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Research Article

On the Oscillations of Nonlinear Third Order Neutral Differential Equations

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Abstract

This paper is concerned with the oscillation of solutions of a class of third order nonlinear neutral differential equations. New sufficient conditions guarantee that every solution is either oscillatory or tends to zero are given. The obtained results improve some recent published results in the literature. Some illustrative examples are given.

Keywords: Oscillations; Nonlinear; Integers

Introduction

In this paper, we are concerned with the oscillatory behavior of solutions of third order differential equations of the type

$$\left(a(t)\left[z''(t)\right]^{\gamma}\right)' + \sum_{i=1}^{m} f_i\left(t, x(\sigma_i(t))\right) = 0,\ t \ge t_0, \tag{1}$$

where $z(t) = x(t) + \sum_{j=1}^{n} p_j(t) x(\tau_j(t))$, m,n are positive integers, $t_0 > 0$ and $a(t)$, $p_j(t)$, $\tau_j(t)$, $\sigma_i(t) \in C([t_0,\infty))$, $i = 1,2,\ldots,m$, $j = 1,2,\ldots,n$.

In the sequel, we assume the following conditions:

1. $a(t)$, $p_j(t)$, $\tau_j(t)$, $\sigma_i(t)$ are positive functions, $\gamma \ge 1$ is a quotient of odd positive integers;

2. $\tau_j(t) \le t$, $\lim_{t\to\infty} \tau_j(t) = \infty$, $\lim_{t\to\infty} \sigma_i(t) = \infty$, $0 \le p_j(t) \le p_{0j}$ and $\sum_{j=1}^{n} p_{0j} < 1$, $j = 1,2,..,n$, $j = 1,2,..,n$;

3. $f_i(t,u) \in C([t_0,\infty) \times \mathbb{R}, \mathbb{R})$ satisfies $u f_i(t,u) > 0$ for all $u \ne 0$ and there exist positive continuous functions $q_i(t)$ defined on $[t_0,\infty)$ such that $|f_i(t,u)| \ge q_i(t)|u|^{\gamma}$, $i = 1,2,\ldots,m$.

The study of the oscillatory behavior of solutions of third-order differential equations has recieved great interest in the last few decades. One of the reasons for that is because in the real life, during the study of some physical phenomena, the qualitative behavior of solutions of third-order differential equations can be succesfully used to predict dynamic behavior of solutions of third-order partial differential equations.

Following this trend we are concerned in this paper with the oscillatory behavior of the third-order neutral differential equation (E).

By a solution of (E), we mean a function $x \in C([T_x,\infty),\mathbb{R})$, $T_x \ge t_0$, which has the properities $z \in C^2([T_x,\infty),R)$, $a(t)\left[z''(t)\right]^{\gamma} \in C^1([T_x,\infty),R)$ and satisfies (E) on $[T_x,\infty)$. In this paper, we consider only those solutions x of (E) which satisfy $\sup\{|x(t)| : t \ge T\} > 0$ for all $T \ge T_x$. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_x,\infty)$; otherwise it is called nonoscillatory.

Recently, increasing attention has been devoted to the oscillation of differential equations of the form (E) and some of its exceptions; have been the subject of intensive researchs see for example the papers [1-13] and references cited in. In particular, we mention here the paper of Grace et al. [5] which studied the oscillation of the third order delay differential equation.

$$\left(a(t)\left[x''(t)\right]^{\gamma}\right)' + q(t) f\left(x(\tau(t))\right) = 0, t \ge t_0, \tag{1.1}$$

By comparing with the first order delay equation, where in their comparison principle it is always required that $\tau(t) < t$. More recently, Baculková and Džurina [3] improved their results for the case when

$$\int_{t_0}^{\infty} a^{\frac{-1}{\gamma}}(t)dt = \infty. \tag{1.2}$$

While the same authors, Baculková and Džurina [2] discussed the oscillation behavior of eqn. (1.1) in the case when

$$\int_{t_0}^{\infty} a^{\frac{-1}{\gamma}}(t)dt < \infty. \tag{1.3}$$

Zhong et al. [13] adapted Grace et al.'s method and extended some of their results to the neutral differential equation

$$\left(a(t)\left[\left(x(t) + p(t)x(\sigma(t))\right)''\right]^{\gamma}\right)' + q(t) f\left(x(\tau(t))\right) = 0.$$

However, the results [13] cannot be applied when

$$\int_{t_0}^{\infty} a^{\frac{-1}{\gamma}}(t)dt < \infty \text{and } \tau(t) \ge t.$$

In this paper are cocerned with this gap for the more general equation (E) by applying a technique similar to that given by those of refs. [8] and [10].

Preliminaries

Lemma 1

Let $x(t)$ be a positive solution of eqns. (E) and (1.2) holds. Then there are only one of the following two cases:

$$(I)\ z(t) > 0, z'(t) > 0, z''(t) > 0 \text{and} \left(a(t)\left[z''(t)\right]^{\gamma}\right)' < 0;$$

$$(II)\ z(t) > 0, z'(t) < 0, z''(t) > 0 \text{and} \left(a(t)\left[z''(t)\right]^{\gamma}\right)' < 0,$$

for $t \ge t_1$, where $t_1 \ge t_0$ is sufficiently large.

Proof: The proof is similar to the proof of Lemma 1 [1] and so it is omitted.

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Lemma 2

Let $x(t)$ be a positive solution of (E). Suppose further that (1.2) holds and the corresponding $z(t)$ satisfies case (II) in Lemma 1. If

$$\int_{t_0}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\sum_{i=1}^{m}\int_{u}^{\infty} q_i(s)ds\right]^{\gamma} dudv=\infty, \tag{2}$$

then $\lim_{t\to\infty} x(t)=0$.

Proof: Assume that $x(t)$ is a positive solution of (E). It is clear that there exists a finite limit, say $\lim_{t\to\infty} z(t)=l\geq 0$. We claim that l=0. If not, then for any $\varepsilon>0$, there exists a $t_1\geq t_0$ such that $l<z(t)<l+\varepsilon$. By choosing $0<\varepsilon<l(1-\sum_{j=1}^{n}p_{0j})/\sum_{j=1}^{n}p_{0j}$, we get

$$x(t)=z(t)-\sum_{j=1}^{n}p_j(t)x(\tau_j(t))$$

$$>l-\sum_{j=1}^{n}p_j(t)z(\tau_j(t))$$

$$>l-\sum_{j=1}^{n}p_{0j}(l+\varepsilon)=N(l+\varepsilon)>Nz(t) \tag{2.1}$$

where $N=(l-\sum_{j=1}^{n}p_{0j}(l+\varepsilon))/(l+\varepsilon)$. This with (E) in view of (A_3) leads to

$$0=\left(a(t)\left[z''(t)\right]^{\gamma}\right)'+\sum_{i=1}^{m} f_i\left(t,x(\sigma_i(t))\right) \tag{2.2}$$

$$\geq\left(a(t)\left[z''(t)\right]^{\gamma}\right)'+N\sum_{i=1}^{m}q_i\left(t\right)z^{\gamma}(\sigma_i(t)) \tag{2.3}$$

Integrating from t to ∞ and using the fact that $z(t)>l$, we obtain

$$0\geq -a(t)\left[z''(t)\right]^{\gamma}+\left(Nl\right)^{\gamma}\sum_{i=1}^{m}\int_{t}^{\infty}q_i(s)ds.$$

i.e.,

$$z''(t)\geq\left(Nl\right)\left[\frac{1}{a(t)}\sum_{i=1}^{m}\int_{t}^{\infty}q_i(s)ds\right]^{\frac{1}{\gamma}}. \tag{2.4}$$

Again by integrating eqn. (2.4) from t to ∞, we get

$$-z'(t)\geq Nl\int_{t}^{\infty}\left[\frac{1}{a(u)}\sum_{i=1}^{m}\int_{u}^{\infty}q_i(s)ds\right]^{\frac{1}{\gamma}}du. \tag{2.5}$$

Integrating eqn. (2.5) from t_1 ($t_1\geq t_0$) to ∞, it follows that

$$\frac{z(t_1)}{Nl}\geq\int_{t_1}^{\infty}\int_{v}^{\infty}\left[\frac{1}{a(u)}\sum_{i=1}^{m}\int_{u}^{\infty}q_i(s)ds\right]^{\frac{1}{\gamma}}dudv.$$

This contradicts eqn. (2.1). Hence l=0. But since $0\leq x(t)\leq z(t)$, then $\lim_{t\to\infty} z(t)=\lim_{t\to\infty} x(t)=0$. This completes the proof.

Now we outline the following two lemmas [1].

Lemma 3

Assume that $u(t)>0$, $u'(t)>0$ and $u''(t)\leq 0$, for $t\geq t_0$. If $\sigma\in C\left([t_0,\infty),(0,\infty)\right)$, $\sigma(t)\leq t$ and $\lim_{t\to\infty}\sigma(t)=\infty$,

Proof: Then for every, there exists a $T_\alpha\geq t_0$ such that $u(\sigma(t))/\sigma(t)\geq\alpha u(t)/t$ for $t\geq T_\alpha$.

Lemma 4

Assume that $u(t)>0$, $u'(t)>0$, and $u'''(t)\leq 0$, for $t\geq t_0$. Then for each $\beta\in(0,1)$, there exists a $T_\beta\geq t_0$ such that $u(t)\geq\beta tu'(t)/2$ for $t\geq T_\beta$.

Further, we give the following auxiliary result which is extracted from those [6]and [7].

Lemma 5

Let $\gamma\geq 1$ be a ratio of two odd positive numbers. Then,

$$A^{1+\frac{1}{\gamma}}-\left(A-B\right)^{1+\frac{1}{\gamma}}\leq\frac{B^{\frac{1}{\gamma}}}{\gamma}\left[(\gamma+1)A-B\right],\text{for all} AB\geq 0 \tag{2.6}$$

and

$$C^{1+\frac{1}{\gamma}}-\frac{\gamma+1}{\gamma}CD^{\frac{1}{\gamma}}\geq\frac{-1}{\gamma}D^{1+\frac{1}{\gamma}},\text{for any} C,D\geq 0. \tag{2.7}$$

Main Results

In this section, we establish new oscillation criteria for eqn. (E) by using a generalized Riccati transformation and integral averaging technique of Philos-type [12]. Let

$$D=\left\{(t,s):t\geq s\geq t_0\right\}\text{and} D_0=\left\{(t,s):t>s\geq t_0\right\}. \tag{3}$$

A function $H\in C^1\left(D,\mathbb{R}\right)$ is said to belong to the class X_γ if

1. $H(t,t)=0$ and $H(t,s)>0$ for all $(t,s)\in D_0$;

2. H has a nonpositive continuous partial derivative $\partial H/\partial s$ on D_0 with respect to the second variable and there exist functions $\rho\in C^1\left([t_0,\infty),(0,\infty)\right)$, $\varphi\in C^1\left([t_0,\infty),(0,\infty)\right)$ and $h\in C\left(D_0,\mathbb{R}\right)$ such that

$$\frac{\partial H(t,s)}{\partial s}+\left[\frac{\rho'(s)}{\rho(s)}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(s)\right]H(t,s)=-h(t,s)\left(H(t,s)\right)^{\frac{\gamma}{\gamma+1}}. \tag{3.1}$$

Note that for $\gamma=1$. X_γ reduces to the class of functions X used [8]. For $\rho=1$ and $\varphi=0$, X_γ reduces to the class of functions W_γ used [9].

Theorem 6

Suppose that the conditions (A_1)-(A_3) hold, $\sigma_i(t)\leq t$, for $i=1,2,\ldots,m$, eqn. (1.2) and (2.1) be satisfied. Assume further that

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left[H(t,s)G(s)-\frac{\left(h\_(t,s)\right)^{\gamma+1}a(s)\rho(s)}{\left(\gamma+1\right)^{\gamma+1}}\right]ds=\infty \tag{3.2}$$

holds for some $c\in(0,1)$ and for some $H\in X_\gamma$, where

$$G(t)=\rho(t)\left[\left(\frac{1}{2}c(1-\sum_{j=1}^{n}p_{0j})\right)^{\gamma m}\sum_{i=1} q_i(t)\left(\frac{(\sigma_i(t))^2}{t}\right)^{\gamma}+a(t)\varphi^{\frac{\gamma+1}{\gamma}}(t)-(a(t)\varphi(t))'\right] \tag{3.3}$$

and $$h\_(t,s)=\max\left\{0,-h(t,s)\right\}. \tag{3.4}$$

Then every solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$.

Proof: Assume that $x(t)$ is a non-oscillatory solution of eqn. (E). Without loss of generality, we may assume that $x(t)$ is eventually positive. Then, there exists a $t_1\geq t_0$ such that $x(t)>0$, $x(\tau_j(t))>0$ and $x(\sigma_i(t))>0$ for $t\geq t_1$ and $i=1,2,..,m$, $j=1,2,..,n$. By Lemma 1, it follows that z satisfies either (I) or (II) for $t\geq t_2$, where $t_2\geq t_1$ is large enough. We consider each of the two cases separately. Assume first that case (I) holds. Hence since $z'(t)>0$, we have

$$x(t)=z(t)-\sum_{j=1}^{n}p_j(t)x(\tau_j(t))$$

$$\geq z(t)-\sum_{j=1}^{n}p_{0j}z(t)=z(t)(1-\sum_{j=1}^{n}p_{0j}) \tag{3.5}$$

In view of (E), we have

$$\left(a(t)\left[z''(t)\right]^{\gamma}\right)'=-\sum_{i=1}^{m}f_i\left(t,x(\sigma_i(t))\right)\leq-\sum_{i=1}^{m}q_i\left(t\right)x^{\gamma}(\sigma_i(t))$$

$$\leq-(1-\sum_{j=1}^{n}p_{0j})^{\gamma}\sum_{i=1}^{m}q_i\left(t\right)z^{\gamma}(\sigma_i(t)) \tag{3.6}$$

Now, consider a generalized Riccati substitution of the form

$$\omega(t)=\rho(t)a(t)\left[\left(\frac{z''(t)}{z'(t)}\right)^{\gamma}+\varphi(t)\right],t\geq t_2. \tag{3.7}$$

Then by eqn. (3.6), we get

$$\omega'(t)\leq\frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)\left(a(t)\varphi(t)\right)'-(1-\sum_{j=1}^{n}p_{0j})^{\gamma}\rho(t)\sum_{i=1}^{m}\frac{q_i(t)z^{\gamma}(\sigma_i(t))}{z'^{\gamma}(t)}$$

$$-\gamma\rho(t)a(t)\left[\frac{\omega(t)}{a(t)\rho(t)}-\varphi(t)\right]^{\frac{\gamma+1}{\gamma}} \tag{3.8}$$

Therefore from Lemma 4 and Lemma 5, it follows that, for any $\alpha\in(0,1)$ and $\beta\in(0,1)$, we have

$$\frac{z(\sigma_i(t))}{z'(t)}=\frac{z(\sigma_i(t))}{z'(\sigma_i(t))}\frac{z'(\sigma_i(t))}{z'(t)}\geq\frac{\alpha\beta}{2}\frac{\left(\sigma_i(t)\right)^2}{t},i=1,2,..,m.$$

Thus

$$\left(\frac{z(\sigma_i(t))}{z'(t)}\right)^{\gamma}\geq\left[\frac{\alpha\beta}{2}\frac{\left(\sigma_i(t)\right)^2}{t}\right]^{\gamma},i=1,2,..,m. \tag{3.9}$$

Combining eqns. (3.8) and (3.9), we get

$$\omega'(t)\leq\frac{\rho'(t)}{\rho(t)}\omega(t)-\left[\frac{1}{2}c(1-\sum_{j=1}^{n}p_{0j})\right]^{\gamma}\rho(t)\sum_{i=1}^{m}q_i(t)\left(\frac{\left(\sigma_i(t)\right)^2}{t}\right)^{\gamma}$$

$$+\rho(t)\left(a(t)\varphi(t)\right)'-\gamma\rho(t)a(t)\left[\frac{\omega(t)}{a(t)\rho(t)}-\varphi(t)\right]^{\frac{\gamma+1}{\gamma}}, \tag{3.10}$$

where $c=\alpha\beta$. Applying the inequality eqn. (2.6) of Lemma 3 with $A=\frac{\omega(t)}{a(t)\rho(t)}$ and $B=\varphi(t)$, we get

$$\left[\frac{\omega(t)}{a(t)\rho(t)}-\varphi(t)\right]^{\frac{\gamma+1}{\gamma}}\geq\left[\frac{\omega(t)}{a(t)\rho(t)}\right]^{\frac{\gamma+1}{\gamma}}-\frac{\varphi^{\frac{1}{\gamma}}(t)}{\gamma}\left[\frac{(\gamma+1)\omega(t)}{a(t)\rho(t)}-\varphi(t)\right]. \tag{3.11}$$

This with eqn. (3.3) yields

$$\omega'(t)\leq\frac{\rho'(t)}{\rho(t)}\omega(t)-\left[\frac{1}{2}c(1-\sum_{j=1}^{n}p_{0j})\right]^{\gamma}\rho(t)\sum_{i=1}^{m}q_i(t)\left(\frac{\left(\sigma_i(t)\right)^2}{t}\right)^{\gamma}$$

$$+\rho(t)\left(a(t)\varphi(t)\right)'-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(a(t)\rho(t))^{\frac{1}{\gamma}}}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\omega(t)$$

$$-a(t)\rho(t)\varphi^{\frac{\gamma+1}{\gamma}}(t)$$

$$=-\rho(t)\left[\left(\frac{1}{2}c(1-\sum_{j=1}^{n}p_{0j})\right)^{\gamma}\sum_{i=1}^{m}q_i(t)\left(\frac{(\sigma_i(t))^2}{t}\right)^{\gamma}+a(t)\varphi^{\frac{\gamma+1}{\gamma}}(t)-(a(t)\varphi(t))'\right]$$

$$+\left[\frac{\rho'(t)}{\rho(t)}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\right]\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(a(t)\rho(t))^{\frac{1}{\gamma}}}.$$

i.e.

$$\omega'(t)\leq-G(t)+\psi(t)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{(a(t)\rho(t))^{\frac{1}{\gamma}}}, \tag{3.12}$$

where $\psi(t)=\left[\frac{\rho'(t)}{\rho(t)}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\right]$. Replacing t in the place of s eqn. (3.12), multiplying both sides by $H(t,s)$ and integrating with respect to s using eqns. (3.1) and (3.4), we get

$$\int_T^t H(t,s)G(s)ds\leq H(t,T)\omega(T)+\int_T^t\frac{\partial H(t,s)}{\partial s}\omega(s)ds+\int_T^t H(t,s)\psi(s)\omega(s)ds$$

$$-\int_T^t\frac{\gamma H(t,s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)ds$$

$$=H(t,T)\omega(T)+\int_T^t\left[-h(t,s)\left(H(t,s)\right)^{\frac{\gamma}{\gamma+1}}\omega(s)-\frac{\gamma H(t,s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\right]ds$$

$$\leq H(t,T)\omega(T)+\int_T^t\left[h\_(t,s)\left(H(t,s)\right)^{\frac{\gamma}{\gamma+1}}\omega(s)-\frac{\gamma H(t,s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\right]ds. \tag{3.13}$$

Now define

$$C=\left(\frac{\gamma H(t,s)\omega^{\frac{\gamma+1}{\gamma}}(s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\right)^{\frac{\gamma}{\gamma+1}}\text{and}D=\left(\frac{h\_(t,s)(\gamma a(s)\rho(s))^{\frac{1}{\gamma+1}}}{(\gamma+1)}\right)^{\gamma}.$$

Applying the inequality eqn. (2.7) of Lemma 3, we get

$$h\_(t,s)\left(H(t,s)\right)^{\frac{\gamma}{\gamma+1}}\omega(s)-\frac{\gamma H(t,s)\omega^{\frac{\gamma+1}{\gamma}}(s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\leq\frac{(h\_(t,s))^{\gamma+1}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}}. \tag{3.14}$$

Thus by eqns. (3.13) and (3.14), we have

$$\frac{1}{H(t,T)}\int_T^t\left[H(t,s)G(s)-\frac{(h\_(t,s))^{\gamma+1}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}}\right]ds\leq\omega(T).$$

This contradicts eqn. (3.2). Now consider the case (II) in Lemma 1. Then by Lemma 2, we have $\lim_{t\to\infty}x(t)=0$. This completes the proof.

In the following result, we consider the case when (1.3) holds, where in this case, there is one more possibility other than (I) and (II) mentioned in Lemma 1.

Theorem 7

Let all the assumptions of Theorem 6 be satisfied. Suppose further that the condition (1.2) is replaced by (1.3) for some $c\in(0,1)$ and for some $H\in X_{\gamma}$, (3.2) holds. If for $\rho(t)=1$, $\varphi(t)=0$, there exists a function $H_1(t,s)\in X_{\gamma}$ such that,

$$\limsup_{t\to\infty}\frac{1}{H_1(t,t_2)}\int_{t_2}^{t}\left[(1-\sum_{j=1}^{n}p_{0j})^{\gamma}H_1(t,s)\sum_{i=1}^{m}q_i(s)\left(\sigma_i(s)-t_1\right)^{\gamma}-\frac{(h_1(t,s))^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}}\right]ds>0 \tag{3.15}$$

for all sufficiently large $t_1\geq t_0$ and $t_2\geq t_1\geq t_0$, then every solution x of (E) is either oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof: Assume that $x(t)$ is a positive solution of (E). Based on the condition eqn. (1.3), there exist three possible cases (I), (II) (as those of Lemma 1), and the following third possibility

$$(III)\ z(t)>0,z'(t)>0,z''(t)<0,\left(a(t)\left[z''(t)\right]^{\gamma}\right)'<0,\ \text{for}\ t\geq t_1,t_1\ \text{is large enough.}$$

Firstly if case (I) and case (II) hold, respectively, we can obtain the conclusion of Theorem 7 by applying the proof of theorem 6.

Now assume the case (III) holds. Then, $z''(t)<0$ for all $t \ge t_1$. Define the function

$$\phi(t)=\frac{a(t)\left(z''(t)\right)^{\gamma}}{z'^{\gamma}(t)},\ t\ge t_1. \tag{3.16}$$

It is clear that $\phi(t)<0$ for $t\ge t_1$, and

$$\phi'(t)=\frac{\left(a(t)\left(z''(t)\right)^{\gamma}\right)'}{z'^{\gamma}(t)}-\frac{\gamma a(t)\left(z''(t)\right)^{\gamma+1}}{z'^{\gamma+1}(t)}. \tag{3.17}$$

Since $z'(t)>0$, then by eqns. (3.5) and (3.17) in view of (E), we get

$$'(t)\le -(1-{}_{j=1}^{n} p_{0j})\ {}_{i=1}^{m} q_i(t)\left(\frac{z(\sigma(t))}{z(t)}\right)-\frac{\gamma\phi\ (t)}{a\ (t)}. \tag{3.18}$$

In view of (III), we have

$$z(t)\ge (t-t_1)z'(t). \tag{3.19}$$

Hence,

$$\left(\frac{z(t)}{t-t_1}\right)'\le 0. \tag{3.20}$$

This implies that

$$\frac{z(\sigma_i(t))}{z(t)}\ge\frac{\sigma_i(t)-t_1}{t-t_1},\ i=1,2,..,m. \tag{3.21}$$

This with eqn. (3.19), leads to

$$\frac{z(\sigma_i(t))}{z'(t)}\ge\left(\sigma_i(t)-t_1\right).$$

Substituting in eqn. (3.18), we get

$$\phi'(t)\le -(1-{}_{j=1}^{n} p_{0j})^{\gamma m}{}_{i=1} q_i(t)\left(\sigma_i(t)-t_1\right)^{\gamma}-\frac{\gamma\phi^{\frac{\gamma+1}{\gamma}}(t)}{a^{\frac{1}{\gamma}}(t)}. \tag{3.22}$$

Interchanging t with s in eqn. (3.22), multiplying both sides by $H_1(t,s)$ and integrating with respect to s from t_2 to t ($t_2\ge t_1$). Then in view of the properties of $H(t,s)$ with $\rho(t)=1$, $\varphi(t)=0$ it follows that,

$${}_{t_2}^{t} H_1(t,s)(1-{}_{j=1}^{n} p_{0j})^{\gamma m}{}_{i=1} q_i(s)\left(\sigma_i(s)-t_1\right)^{\gamma}\le H_1(t,t_2)\phi(t_2)+{}_{t_2}^{t}\frac{\partial H_1(t,s)}{\partial s}\phi(s)ds$$

$$-{}_{t_2}^{t}\frac{\gamma\phi^{\frac{\gamma+1}{\gamma}}(s)H_1(t,s)}{a^{\frac{1}{\gamma}}(s)}ds$$

$$\le H_1(t,t_2)\phi(t_2)+{}_{t_2}^{t}-h_1(t,s)\left(H_1(t,s)\right)^{\frac{\gamma}{\gamma+1}}\phi(s)ds$$

$$+{}_{t_2}^{t}\left[-\frac{\gamma H_1(t,s)}{a^{\frac{1}{\gamma}}(s)}\left(-\phi(s)\right)^{\frac{\gamma+1}{\gamma}}\right]ds \tag{3.23}$$

Now define

$$C_1^{\frac{\gamma+1}{\gamma}}=\frac{\gamma H_1(t,s)\left(-\phi(s)\right)^{\frac{\gamma+1}{\gamma}}}{a^{\frac{1}{\gamma}}(s)}\text{and } D_1^{\frac{1}{\gamma}}=\frac{h_1(t,s)\left(\gamma a(s)\right)^{\frac{1}{\gamma+1}}}{(\gamma+1)}.$$

Applying the inequality eqn. (2.7), it follows that

$$-h_1(t,s)\left(H_1(t,s)\right)^{\frac{\gamma}{\gamma+1}}\phi(s)-\frac{\gamma H_1(t,s)}{a^{\frac{1}{\gamma}}(s)}\left(-\phi(s)\right)^{\frac{\gamma+1}{\gamma}}\le\frac{\left(h_1(t,s)\right)^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}}. \tag{3.24}$$

This with eqn. (3.23) leads to

$$\frac{1}{H_1(t,t_2)}{}_{t_2}^{t}\left[(1-{}_{j=1}^{n} p_{0j})^{\gamma}H_1(t,s)^{m}_{i=1}q_i(s)\left(\sigma_i(s)-t_1\right)^{\gamma}-\frac{\left(h_1(t,s)\right)^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}}\right]ds\le\phi(t_2),$$

which contradicts eqn. (3.15). This completes the proof.

Theorem 8

Assume that the conditions (A_1)-(A_3) hold. Suppose that $i=1,2,..,m$, for $i=1,2,..,m$, eqn. (1.2), (2.1) and (3.4) hold. If for some $\beta\in(0,1)$ and for some $H\in X_{\gamma}$,

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}{}_{t_0}^{t}\left[H(t,s)G_1(s)-\frac{\left(h\_(t,s)\right)^{\gamma+1}a(s)\rho(s)}{(\gamma+1)^{\gamma+1}}\right]ds=\infty \tag{3.25}$$

where

$$G_1(t)=\rho(t)\left[\left(\frac{1}{2}\beta t(1-{}_{j=1}^{n} p_{0j})\right)^{\gamma m}_{i=1} q_i(t)+a(t)\varphi^{\frac{\gamma+1}{\gamma}}(t)-(a(t)\varphi(t))'\right], \tag{3.26}$$

then every solution x of eqn. (E) is either oscillatory or satisfies $\lim_{t\to\infty}x(t)=0$.

Proof: Assume that $x(t)$ is a non-oscillatory solution of (E). Without loss of generality, we may assume that $x(t)$ is eventually positive. Going through as in the proof of Theorem 6, we arrive eqn. (3.8). Since $z'(t)>0$ and $\sigma_i(t)\ge t$, for $i=1,2,..,m$, we obtain

$$\omega'(t)\le\frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)\left(a(t)\varphi(t)\right)'-(1-{}_{j=1}^{n} p_{0j})^{\gamma}\rho(t)^{m}_{i=1}\frac{q_i(t)z^{\gamma}(t)}{z'^{\gamma}(t)}$$

$$-\gamma\rho(t)a(t)\left[\frac{\omega(t)}{\rho(t)a(t)}-\varphi(t)\right]^{\frac{\gamma+1}{\gamma}} \tag{3.27}$$

Then from Lemma 5 It follows for any $\beta\in(0,1)$ that,

$$\frac{z(t)}{z'(t)}\ge\frac{\beta t}{2}.$$

i.e.,

$$\left(\frac{z(t)}{z'(t)}\right)^{\gamma}\ge\left(\frac{\beta t}{2}\right)^{\gamma}. \tag{3.28}$$

Combining eqns. (3.27) and (3.28), we get

$$\omega'(t)\le\frac{\rho'(t)}{\rho(t)}\omega(t)+\rho(t)\left(a(t)\varphi(t)\right)'-(\frac{1}{2}\beta t(1-{}_{j=1}^{n} p_{0j}))^{\gamma}\rho(t)^{m}_{i=1}q_i(t)$$

$$-\gamma\rho(t)a(t)\left[\frac{\omega(t)}{\rho(t)a(t)}-\varphi(t)\right]^{\frac{\gamma+1}{\gamma}} \tag{3.29}$$

Using the inequality eqns. (2.6) to (3.29), we conclude that

$$\omega'(t)\le-\rho(t)\left[\left(\frac{1}{2}\beta t(1-{}_{j=1}^{n} p_{0j})\right)^{\gamma m}_{i=1} q_i(t)+a(t)\varphi^{\frac{\gamma+1}{\gamma}}(t)-(a(t)\varphi(t))'\right]$$

$$+\left[\frac{\rho'(t)}{\rho(t)}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\right]\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}$$

$$=-G_1(t)+\psi(t)\omega(t)-\frac{\gamma\omega^{\frac{\gamma+1}{\gamma}}(t)}{\left(\rho(t)a(t)\right)^{\frac{1}{\gamma}}}, \tag{3.30}$$

where $\psi(t)=\left[\frac{\rho'(t)}{\rho(t)}+(\gamma+1)\varphi^{\frac{1}{\gamma}}(t)\right]$. Replacing t with $s$eqn. (3.30), multiplying both sides by $H(t,s)$ and integrating with respect to s from T_1 ($T_1 \geq t_0$) to t. In view of the fact that $H(t,t)=0$, it follows eqns. (3.1) and (3.4) that

$$\int_{T_1}^{t} H(t,s)G_1(s)ds \leq H(t,T_1)\omega(T_1)+\int_{T_1}^{t}\left[h\_(t,s)\left(H(t,s)\right)^{\frac{\gamma}{\gamma+1}}\omega(s)-\frac{\gamma H(t,s)}{(a(s)\rho(s))^{\frac{1}{\gamma}}}\omega^{\frac{\gamma+1}{\gamma}}(s)\right]ds. \quad (3.31)$$

Again following the proof of Theorem 6, we get a contradiction eqn. (3.25). Assume that case (II) holds. By virtue of Lemma 2, $\lim_{\to\infty} x(t)=0$, and thus the proof is completed.

Theorem 9

Let all the assumptions of Theorem 8 be satisfied. Suppose that the condition eqn. (1.2) is replaced by eqn. (1.3) and for some $\beta \in (0,1)$ and some $H \in X_\gamma$, eqn. (3.25) holds. Suppose further that for $\rho(t)=1$, $\varphi(t)=0$, there exists a function $H_2(t,s) \in X_\gamma$ such that, for all sufficiently large $t_1 \geq t_0$ and $t_2 \geq t_1 \geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H_2(t,t_2)}\int_{t_2}^{t}\left[(1-\sum_{j=1}^{n} p_{0j})^{\gamma}(s-t_1)^{\gamma}H_2(t,s)\sum_{i=1}^{m}q_i(s)-\frac{(h_2(t,s))^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}}\right]ds>0. \quad (3.32)$$

Then every solution x of eqn. (E) is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$.

Proof: Assume that $x(t)$ is a positive solution of eqn. (E). Based on the condition eqn. (1.3), there exist the three possible cases (I), (II) and (III) for $t \geq t_1$, t_1 is large enough. Assume that case (I) and case (II) hold, respectively. We can obtain the conclusion of Theorem 9 by applying the proof of Theorem 8. Now assume that case (III) holds. Consider again the function $\phi(t)$ defined by eqn. 3.16. Then we can easily deduce eqn. 3.18. Since $z'(t)>0$ and $\sigma_i(t) \geq t$, $i=1,2,..,m$, we obtain

$$\phi'(t) \leq -(1-\sum_{j=1}^{n} p_{0j})^{\gamma}\sum_{i=1}^{m}q_i(t)\left(\frac{z(t)}{z'(t)}\right)^{\gamma}-\frac{\gamma\phi^{\frac{\gamma+1}{\gamma}}(t)}{a^{\frac{1}{\gamma}}(t)}. \quad (3.33)$$

In view of (III), we see that

$$\left(\frac{z(t)}{z'(t)}\right)^{\gamma} \geq (t-t_1)^{\gamma}. \quad (3.34)$$

This with eqn. (3.33) leads to

$$\phi'(t) \leq -(1-\sum_{j=1}^{n} p_{0j})^{\gamma}\sum_{i=1}^{m}q_i(t)(t-t_1)^{\gamma}-\frac{\gamma\phi^{\frac{\gamma+1}{\gamma}}(t)}{a^{\frac{1}{\gamma}}(t)}.$$

Going through as in the proof of Theorem 7, we can easily deduce that

$$\frac{1}{H_2(t,t_2)}\int_{t_2}^{t}\left[(1-\sum_{j=1}^{n} p_{0j})^{\gamma}(s-t_1)^{\gamma}H_2(t,s)\sum_{i=1}^{m}q_i(s)-\frac{\left(h_2(t,s)\right)^{\gamma+1}a(s)}{(\gamma+1)^{\gamma+1}}\right]ds \leq \phi(t_2),$$

which contradicts eqn. 3.32. This completes the proof.

Example 1

Consider the differential equation

$$\left(\left(\left(x(t)+\frac{1}{e^{\frac{1}{5}}}x(t-\frac{1}{5})+\frac{1}{5e^{\frac{1}{2}}}x(t-\frac{1}{2})\right)''\right)^{3}\right)'+\frac{3993}{125}x^{3}(t)=0, t \geq 1. \quad (3.35)$$

Choosing $\rho(t)=\frac{1}{t^3}$, $\varphi(t)=\frac{1}{8t^3}$ and $H(t,s)=(t-s)^2$. It is clear that all assumptions of Theorem 6 are satisfied. Hence, every solution x of eqn. (3.35) is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$. As a matter of fact, one such solution is $x(t)=e^{-t}$.

Example 2

Consider the differential equation

$$\left(t^3\left(x(t)+\frac{1}{16}x(\frac{t}{2})+\frac{1}{108}x(\frac{t}{3})\right)''\right)'+42x(t)=0, t \geq 1. \quad (3.36)$$

By choosing $\rho(t)=\frac{1}{t}$, $\varphi(t)=\frac{1}{2t}$, $H(t,s)=(t-s)$, $H_1(t,s)=\frac{(t-s)^2}{s^2}$ and $t_2=t_1=t_0=1$. It is clear that all assumptions of Theorem 7 are satisfied. Hence, every solution x of eqn. 3.36 is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$, when $c>\frac{108}{2807}$. In fact such solution of eqn. 3.36 is $x(t)=\frac{1}{t^3}$.

Example 3

Consider the differential equation

$$\left(t\left(x(t)+p_1x(\frac{t}{2})\right)''\right)'+\frac{\lambda}{t^2}x(t)=0, \lambda>0, t \geq 1. \quad (3.37)$$

By choosing $\rho(t)=1$, $\varphi(t)=0$ and $H(t,s)=(t-s)^2$. It is clear that all assumptions of Theorem 6 and Theorem 8 are satisfied. Hence, every solution x of eqn. 3.37 is either oscillatory or satisfies $\lim_{t\to\infty} x(t)=0$, provided that $\lambda>0$. We note that the authors [10] proved that eqn. 3.37 is oscillatory if $\lambda>1/(4k(1-p_1))$ for some $k \in (1/4,1)$ and so our result improves those [10].

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