for $3\le k<P_{g+1}$.

We have the best asymptotic formula [8]:

$$\pi_{k-1}(N,3)=|\{(j1)P_2(j-2)P_1=\text{prime}, 3\le j\le k, P_1, P_2\le N\}|$$

$$=\frac{J_3(\omega)\omega^{k-2}}{\varphi^k(\omega)}\frac{N^2}{\log^k N}(1+o(1)), \quad (23)$$

From eqn (23) we have the best asymptotic formula:

$$\pi_{k-1}(N,3)=\prod_{2\le P\le k}\frac{P^{k-2}}{(P-1)^{k-1}}\prod_{k<P}\frac{P^{k-2}(P-k+1)}{(P-1)^{k-1}}\frac{N^2}{\log^k N}(1+o(1)). \quad (24)$$

From eqn (24) we are able to find the smallest solution $\pi_{k-1}(N_0,3)>1$ for large $k<P_{g+1}$.

Example 4. Suppose $k=3$ and $P_{g+1}>3$. From eqn (19) we have:

$P_3=2P_2-P_1$. (25)

From eqn (22) we have:

$$J_3(\omega)=\prod_{3\le P}(P-1)(P-2)\to\infty \text{ as } \omega\to\infty, \quad (26)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_2(N,3)=2\prod_{3\le P}\left(1-\frac{1}{(P-1)^2}\right)\frac{N^2}{\log^3 N}(1+o(1))=1.32032\frac{N^2}{\log^3 N}(1+o(1)). \quad (27)$$

Example 5: Suppose $k=4$ and $P_{g+1}>4$. From eqn (19) we have:

$P_3=2P_2-P_1$, $P_4=3P_2-2P_1$. (28)

From eqn (22) we have:

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-3)\to\infty \text{ as } \omega\to\infty, \quad (29)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3 and P_4 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_3(N,3)=\frac{9}{2}\prod_{5\le P}\frac{P^2(P-3)}{(P-1)^3}\frac{N^2}{\log^4 N}(1+o(1)). \quad (30)$$

Example 6: Suppose $k=5$ and $P_{g+1}>5$. From eqn (19) we have:

$P_3=2P_2-P_1$, $P_4=3P_2-2P_1$, $P_5=4P_2-3P_1$. (31)

From eqn (22) we have:

$$J_3(\omega)=2\prod_{5\le P}(P-1)(P-4)\to\infty \text{ as } \omega\to\infty, \quad (32)$$

We prove that there exist infinitely many primes P_1 and P_2 such that P_3, P_4 and P_5 are primes. From eqn (24) we have the best asymptotic formula:

$$\pi_4(N,3)=\frac{27}{2}\prod_{5\le P}\frac{P^3(P-4)}{(P-1)^4}\frac{N^2}{\log^5 N}(1+o(1)). \quad (33)$$

Corollary 2: Arithmetic progression with three prime variables.

From eqn (18) we obtain the arithmetic progression with three prime Lie Theory variables: P_1, P_2 and P_3.

$P_4=P_3+P_2-P_1$, $P_j=P_3+(j-3)P_2-(j-3)P_1$, $4\le j\le k<P_{g+1}$ (34)

We have Jiang function:

$$J_4(\omega)=\prod_{3\le P}((P-1)^3-X(P)), \quad (35)$$

$X(P)$denotes the number of solutions for the following congruence:

$$\prod_{j=4}^{k}(q_3+(j-3)q_2-(j-3)q_1)\equiv 0(\text{mod } P), \quad (36)$$

Where $q_i=1,2,\ldots,P-1$, $i=1,2,3$.

Example 7: Suppose $k=4$ and $P_{g+1}>4$. From eqn (34) we have:

$P_4=P_3+P_2-P_1$. (37)

From eqns (35) and (36) we have:

$$J_4(\omega)=\prod_{3\le P}(P-1)(P^2-3P+3)\to\infty \text{ as } \omega\to\infty, \quad (38)$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that P_4 are primes. We have the best asymptotic formula:

$$\pi_2(N,4)=2\prod_{3\le P}\left(1+\frac{1}{(P-1)^3}\right)\frac{N^3}{\log^4 N}(1+o(1)). \quad (39)$$

For $k\ge5$ from eqns (35) and (36) we have Jiang function:

$$J_4(\omega)=\prod_{3\le P<(k-1)}(P-1)^2\times\prod_{(k-1)\le P}(P-1)[(P-1)^2-(P-2)(k-3)]\to\infty \text{ as } \omega\to\infty. \quad (40)$$

We prove that there exist infinitely many primes P_1 and P_2 and P_3 such that $P_4,\ldots,P_k$ are primes for $5\le k<P_{g+1}$.

We have the best asymptotic formula:

$$_{k-2}(N, 4)= \left|\{P_3+(j-3)P_2-(j-3)P_1=\text{prime},\ 4 \le j \le k,\ P_1,P_2,P_3 \le N\}\right|$$

$$=\frac{J_4(\omega)\omega^{k-3}}{\varphi^k(\omega)}\frac{N^3}{\log^k N}(1+o(1)). \quad (41)$$

From eqn (41) we have:

$$\pi_{k-2}(N,4) = \prod_{2\le P<(k-1)}\frac{P^{k-3}}{(P-1)^{k-2}}\prod_{(k-1)\le P}\frac{P^{k-3}[(P-1)^2-(P-2)(k-3)]}{(P-1)^{k-1}}\frac{N^3}{\log^k N}(1+o(1)). \quad (42)$$

From eqn (42) we are able to find the smallest solution $\pi_{k-2}(N_0, 4)>1$ for large $k<P_{g+1}$.

Corollary 3: Arithmetic progression with four prime variables.

From eqn (18) we obtain the arithmetic progression of algebra with four prime variables: P_1, P_2, P_3 and P_4

$$P_5=P_4+2P_3-3P_2+P_1,\ P_j=P_4+(j-3)P_3-(j-2)P_2+P_1,\ 5 \le j \le k < P_{g+1} \quad (43)$$

We have Jiang function:

$$J_5(\omega) = \prod_{3\le P}\left[(P-1)^4 - X(P)\right], \quad (44)$$

$X(P)$ denotes the number of solutions for the following congruence:

$$\prod_{j=5}^{k}[q_4+(j-3)q_3-(j-2)q_2+q_1]\equiv 0 \pmod P, \quad (45)$$

Where,

$q_i=1,\ldots,P-1, i=1,2,3,4$

Example 8: Suppose k=5 and $k<P_{g+1}>5$. From eqn (43) we have:

$$P_5=P_4+2P_3-3P_2+P_1. \quad (46)$$

From eqns (44) and (45) we have:

$$J_5(\omega) = 12\prod_{5\le P}(P-1)(P^3-4P^2+6P-4)\to\infty \text{ as } \omega\to\infty. \quad (47)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 are primes.

We have the best asymptotic formula:

$$\pi_2(N,5) = \frac{J_5(\omega)\omega}{\varphi^5(\omega)}\frac{N^4}{\log^5 N}(1+o(1)). \quad (48)$$

Example 9: Suppose k=6 and $P_{g+1}>6$. From eqn (43) we have:

$$P_5=P_4+2P_3-3P_2+P_1,\ P_6=P_4+3P_3-4P_2+P_1. \quad (49)$$

From eqns (44) and (45) we have:

$$J_5(\omega) = 10\prod_{5\le P}(P-1)(P^3-5P^2+10P-9)\to\infty \text{ as } \omega\to\infty. \quad (50)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that P_5 and P_6 are primes.

We have the best asymptotic formula:

$$\pi_3(N,5) = \frac{J_5(\omega)\omega^2}{\varphi^6(\omega)}\frac{N^4}{\log^6 N}(1+o(1)). \quad (50)$$

For k≥7 from eqns (44) and (45) we have Jiang function:

$$J_5(\omega) = 6\prod_{5\le P\le(k-4)}(P-1)(P^2-3P+3)$$

$$\times\prod_{(k-4)<P}\left\{(P-1)^4-(P-1)^2\left[(P-3)(k-4)+1\right]-(P-1)(2k-9)\right\}\to\infty \text{ as } \omega\to \quad (51)$$

We prove there exist infinitely many primes P_1, P_2, P_3 and P_4 such that $P_5,\ldots,P_k$ are primes.

We have best asymptotic formula:

$$\pi_{k-3}(N, 5)= \left|\{P_4+(j-3)P_3-(j-2)P_2+P_1=\text{prime},\ 5 \le j \le k,\ P_1,\ldots,P_4 \le N\}\right|$$

$$=\frac{J_5(\omega)\omega^{h-4}}{\varphi^k(\omega)}\frac{N^4}{\log^k N}(1+o(1)). \quad (52)$$

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