Journal of Physical Mathematics
Moulahi and Dlala, J Phys Math 2017, 8:4
DOI: 10.4172/2090-0902.1000248
Research Article
Open Access

Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle

Moulahi A[1]* and Dlala M[2]

[1]Department of Mathematics, college of Sciences, Qassim university, Kingdom of Saudi Arabia
[2]Department of Mathematics, Faculty of Sciences of Monastir, University of Monastir, Monastir, Tunisia

Abstract

We consider a stabilization problem for a coupled wave equations on an exterior of bounded domain $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ with interior stabilization. Under a geometrical control condition (BLR condition), for any initial data in the energy space, we show a result of exponential stability in odd dimensional case and polynomial stability in the case of even dimension.

Keywords: Defect measure; stabilization; Energy; Resolvent; Low and High frequencies

Introduction

In this paper we study the stabilization of a coupled wave equations. More precisely, we consider the following initial and boundary value problem :

$$\partial_t^2 u_1 - \Delta u_1 + a(x)\partial_t u_1 + \partial_t u_2 = 0 \; in \; \Omega \times (0,+\infty), \tag{1}$$

$$\partial_t^2 u_2 - \alpha \Delta u_2 - \partial_t u_1 = 0 \; in \; \Omega \times (0,+\infty), \tag{2}$$

$$u_1 = 0 \; on \; \partial\Omega \times (0,+\infty), \tag{3}$$

$$u_2 = 0 \; on \; \partial\Omega \times (0,+\infty), \tag{4}$$

$$u_1(x,0) = u_1^0(x), \partial_t u_1(x,0) = u_1^1(x) \; in \; \Omega, \tag{5}$$

$$u_2(x,0) = u_2^0(x), \; \partial_t u_2(x,0) = u_2^1(x) \; in \; \Omega, \tag{6}$$

where $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ and $\mathcal{O}$ an open bounded set of $\mathbb{R}^d$ with smooth boundary $\partial\Omega = \partial\mathcal{O}$, $a(x) \in C_0^\infty(\Omega)$ is a positive functions and α is a positive constant.

The study of systems like (1)-(6) (and more generally coupled PDEs systems) is motivated by several physical considerations. In fact, There are many applied problems that can be modeled using coupled partial differential equations, for instance in heating processes, magnetohydrodynamics, quantum mechanics, optics, fluid dynamics....

Among the nowadays many contributions, using different methods and techniques, are given, and relevant reference therein [1,2].

One of the earliest tools in the stabilization analysis of partial differential equation is the micro-local default action of Gérard [3], Tartar [4].

Such techniques have been used firstly to study and to explicit the value of the best decay rate of damped waves equation [5], reduce the boundary and the regularity of the initial data or to show that the geometric condition for control by the board is required [6,7].

Similar works, based on the use of microlocal defect measures in the spirit of the article, have been achieved [5]. In the large time behavior of solutions of the wave equation were studied. The microlocal defect measures have been used to provide estimates of energy was shown in particular how these demonstrate the results of exact controllability, observation and stabilization [8]. without any assumption on the dynamics, the logarithmic decay of the local energy with respect to any Sobolev norm larger than the initial energy is proved [6]. In the two and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions were studied. In two space dimensions they proved a sufficient (and almost necessary) condition for the uniform decay under an assumption on the boundary of the domain and in three space dimensions sufficient conditions for the uniform decay are given [9].

Also, These techniques are used to study the stabilization of the wave equation in a domain with exterior Dirichlet condition [1], for the equation of damped waves equation in an outside field and under an "Exterior Geometric Control" condition inspired from the so-called microlocal condition of Bardos et al. [10] then for the stabilization of electromagnetic waves on an exterior bounded obstacle in 2D and 3D is treated, and under an exterior geometric control condition the behavior of the solution for large time is studied [11-13].

Later, in three dimension space and under a microlocal geometric condition, the rate of decay of the local energy for solutions of the Lamé system on exterior domain, with localized nonlinear damping was given in ref. [14].

Recently these techniques are also used to study the stabilization of different coupled equations and different results have been established in this domain, some results are given, by Duyckaerts [15] the exponential and the polynomial stabilization of a coupled hyperbolic-parabolic system of thermoelasticity are addressed with microlocal techniques, explained by Atallah-Baraket and Kammerer [16] the energy decay of thermoelasticity system with a degenerated second order operator in the Heat equation was studied, a stabilization problem for a coupled wave equations on a compact Riemanian manifold under a geometrical control condition was examined and a logarithmic decay result of the energy is given [13]. And finally in the exact controllability problem on a compact manifold for two coupled wave equations, with a control function acting on one of them only was treated [17].

***Corresponding author:** Moulahi A, Proferssor, College Of Business and Economics, Qassim university, Kingdom of Saudi Arabia, Tel:154545445; E-mail: ammar.moulahi@fsm.rnu.tn

Received August 03, 2017; **Accepted** October 02, 2017; **Published** October 05, 2017

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

Our aim in this work is to establish the energy decay and to give the best rate of convergence of a coupled damped wave equation On an exterior bounded obstacle. We prove this result in a geometric hypothesis and by using the arguments of the analysis microlocal.

The organization of this paper is as follows. In section 2, we give the main result and recalled some preliminary results. In section 3, we will study the poles of the resolvent, in the first, by means of conventional techniques is given a location on the low frequencies and by the defect measures theory we study the high frequencies. In section 4, the main results concerning the stability of systems are established. startsection section1@-3.5ex plus -1ex minus -.2ex2.3ex plus .2ex Preliminaries and Main result

Let $u=(u_1,u_2)$ then the system of equations (1)-(6) is equivalent to the following system

$$\begin{cases}\partial_t^2 u - D_\alpha u + K_a\partial_t u = 0 \ \ in \ \ \Omega\times(0,+\infty),\\ u=0 \ \ on \ \ \partial\Omega\times(0,+\infty),\\ u(\cdot,0)=u^0 \ \ \partial_t u(\cdot,0)=u^1, \ \ in \ \ \Omega,\end{cases} \tag{7}$$

where

$$D_\alpha = \begin{pmatrix}\Delta & 0\\ 0 & \alpha\Delta\end{pmatrix}, \quad K_a = \begin{pmatrix}2a(x) & 1\\ -1 & 0\end{pmatrix},$$

$$u^0=(u_1^0,u_2^0) \ and \ u^1=(u_1^1,u_2^1).$$

Due to the nonlinear semi-group theory, it is well known that the problem (7) has an unique solution, obtained by using the Lummer-Philips theorem for an unbounded operator [18].

We consider the Hilbert space $H=\left(H_0^1(\Omega)\right)^2\oplus(L^2(\Omega))^2$, which is the closure of $\left(C_0^\infty(\Omega)\right)^2\times\left(C_0^\infty(\Omega)\right)^2$ with respect to the norm

$$\int_\Omega |\nabla f_1|^2+\alpha|\nabla f_2|^2+|g_1|^2+|g_2|^2\,dx.$$

We define

$$A_a^\alpha = \begin{pmatrix}0 & id\\ D_\alpha & -K_a\end{pmatrix} \tag{8}$$

and

$$\begin{aligned}\mathcal{D}(A_a^\alpha) &= \left\{(u_1,u_2)\in\{(u_1,u_2)\in H \ and \ (u_2, D_\alpha u_1 - K_a u_2)\in H\right\}\\ &= \left((H_0^1(\Omega))\cap H^2(\Omega)\right)^2\oplus\left(H_0^1(\Omega)\right)^2\Big\}.\end{aligned} \tag{9}$$

We can write the problem (7) as the following form

$$\begin{aligned}\partial_t u_a^\alpha(t)f &= A_a^\alpha u_a^\alpha(t)f\\ u_a^\alpha(0)f &= f\end{aligned} \tag{10}$$

where $u_a^\alpha(t)f=\left(u,\partial_t u\right)$. The problem (7) and (10) are equivalents if and only if that A_a^α has a domain $D(A_a^\alpha)$.

The problem (7) has an unique solution, obtained by using the Hille-Yosida theorem for an unbounded operator.

Let $u(x,t)=(u_1,u_2)(x,t)$ solution of (7) and we set $\nabla_x^\alpha u=(\nabla_x u_1,\sqrt{\alpha}\nabla_x u_2)$. We define the energy functional at the time t by

$$\begin{aligned}E(u,t) &= \frac{1}{2}\int_\Omega\left(|\partial_t u|^2+|\nabla_x^\alpha u|^2\right)\\ &= \frac{1}{2}\int_\Omega\left(|\partial_t u_1|^2+|\partial_t u_2|^2+|\nabla_x u_1|^2+\alpha|\nabla_x u_2|^2\,dx\right)\end{aligned} \tag{11}$$

that satisfy the following estimation

$$E(u,0)-E(u,t)=\int_0^t\int_\Omega a(x)\,|\partial_s u_1(x,s)|^2\,dxds. \tag{12}$$

Let $R>0$ such that $\overline{\mathcal{O}}\subset B_R=\left\{x\in R^n, \|x\|<R\right\}$, we set $\Omega_R=\Omega\cap B_R$. For $u=(u_1,u_2)$ solution of (7), we denote $E_R(u)(t)$ the local energy of at instant $t>0$ define by

$$E_R(u)(t)=\int_{\Omega_R}\left(|\nabla u_1|^2+|\partial_t u_1|^2+\alpha|\nabla u_2|^2+|\partial_t u_2|^2\right)dx. \tag{13}$$

Now, according to the research of Moulahi [19], we recall that at the boundary point $(t,x)\in\partial\widetilde{\Omega}$ ($\widetilde{\Omega}=R\times\Omega$). Let $(t,\eta)\neq(0,0)$ be a tangential direction to $(t,x)\in\partial\widetilde{\Omega}$; that is $\eta.\nu(x)=0$, $\nu(x)$ being the exterior normal to $\partial\Omega$ at x. with the assumption $\alpha\neq1$ we can consider (τ,η) as an element of $T^*_{(t,x)}(\partial\widetilde{\Omega})$, and to look for its inverse image is the both characteristic sets means to look for $\lambda\in R$ such that

$$p_1(t,x;\tau,\eta+\lambda\nu(x))=0, \ \ p_\alpha(t,x;\tau,\eta+\lambda\nu(x))=0.$$

That is

$$p_1(t,x;\tau,\eta+\lambda\nu(x))=|\eta|^2+\lambda^2-\tau^2, \ \ p_\alpha(t,x;\tau,\eta+\lambda\nu(x))=\alpha(|\eta|^2+\lambda^2)-\tau^2$$

and we write

$$\lambda=\pm\sqrt{\tau^2-|\eta|^2}, \ \ or \ \ \lambda=\pm\sqrt{\frac{\tau^2}{\alpha}-|\eta|^2}.$$

Hence, for the existence of such real λ, one of the two relations

$$r_1=\tau^2-\eta^2\geq0 \ \ or \ \ r_\alpha=\tau^2-\alpha\eta^2\geq0$$

mast be fulfilled. From the geometrical point of view there are some possibilities for a tangential direction $\zeta=(\tau,\eta)\neq(0,0)$ with different number of inverse image with respect to the projection $T^*\overline{\widetilde{\Omega}}_{|\partial\widetilde{\Omega}}\to T^*(\partial\widetilde{\Omega})$. We introduce the characteristic transversal manifold:

$$Char\mathcal{T}=Char\mathcal{T}_\Omega\cup Char\mathcal{T}_{\partial\Omega},$$

where

$$Char\mathcal{T}_\Omega=\left\{(t,x;\tau,\xi),\ \tau^2-\alpha|\xi|^2=0, t>0\right\}$$

$$Char\mathcal{T}_{\partial\Omega}=\left\{(t,y;\tau,\eta),\ y\in\partial\Omega,\ t>0, r_\alpha\geq0\right\}$$

and the characteristic longitudinal manifold of the wave coupled system is

$$Char\mathcal{L}=Char\mathcal{T}_\Omega\cup Char\mathcal{T}_{\partial\Omega},$$

where

$$Char\mathcal{L}_{\partial\Omega}=\left\{(t,y;\tau,\eta),\ y\in\partial\Omega,\ t>0, r_1\geq0\right\}$$

$$Char\mathcal{L}_{\partial\Omega}=\left\{(t,y;\tau,\eta),\ y\in\partial\Omega,\ t>0, r_1\geq0\right\}$$

the characteristic manifold of the system is

$$char\mathcal{P}_\alpha=char\mathcal{P}_{\alpha\Omega}\cup char\mathcal{P}_{\alpha\partial\Omega}$$

and the assumption on the coupled wave ($\alpha\neq1$) one obtains

$$char\mathcal{P}_{\alpha\Omega}=Char\mathcal{T}_\Omega\cup Char\mathcal{L}_\Omega$$

and

$$Char\mathcal{P}_{\alpha\partial\Omega}=Char\mathcal{T}_{\partial\Omega} \ if \ \alpha>1$$

either

$$Char\mathcal{P}_{\alpha\partial\Omega}=Char\mathcal{L}_{\partial\Omega} \ if \ \alpha<1.$$

According, we recall the following definition [12,14]

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

Definition 0.1

Let $\eta \in T^*\partial\Omega$, we say that

1. η is a elliptic (or $\eta \in \mathcal{E}$) if and only if $\eta \notin Char\mathcal{P}_{\alpha\partial\Omega}$.

2. η is a hyperbolic for the longitudinal wave (or $\eta \in \mathcal{H}_L$) if and only if $r_1>0$

3. η is a glancing for the longitudinal wave (or $\eta \in \mathcal{G}_L$) if and only if $r_1=0$.

4. η is a hyperbolic for the transversal wave (or $\eta \in \mathcal{H}_T$) if and only if $r_\alpha>0$.

5. η is a glancing for the transversal wave (or $\eta \in \mathcal{G}_T$) if and only if $r_\alpha=0$

Now, we are going to make a description of a generalized bicharacteristic path and refer to the research of Lebeau G [5] for more details. The generalized bicharacteristic flow lives in $Char\mathcal{P}_\alpha \subset T^*\tilde{\overline{\Omega}}$ and for $\rho \in Char\mathcal{P}_\alpha$, we denote by $G(s,\rho)$ the generalized bicharacteristic path starting from ρ. Since $char\mathcal{P}_\alpha$ is the disjoint union of $char\mathcal{P}_{\alpha\Omega}$ $\mathcal{H}_T$ and $\mathcal{G}_T$ if $\alpha>1$ or $Char\mathcal{P}^{\alpha}_{\alpha\,\Omega}$, $\mathcal{H}_L$ and $\mathcal{G}_L$ if $\alpha<1$. We shall consider separately the case where ρ belongs to each one of these sets. Moreover all the description below holds for $|s|$ small, in the following we assume $\alpha>1$.

Case 1. $\rho \in char\mathcal{P}_{\alpha\Omega}$ Here $\rho=(x,t;\xi,\tau)$ where $x \in \Omega$, $t \in (0,T)$ $\mathrm{dept}_{1,\alpha}(x,t;\xi,\tau)=0$. Then for $|s|$ small, we have

$$G(s,\rho)=\left(x(s),t(s),\tau,\xi\right) \subset T^*(R\times\Omega).$$

Where $(x(s),\xi)$ is the characteristic starting from the point (x,ξ) of

- $p_1=-\tau^2+|\xi|^2$ if $\rho \in Char\mathcal{L}_\Omega$,
- $p_\alpha=-\tau^2+\alpha|\xi|^2$ if $\rho \in Char\mathcal{T}_\Omega$.

Case 2. $\rho \in E(char\mathcal{P}_\alpha)_{\partial\Omega}$ (i.e $0 \le p_1$)

Here $\rho=(x(s),t(s),\eta(s),\tau(s))$ where $x \in \partial\Omega$, $t \in (0,T)$ and the equation $p_{1,\alpha}(x,t,\eta+\xi_n,\tau)=0$ has roots $\xi_n=\lambda\nu(x)$ described in ref. [12] and we have one of the two relation

$$r_1=\tau^2-\eta^2,\quad r_\alpha=\tau^2-\alpha\eta^2.$$

For $s>0$ (resp. $s<0$), let $G^-(s,\rho)=(x^-(s),t(s),\xi^-,\tau(s))$ (resp. $G^-(s,\rho)=(x^-(s),t(s),\xi^-,\tau(s))$ be the outgoing (resp. incoming) bicharacterestic of $\mathcal{P}_\alpha$. The generalized bicharacteristic path is such that $G(0,\rho)=\rho$ and

$$G(s,\rho)=\begin{cases} G^+(s,\rho) & 0<s<\varepsilon \\ G^-(s,\rho) & -\varepsilon<s<0. \end{cases}$$

Four possibilities may occur

1. $$\begin{cases} x^+(s) &= x+2\alpha s\xi^+,\ 0<s<\varepsilon, \\ x^-(s) &= x+2\alpha s\xi^-,\ -\varepsilon<s<0, \end{cases}$$

where $\xi^+=\eta-\dfrac{\sqrt{r_\alpha}}{\sqrt{\alpha}}\nu(x)$ and $\xi^-=\eta+\dfrac{\sqrt{r_\alpha}}{\sqrt{\alpha}}\nu(x)$.

In particular, if $0<r$ one has $x(s)\in\Omega$ for small $|s|\neq 0$

2. If $0 \le p_\alpha$ (i.e., $\eta \in G_\alpha \cup \mathcal{H}_\alpha \subset \mathcal{H}_1$):

$$\begin{cases} x^+(s) &= x+2s\xi^+,\ 0<s<\varepsilon, \\ x^-(s) &= x+2s\xi^-,\ -\varepsilon<s<0, \end{cases}$$

where $\xi^+=\eta-\sqrt{r_1}\nu(x)$ and $\xi^-=\eta+\sqrt{r_1}\nu(x)$.

$$\begin{cases} x^+(s) &= x+2s\xi^+,\ 0<s<\varepsilon, \\ x^-(s) &= x+2\alpha s\xi^-,\ -\varepsilon<s<0, \end{cases}$$

where $\xi^+=\eta-\sqrt{r_1}\nu(x)$ and $\xi^-=\eta+\dfrac{\sqrt{r_\alpha}}{\sqrt{\alpha}}\nu(x)$.

$$\begin{cases} x^+(s) &= x+2\alpha s\xi^+,\ 0<s<\varepsilon, \\ x^-(s) &= x+2s\xi^-,\ -\varepsilon<s<0, \end{cases}$$

where $\xi^+=\eta-\dfrac{\sqrt{r_\alpha}}{\sqrt{\alpha}}\nu(x)$, $\xi^-=\eta+\sqrt{r_1}\nu(x)$.

We can see that the nature of the generalized bicharacteristic path changes when hitting the boundary, since it moves from $char\mathcal{L}$ to $char\mathcal{T}$ in 2 ii- and conversly from $char\mathcal{T}$ to $char\mathcal{L}$ in 2 iii-. Following ref. [14], we have:

Definition 0.2

We will call generalized bicharacteristic path any curve which consists of generalized bicharacteristics of $\mathcal{P}_\alpha$ with possibility of moving from a characteristic manifold to another, at each of $\partial\tilde{\Omega}$, in the way indicated above.

In order to state the main results of this paper, we give the definition of outside geometric control condition (OGCC) introduced [1] inspired of [5].

Definition 0.3

Let $R>0$ such that $\overline{\mathcal{O}} \subset B_R$, $T_R>0$ and $\omega=\{a>0\}$ We shall say that (ω,T_R) satisfy the outgoing geometric control condition (OGCC) above B_R if every generalized geodesic path [1] γ derived, at time $t=0$ a point in $T^*_b(R_+\times\Omega_R)$ satisfies the following conditions

- γ leave $\mathbb{R}\times B_R$ before the time T_R.
- γ meet the region $R_+\times\omega$ between the times 0 and T_R.

Let $t>0$, we set

$$C_1(t)=\inf_{\rho_0}\frac{1}{t}\int_0^t a(x_1(s,\rho_0))ds,\ C_2(t)=\inf_{\rho_0}\frac{1}{t}\int_0^t a(x_2(s,\rho_0))ds.$$

that satisfies

$$tC_i(t)+sC_i(s) \le (t+s)C_i(t+s),\ i=1,2.$$

We denote

$$\begin{aligned} C(t) &= \min\left(C_1(t),C_2(t)\right) \\ &= \min\left(\inf_{\rho_0}\frac{1}{t}\int_0^t a(x_1(s,\rho_0))ds,\ \inf_{\rho_0}\frac{1}{t}\int_0^t a(x_2(s,\rho_0))ds\right) \end{aligned} \quad (14)$$

that is a additive function and we set $C(\infty)=\lim_{t\to+\infty}C(t)$. We have $C(t)\le C(\infty)$ for all t.

Theorem 0.4 Assume $\alpha\neq 1$ and under the hypothesis of (OGCC) above the B_R, for any $\delta<\rho=2min\left(D(0),C(\infty)\right)$, there exists $c>0$ such that for all $g \in H$ supported in B_R we have the following estimate of the energy

$$\begin{aligned} E_R(u)(t) &\le e^{-\delta t}E(u)(0)\ \forall t\ge 0\ if\ d\ odd \\ &or \\ E_R(u)(t) &\le \frac{1}{t^d}E(u)(0)\ \forall t\ge 0\ if\ d\ even \end{aligned} \quad (15)$$

startsection section1@-3.5ex plus -1ex minus -.2ex2.3ex plus .2ex

[1] A generalized geodesic path is a projection of a generalized bicharacteristic path on $\overline{\Omega}$.

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

Location of the outgoing resolvent poles

We consider the operator $\mathcal{R}_a^\alpha(\lambda)$ define by the following expression

$$\mathcal{R}_a^\alpha(\lambda)f = \int_0^{+\infty} e^{-i\lambda t} u_a^\alpha(t) f dt \ \ for \ \ Im\lambda < 0 \tag{16}$$

A_a^α is dissipative operator, by the Hille-Yosida theorem, generate a contraction semigroup $\left(u_a^\alpha(t)\right)_{t\geq 0}$.

Then, it is clear that the relation (16) define a bounded family of operators from $(L^2(\Omega))^2$ *onto* $H(\Omega)$ and it is holomorphic in $\{Im\lambda<0\}$

Moreover, we have the following characterization of the resolvent $\mathcal{R}_a^\alpha$:

Lemma 0.5 For all $f\in\left(L^2(\Omega)\right)^2$ with support in B_R and for all $\lambda\neq 0$ and $Im\lambda\leq 0$ we have $\mathcal{R}_a^\alpha(\lambda)f$ is the unique solution satisfies the outgoing radiation condition (OGRC) of the following problem*:*

$$\begin{array}{rcl} -D_\alpha\Psi - \lambda^2\Psi + i\lambda K_a\Psi &=& f in \ \Omega, \\ \Psi &=& 0 \ on \ \partial\Omega, \end{array} \tag{17}$$

Firstly, we recall that $u=(u_1, u_2)$ satisfy the outgoing radiation condition if the following identity satisfy

$$\lim_{R\to\infty}\int_{|x|=R} |\partial_\nu u_1 + i\lambda u_1|^2 + |\sqrt{\alpha}\partial_\nu u_2 + i\lambda u_2|^2 \, d\sigma(x) = 0. \tag{18}$$

Now, let ψ the difference between two solution of (17). Then, ψ satisfy the homogenous problem with Dirichelet boundary. By integration on Ω_R for R large enough, we have

$$i\overline{\lambda}\langle\overline{\psi}, D_\alpha\Psi\rangle + i\overline{\lambda}\lambda^2\langle\overline{\psi},\psi\rangle - i\overline{\lambda}\lambda\langle\overline{\Psi}, K_a\psi\rangle = 0$$

this implies that

$$\begin{array}{c} i\overline{\lambda}\lambda^2\int_{\Omega_R}|\psi_1|^2\,dx + i\overline{\lambda}\lambda^2\int_{\Omega_R}|\psi_2|^2\,dx \\ -i\overline{\lambda}\int_{\Omega_R}|\nabla\psi_1|^2 + \alpha|\nabla\psi_2|^2\,dx + \overline{\lambda}\lambda\int_{\Omega_R}a(x)|\psi_1|^2\,dx \\ -2\overline{\lambda}Im\lambda\int_{\Omega_R}\overline{\psi_2}\cdot\psi_1 \\ i\overline{\lambda}\int_{\partial\Omega_R}\partial_\nu\psi_1\cdot\overline{\psi_1}d\sigma(x) + i\overline{\lambda}\int_{\partial\Omega_R}\partial_\nu\psi_2\cdot\overline{\psi_2}d\sigma(x) = 0 \end{array} \tag{19}$$

In particular, given that the real part of (19) is zero, gives

$$\begin{array}{c} -2Im\left(\lambda\int_{|x|=R}\partial_r\overline{\psi_1}\cdot\psi_1 d\sigma + \lambda\alpha\int_{|x|=R}\partial_r\overline{\psi_2}\cdot\psi_2 d\sigma\right) = -2|\lambda|^2\int_{\Omega_R}a(x)|\psi_1|^2 \\ -2Im\lambda\int_{\Omega_R}|\nabla\psi_1|^2 + \alpha|\nabla\psi_2|^2 - 2Im\lambda|\lambda|^2\int_{\Omega_R}|\psi_1|^2 + |\psi_2|^2\,dx. \end{array} \tag{20}$$

Since

$$\begin{array}{rcl} |\partial_\nu^\alpha\Psi + i\lambda\Psi|^2 &=& |\partial_\nu\psi_1|^2 + \alpha|\partial_\nu\psi_2|^2 + |\lambda|^2|\psi_1|^2 + |\lambda|^2|\psi_2|^2 \\ &-& 2Im\langle\lambda\psi_1\cdot\overline{\partial_\nu\psi_1}\rangle - 2\alpha Im\langle\lambda\psi_2\cdot\overline{\partial_\nu\psi_2}\rangle. \end{array}$$

Using the outgoing radiation condition, we get

$$\begin{array}{c} \lim\limits_{R\to+\infty}\left\{-2|\lambda|^2\int_{\partial\Omega}a(x)|\psi_1|^2\,dx - 2Im\int_{\Omega_R}|\nabla\psi_1|^2 + \alpha|\nabla\psi|^2\,dx\right. \\ -2Im\lambda|\lambda|^2\int_{\Omega_R}|\psi_1|^2 + \alpha|\psi_2|^2 + \int_{|x|=R}|\partial_\nu\psi_1|^2 + |\lambda|^2|\psi_1|^2\,d\sigma(x) \\ \left. + \int_{|x|=R}|\partial_\nu\psi_2|^2 + |\lambda|^2|\psi_2|^2\,d\sigma(x)\right\} = 0. \end{array} \tag{21}$$

Therefore, if $Im\lambda<0$ then we have $\int_\Omega|\Psi|^2\,dx = 0$ that implies $\Psi=0$ in Ω. Assuming that $Im\lambda=0$ and $\lambda\neq 0$ the equation (19) and

$$Im\left[\int_{|x|=R}\partial_\nu\psi_1\cdot\overline{\psi_1}d\sigma(x) + \int_{|x|=R}\partial_\nu\psi_2\cdot\overline{\psi_2}d\sigma(x)\right] + |\lambda|^2\int_{\partial\Omega}a(x)|\psi_1|^2\,d\sigma(x) = 0$$

and combining the radiation condition, we conclude that $\Psi_{\{a>0\}}\equiv 0$. Moreover, if $meas\left(\Omega\cap\{a>0\}\right)>0$, it is easily to see that $\Psi\equiv 0$ Which proves the lemma.

In the following, we study the outgoing resolvent $\mathcal{R}_a^\alpha(\lambda)$ on the real axis. We show firstly that it has no real pole and secondly it is bounded in the neighborhood of 0 in any angular sector does not meet the imaginary axis *iR*.

Boundedness of the Resolvent Near Zero

$$\begin{array}{rcl} \left(-D_\alpha - \lambda^2 id + i\lambda K_a\right)u &=& f \ in \ \Omega, \\ u &=& 0 \ on \ \partial\Omega. \end{array} \tag{22}$$

Before beginning the study of holomorphic of the resolvent $R_a^\alpha(\lambda)$, Let us note that we can see (17) as a perturbation of the following problem in a free space

$$\left(-D_\alpha - \lambda^2 id + i\lambda J\right)w = g \ in \ \ R^d, \ g=(g_1,g_2)\in\left(L^2(\Omega_R)\right)^2. \tag{23}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

The solution of the eqn. (23) is given by $w=\mathcal{R}_0^\alpha(\lambda)g$ with $\mathcal{R}_0^\alpha(\lambda)$ is the free outgoing resolvent given by

$$\mathcal{R}_0^\alpha(\lambda)g = \int_{R^d}\Gamma_+^\alpha(|x-y|,\lambda)g(x)dx \tag{24}$$

$$\Gamma_+^\alpha(|x-y|,\lambda) = \begin{pmatrix} \gamma_+(r,\lambda) & \dfrac{i}{(1-\frac{1}{\sqrt{\alpha}})}\{\gamma_+(r,\frac{\lambda}{\sqrt{\alpha}}) - \gamma_+(r,\lambda)\} \\ \dfrac{-i}{(1-\frac{1}{\sqrt{\alpha}})}\{\gamma_+(r,\frac{\lambda}{\sqrt{\alpha}}) - \gamma_+(r,\lambda)\} & \dfrac{1}{\alpha}\gamma_+(r,\frac{\lambda}{\sqrt{\alpha}}) \end{pmatrix}$$ where

and $\gamma_+(r,\lambda) = -\frac{i}{4}\left(\frac{\lambda}{2\pi r}\right)^{\frac{d}{2}-1}H^{(2)}(\lambda r)$ is the Hankel function and furthermore $\gamma_+(r,\lambda)\simeq r^{\frac{1}{2}(d-1)}e^{ir\lambda}$ for r large [6].

Now, let

$$u = w - \Theta v$$

where

$$\begin{array}{rcl} D_\alpha v + \xi^2 v - i\xi K_a v &=& 0 \ in \ \Omega_R \\ v &=& w \ on \ \partial\Omega \\ v &=& 0 \ on \ \{x\in R^d, |x|=R\} \end{array}$$

and $\Theta\in C_0^\infty$ equal to 1 on a neighborhood of $\partial\Omega$ with support in B_R. The parameter ξ being chosen and subsequently fixed the following discussion. And w is completely determined by g and v is completely determined by w. The problem then is to determine the function g for which the function u verifies (22).

$$\begin{array}{rcl} f &=& (-D_\alpha - \lambda^2 id + i\lambda K_a)u \\ &=& \left(-D_\alpha - \lambda^2 id + i\lambda K_0\right)w - (-D_\alpha - \lambda^2 id + i\lambda K_a)\Theta v - i\lambda(K_0 - K_a)w \\ &=& g - \mathcal{T}_\lambda g \end{array}$$

where

$$\mathcal{T}_\lambda g = -(D_\alpha\Theta)v - \nabla^\alpha\Theta\nabla v - \left(\lambda^2 - \xi^2 + i(\lambda-\xi)K_a\right)(\Theta v) - i\lambda(K_0 - K_a)w$$

Lemma 0.6: We have

1. $\mathcal{T}_\lambda$ is a bounded operator on $\left(L^2(\Omega_R)\right)^2$ for any $\lambda\in C\setminus\{0\}$

2. $\mathcal{T}_\lambda$ is a holomorphic function at λ in C on the Riemannian Logarithmic surface.

Proof. Let $\left(H^k(\Omega)\right)^2$ the Sobolev space functions with the following norm

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

$$\| g \|_{k'} = \left\{ \sum_{|l|\le k} \int_{\Omega_R} |\partial^l g_1|^2 + |\partial^l g_2|^2 \right\}^{\frac{1}{2}}.$$

By (24) and the oscillatory integral theory we can see that

$$\| \Theta w \|_{2'} \le C_\lambda \| g \|_{0'} \tag{25}$$

Where C_λ is bounded uniformly on any compact Riemanian Logarithmic surface [20]. Now we set $\phi = v - \Theta w$ satisfy the following problem

$$\begin{aligned} D_\alpha \phi + \xi^2 \phi - i\xi K_a \phi &= -D_\alpha(\Theta w) - \xi^2 \Theta w \text{ in } \Omega_R, \\ \phi &= 0 \; on \quad \partial\Omega, \\ \phi &= 0 \; on \quad \{x; |x| = R\}. \end{aligned} \tag{26}$$

by the ellipticity argument we deduce that

$$\| \phi \|_2' \le C_\lambda \| \Theta w \|_2' \tag{27}$$

we obtain by eqn. (25)

$$\| \phi \|_2' \le \widetilde{C}_\lambda C_\lambda \| g \|_0' . \tag{28}$$

Moreover $\mathcal{T}_\lambda$ contains only derivations of order less than or equal to 1 of φ,

$$\| \mathcal{T}_\lambda g \|_1' \le \| \phi \|_2' \le \| g \|_0' \tag{29}$$

by the Rillich identity, we deduce that $\mathcal{T}_\lambda$ is compact operator on $\left(L^2(\Omega)\right)^2$ and this implies that $\mathcal{R}_a^\alpha(\lambda)$ is meromorphic on C (resp. Riemannian logarithmic surface) if d odd (resp. d is even).

Low Frequencies

First we prove that the resolvent $R_{a,\chi}^\alpha(\lambda)$ have not poles in the real axis and it is bounded in an angular sector contain the real axis at a neighborhood of zero. For this we begin by the following result

Now, we prove that in a neighborhood of the zero, the resolvent $R_{a,\chi}^\alpha(\lambda)$ is bound in an angular sector contain a real axis. The same result has proved by Morawetz CS [21] in the standard Laplacian case where the dimensional space d=2,3 with Neumann or Dirichelet boundary condition and generalized by Burq N in the Dirichlet case [6].

Proposition 0.7: *Let* $\gamma = e^{i\theta}$ $\theta \in [-\frac{\pi}{4}, \frac{5\pi}{4}]$ *an* $d\Lambda_\gamma$ *the angular sector opening* $\frac{\pi}{2}$ *symmetric around of* γ^{-1}

$$\Lambda_\gamma = \left\{ \lambda \in C^*; Re(\gamma\lambda) \ge |Im(\gamma\lambda)| \right\}.$$

Then $R_{a,\chi}^\alpha$ uniformly bounded in Λ_γ.

Proof. Let $f \in \left(L^2(\Omega)\right)^2$ with compact support in B_R. By the previous lemma, the function $\Phi = R_a^\alpha(\lambda) f$ is the unique solution satisfying the (OGRC)of the problem

$$\begin{aligned} \left[-D_\alpha - \lambda^2 + i\lambda K_a\right]\Phi &= f \;\; in \;\; \Omega \\ \Phi &= 0 \;\; on \;\; \partial\Omega \end{aligned} \tag{30}$$

Let $\lambda \in \Lambda_\gamma$ and u solution satisfy the outgoing radiation condition of the following system

$$\begin{aligned} \left(D_\alpha + \lambda^2 - i\lambda K_a\right)u &= g \; in \, \Omega, \\ u &= 0 \; on \; \partial\Omega, \end{aligned} \tag{31}$$

where $suppg \subset \{|x| < R\}$. We choose a function $\chi \in C^\infty(R)$ equal to 0 for $|x|$<R and to1 for $|x|$>2R. We follow the proof of ref. [6], we obtain for $\lambda \in \Lambda_\gamma$

$$\begin{aligned} Re\int_\Omega e^{-4\gamma\lambda r}\overline{u}\cdot g &= Re\int_\Omega e^{-4\gamma\lambda r}\overline{u}\cdot\left(D_\alpha + \lambda^2 - i\lambda K_a\right)u \\ &= Re\Big[\int_\Omega e^{-4\gamma\lambda r}\left(-|\nabla u_1|^2 - \alpha|\nabla u_2|^2 + \lambda^2|u_1|^2 + \lambda^2|u_2|^2\right) \\ &- i\lambda e^{-4\gamma\lambda r}\left(a(x)|u_1|^2 + \overline{u_2}\cdot u_1 - \overline{u_1}\cdot u_2\right) \\ &+ \int_\Omega \gamma\lambda\partial_i r\partial_j u_1\cdot\overline{u}_1 e^{-4\gamma\lambda r} + \alpha\gamma\lambda\partial_j r\partial_i u_2\cdot\overline{u}_2 e^{-4\gamma\lambda r}\Big] \end{aligned} \tag{32}$$

This implies that

$$\begin{aligned} \int_\Omega e^{-4Re(\gamma\lambda)}\left[|\nabla u_1|^2 + \alpha|\nabla u_2|^2\right] &\le C|\lambda|^2\int_\Omega e^{-4Re(\gamma\lambda)r}\left[|u_1|^2 + |u_2|^2\right]dx \\ &+ \left|\int_\Omega e^{-4Re(\gamma\lambda)r}\left[\overline{u}_1\cdot g_1 + \overline{u}_2\cdot g_2\right]dx\right| \\ &+ c|\lambda|\int_{\partial\Omega} e^{-4Re(\gamma\lambda)r}a(x)|u_1|^2\, d\sigma(x) \end{aligned} \tag{33}$$

Then for $|\lambda| \le 1$;

$$\int_{r>3R} e^{-4Re(\gamma\lambda)r}\left(|\nabla u_1|^2 + \alpha|\nabla u_2|^2\right) \le$$

$$\begin{cases} c\int_{\Omega\cap\{r<2R\}} |\nabla u_1|^2 + \alpha|\nabla u_2|^2 + |u_1|^2 + |u_2|^2\, dx & if \quad d > 2 \\ c|\ln\lambda|\int_{\Omega\cap\{r<2R\}} c|\nabla u_1|^2 + \alpha|\nabla u_2|^2 + |u_1|^2 + |u_2|^2\, dx & if \quad d = 2 \end{cases} \tag{34}$$

by (33), (34) and $|\lambda|$<1 we get

$$\begin{aligned} \int_{\Omega\cap\{r<3R\}} |\nabla u_1|^2 + \alpha|\nabla u_2|^2\, dx &\le c|\lambda|^2|\ln\lambda|\int_{\omega\cap\{r<3R\}} |\nabla u_1|^2 + \alpha|\nabla u_2|^2 \\ &+ c\left|\int e^{-4\gamma\lambda r}\left(\overline{u}_1\cdot g_1 + \overline{u}_2\cdot g_2\right)\right| \end{aligned}$$

Since $suppG \subset \Omega \cap \{r < R\}$, $\lambda \in \Lambda_\gamma$ and λ small enough

$$\begin{aligned} \int_{\Omega\cap\{r<3R\}} |\nabla u_1|^2 + \alpha|\nabla u_2|^2\, dx| &\le c\left|\int_{\Omega\cap\{r<3R\}} e^{-4\gamma\lambda r}\left(\overline{u}_1\cdot g_1 + \overline{u}_2\cdot g_2\right)dx\right. \\ &\le c\left(\int_{\Omega\cap\{r<3R\}} e^{-4Re(\gamma\lambda)r}\left(|u_1|^2 + |u_2|^2\right)\right)^{\frac{1}{2}} \\ &\times \left(\int_{\Omega\cap\{r<3R\}} e^{-4Re(\gamma\lambda)r}\left(|g_1|^2 + |g_2|^2\right)\right)^{\frac{1}{2}}. \end{aligned}$$

Using the Hardy- Poincaré inequality for d>2, we obtain

$$\begin{aligned} \left(\int_{\Omega\cap\{r<3R\}} e^{-4Re(\gamma\lambda)r}\left(|\nabla u_1|^2 + \alpha|\nabla u_2|^2\right)\right)dx &\le c\left(\int_{\Omega\cap\{r<3R\}} e^{-4Re(\gamma\lambda)r}\left(|\nabla u_1|^2 + \alpha|\nabla u_2|^2\right)\right)^{\frac{1}{2}} \\ &\times \left(\int_{\Omega\cap\{r<3R\}} e^{-4Re(\gamma\lambda)r}\left(|g_1|^2 + |g_2|^2\right)\right)^{\frac{1}{2}}. \end{aligned}$$

For d=2 is used [6]. So in both cases we give a uniform bound of norm of the resolvent from $(L^2_{com}(\Omega))^2$ into $H^1_{0,loc}$ for λ close to zero and in the Λ_γ. By choosing a finite number of real γ_i it covers a neighborhood of upper half-plane (which is excluded 0)$\cup\Lambda_{\gamma i}$ which leads to the conclusion that the resolvent is bounded near zero and we have the assumption (1.1) in ref. [6].

which implies that one have to λ goes to zero and $|arg(\lambda) + \pi/2| \le \pi$, the following behavior:

Proposition 0.8: $\mathcal{R}_\chi^\alpha$ *does not allow the accumulation point, has no zero on the real axis and admits the following behavior*

$$\mathcal{R}_\chi^\alpha(\lambda) \simeq \begin{pmatrix} R_0(\lambda) & \dfrac{-i}{1-\dfrac{1}{\sqrt{\alpha}}}(R_0(\lambda) - R_0(\dfrac{\lambda}{\sqrt{\lambda}})) \\ \dfrac{i}{1-\dfrac{1}{\sqrt{\alpha}}}(R_0(\lambda) - R_0(\dfrac{\lambda}{\sqrt{\lambda}})) & (\dfrac{\lambda}{\sqrt{\alpha}})^{-1}\mathcal{M}_d + \mathcal{F}_d(\lambda/\sqrt{\alpha}) \end{pmatrix} \quad if \quad d \quad is \quad odd$$

and

$$\mathcal{R}_\chi^\alpha(\lambda) \simeq \begin{pmatrix} \lambda^{d-2}\ln(\lambda)\mathcal{M}_d + \mathcal{F}_d & \dfrac{-i}{1-\dfrac{1}{\sqrt{\alpha}}}(R_0(\lambda) - R_0(\dfrac{\lambda}{\sqrt{\lambda}})) \\ \dfrac{i}{1-\dfrac{1}{\sqrt{\alpha}}}(R_0(\lambda) - R_0(\dfrac{\lambda}{\sqrt{\lambda}})) & \lambda^{d-2}\ln(\lambda)\mathcal{M}_d + \mathcal{F}_d(\lambda/\sqrt{\alpha}) \end{pmatrix} \quad if \quad d \quad is \quad even$$

where *rank* $(\mathcal{M}_d) \le 1$ and $\mathcal{F}_d$ is analytic at λ=0.

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

We begin by the following lemma inspired from [2] which will be useful to the proof of our proposition.

which gives a good uniform bound on the norm of the resolvent from $(L^2(\Omega))^2$ onto $(H^1(\Omega))^2$ for λ close to zero and in the sector.

Proposition 0.9: *$\mathcal{R}_0^\alpha(\lambda)$ and $\mathcal{R}_a^\alpha$ have the same behavior near zero.*

Proof. Let $\mathcal{R}_0^\alpha(\lambda):\left(L^2_{com}(\Omega)\right)^2\to\left(L^2_{com}(\Omega)\right)^2$ the operator define by $\mathcal{R}_0^\alpha(\lambda)f$ is the unique solution of $\left(-D_\alpha-\lambda^2+i\lambda J\right)u=f$ in Ω, $u_{|\Omega}=0$ and u satisfy (OGRC). Let $f\in\left(L^2(\Omega)\right)^2$ supported in B_R $a(x)$ is supported in B_R and $u=\mathcal{R}_0^\alpha(\lambda)f$.

Then we obtain

$$\begin{aligned}\left(-D_\alpha-\lambda^2 id+i\lambda K_a\right)u &= f+i\lambda(K_a-J)u\\ u_{/\partial\Omega} &= 0\end{aligned}\tag{35}$$

And u satisfy the (OGRC). It follows that

$$\begin{aligned}u=\mathcal{R}_0^\alpha(\lambda)f &= \mathcal{R}_a^\alpha\left(f+i\lambda(K_a-J)u\right)\\ &= \mathcal{R}_a^\alpha(\lambda)\left[id+i\lambda(K_a-J)\mathcal{R}_0^\alpha(\lambda)\right]f\end{aligned}$$

so for any $f\in\left(L^2(\Omega)\right)$ supported in B_R, we have

$$\mathcal{R}_\chi^\alpha(\lambda)f=\mathcal{R}_{\chi,a}^\alpha(\lambda)\left[id+i\lambda(K_a-J)R_\chi^\alpha(\lambda)\right]f$$

where $\mathcal{R}_\chi^\alpha=\chi\mathcal{R}_0^\alpha\chi$ is the troncated free resolvent.

Lemma 0.10 *$\lambda R_{a,\chi}^\alpha(\lambda)$ is analytic at λ=0 and $i\lambda(K_a-J)R_\chi^\alpha(\lambda)\to 0$ when λ→0*

Taking into account the Lemma 0.10 we deduce that $\mathcal{R}_0^\alpha(\lambda)$ and $\mathcal{R}_a^\alpha(\lambda)$ have the same behavior near zero.

Studies of High Frequencies

This section is devoted to the proof of Theorem 0.11.

Theorem 0.11: There exists $\delta_0>0$and $\lambda_0>0$ such that the truncated outgoing resolvent $\mathcal{R}_{a,\chi}^\alpha$ extends so as holomorphic in the region

$$G_{\lambda_0}=\{\lambda\in C, Im\le\delta_0 \text{ and } |Re\lambda|>\lambda_0\}.\tag{36}$$

More precisely, there exists c>0 such that for $f\in(L^2(\Omega))^2$, suppf $suppf\subset B_R$ and for all $\lambda\in G_0$e have

$$\|\nabla^\alpha\mathcal{R}_{a,\chi}^\alpha(\lambda)f\|^2_{(L^2_R)^2}+\|\lambda\mathcal{R}_{a,\chi}^\alpha(\lambda)f\|_{(L^2_R)^2}\le c\|f\|_{(L^2_R)^2}.\tag{37}$$

Firstly, we denote that the operator $R_a^\alpha(\lambda)$ defined by $(L^2(\Omega))^2$ in H is meromorphic on C (resp. the Riemann surface of the logarithm) if n is even (resp. odd), holomorphic on $\{Im\lambda<0\}$. Moreover, c, δ_0 and λ_0 don't depend of and we can check that $R_a^\alpha(-\bar\lambda)=\overline{R_a^\alpha(\lambda)}$. This allows us to limit our study to $Re(\lambda)>0$. The proof of (37) is based on a reductio ad absurdum argument. We assume that for any c (in particular for $_n$=c=n∈N, there exists $f_n\in(L^2)^2$ and $suppf_n=(f_{1n},f_{2n})\subset B_R$ such that $Im\lambda_n\to0$ and $Re\lambda\ge n$ (we assume for example $Re\lambda\ge0$)such that

$$\|\nabla^\alpha\mathcal{R}_a^\alpha(\lambda_n)f_n\|^2+\|\lambda_n\mathcal{R}_a^\alpha(\lambda_n)f_n\|^2_{(L^2)^2}\ge n\|f_n\|^2_{(L^2)^2}\tag{38}$$

We note that $u_n=(u_n^1,u_n^2)=\mathcal{R}_a^\alpha(\lambda_n)(f_n)$ is normalized by $\|\nabla^\alpha u_n\|^2_{(L^2_R)^2}+\|\lambda_n u_n\|^2_{(L^2_R)^2}<\infty$. We obtain

$$\begin{cases}-D_\alpha u_n-\lambda_n^2u_n+i\lambda K_au_n &= f_n \text{ in } \Omega\\ u_n &= 0 \text{ on } \partial\Omega\\ u_n \text{ satisfy the outgoing radiation condition}\end{cases}$$

$$\begin{aligned}&\|\nabla u_n\|^2_{(L^2_R)^2}+\|\lambda_nu_n\|^2_{(L^2_R)^2}=1,\ \|f_n\|_{(L^2_R)^2}\to0,\\ &\|u_n\|_{(L^2_R)^2}\to0,\ \frac{1}{Re\lambda_n}\to0 \text{ and } Im(\lambda_n)\to0.\end{aligned}\tag{39}$$

Lemma 0.12: *We* have $u_n\rightharpoonup 0$ in $[H^1_{loc}(\Omega)]^2$, $\lambda_nu_n\rightharpoonup0$ in $[L^2_{loc}(\Omega)]^2$.

Proof. By (39), we obtain that $u_n\rightharpoonup 0$ in $(H^1_R(\Omega_R))^2$, where $\Omega_R=\Omega\cap B_R$

Moreover

$$\lambda_nu_n=-\frac{1}{\lambda_n}D_\alpha u_n-\frac{1}{\lambda_n}f_n-iK_au_n.$$

By effecting the scalar product with $\Phi\in\left(C_0^\infty(\Omega)\right)^2$,

$$\begin{aligned}\langle\lambda_nu_n,\Phi\rangle &= \langle-\frac{1}{\lambda_n}D_\alpha u_n-\frac{1}{\lambda_n}f_n,\Phi\rangle-i\langle K_au_n,\Phi\rangle\\ &= \langle\frac{1}{\lambda_n}\nabla u_n^1,\nabla\Phi_1\rangle+\alpha\langle\frac{1}{\lambda_n}\nabla u_n^2,\nabla\Phi_2\rangle-\langle\frac{1}{\lambda_n}f_n,\Phi\rangle-i\langle K_au_n,\Phi\rangle\end{aligned}$$

we get

$$\|\lambda u_n\|_{(L^2(\Omega_R))^2}\le\|\frac{1}{\lambda_n}\nabla^\alpha u_n\|_{(L^2(\Omega_R))^2}+\|\frac{1}{\lambda_n}f_n\|_{(L^2(\Omega_R))^2}+C\|u_n\|^2_{(L^2(\Omega_R))^2}$$

this implies$\lambda_nu_n\to0$ in $(L^2(\Omega_R))^2$.

Let $\chi\in C_0^\infty(\Omega,M_2(R))$ equal to the *id* near the boundary and supported in B_R. We set $w_n=(id-\chi)u_n$.

We can see that

$$-D_\alpha w_n-\lambda^2w_n+i\lambda K_aw_n=[D_\alpha,\chi]u_n+(id-\chi)f_n \text{ in } R^d$$

then $w_n=\mathcal{R}^\alpha(\lambda_n)g_n$ where $\mathcal{R}^\alpha$ is the outgoing free resolvent of the $-D_\alpha-\lambda I_2+i\lambda J$ operator and $g_n=[D_\alpha,\chi]u_n+(id-\chi)f_n$ bounded in $\left(L^2(R^d)\right)^2$, supported in B_R.

We have

$$\mathcal{R}^\alpha(\lambda_n)g_n=\int_0^{+\infty}e^{-i\lambda_nt}\Lambda^\alpha(t)g_ndt$$

where $\Lambda^\alpha(t)$ is the free propagator.

So, by part integration and noticing the locate energy go to zero at +∞

$$\lambda_n\mathcal{R}^\alpha(\lambda_n)g_n=-i\int_0^{+\infty}e^{-i\lambda_nt}\partial_t\Lambda^\alpha(t)g_ndt$$

As $Im\lambda_n\le\delta_0$ for $Re\lambda_n$ large enough for R′>Rwe have

$$\begin{aligned}\|w_n\|^2_{H^1(\Omega_{R'})} &= \|\nabla w_n^1\|^2_{L^2(\Omega_{R'})}+\|\sqrt{\alpha}\nabla w_n^2\|^2_{L^2(\Omega_{R'})}+\|\lambda_nw_n^1\|^2_{L^2(\Omega_{R'})}+\|\lambda_nw_n^2\|^2_{L^2(\Omega_{R'})}\\ &\le c\left(\|g_n^1\|^2_{L^2(\Omega_R)}+\|g_n^2\|^2_{L^2(\Omega_R)}\right).\end{aligned}$$

this inequality is deduced from the method of stationary phase for $Re\lambda_n\to0$ ([8]). We can see that is bounded w_n in $[H^1_{loc}(B^c_R)]^2$ and$\|w_n\|\to0$. So $u_n\rightharpoonup0$ in $[H^1_{loc}(\Omega)]^2$ and $\lambda_nu_n\rightharpoonup0$ in $[L^2_{loc}(\Omega)]^2$.

Let $v_n(t,x)=e^{-iRe(\lambda_n)t}u_n(x)$.

The sequence v_n satisfy the following wave equation:

$$\begin{aligned}\partial_t^2v_n-D_\alpha v_n+J\partial_tv_n+\begin{pmatrix}2a(x)-2\delta_0 & 0\\ 0 & -2\delta_0\end{pmatrix}\partial_tv_n &= e^{-itRe(\lambda_n)}\tilde f_n\\ v_n &= 0 \text{ on } \partial\Omega\times R,\end{aligned}\tag{40}$$

where $\tilde f_n=f_n+(Im\lambda_n)^2u_n+2Re\lambda_n\left(Im\lambda_n-\delta_0\right)u_n+Im\lambda_nJu_n+Im\lambda_n\begin{pmatrix}2a(x) & 0\\ 0 & 0\end{pmatrix}u_n$.

Note that

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

$\tilde{f}_n \to 0$ in $[L^2_{loc}(\Omega)]^2$ in fact we have:

$$\| f_n \|_{(L^2_{loc}(\Omega))^2} \to 0$$

$$\| (Im\lambda_n)^2 u_n \|_{(L^2_{loc}(\Omega))^2} \leq | (Im\lambda_n)^2 \| \| u_n \|_{(L^2_{loc}(\Omega))^2} \to 0$$

$$\begin{aligned} \| Re\lambda_n (Im\lambda_n - \delta_0) u_n \|_{(L^2_{loc}(\Omega))^2} &\leq |\frac{Re\lambda_n}{\lambda_n}| | Im\lambda_n - \delta_0 | \| \lambda_n u_n \|_{(L^2_{loc}(\Omega))^2} \\ &\leq | Im\lambda_n - \delta_0 | \| \lambda_n u_n \|_{(L^2_{loc}(\Omega))^2} \to 0 \end{aligned}$$

$$\| Im\lambda_n J u_n \|_{(L^2_{loc}(\Omega))^2} \leq C \| \lambda_n u_n \|_{(L^2_{loc}(\Omega))^2} \to 0$$

$$\| Im\lambda_n \begin{pmatrix} 2a(x) & 0 \\ 0 & 0 \end{pmatrix} u_n \|_{(L^2_{loc}(\Omega))^2} \leq M \| \lambda_n u_n \|_{(L^2_{loc}(\Omega))^2} \to 0$$

We can associate a microlocal defect measure μ in $\left(H^1_{loc}(\Omega \times R)\right)^2$, the support of is a subset characteristic of the variety. On the other hand, $supp\mu \cap B_R \times (0,+\infty) \neq \varnothing$ because if not μ=0 on B_R,which contradicts the fact that $\| v_n \|_{H^1} = 1$.

Lemma 0.13: For all $x \notin B_R$,we have:

$$Supp\mu \subset \{(t,x,\tau,\xi); \{|\xi|^2 = \tau^2; or\ \alpha |\xi|^2 = \tau^2\} \wedge x.\xi > 0\}$$

Proof. Indeed, let ω a borel set of $T^*(B_R) \times [0,1]$ such that $\mu(\omega) \neq 0$ On $\omega_1 \cup \omega_2$ where ω_1 and ω_1 are two defined by:

$$\omega_1 = \{\rho \in \omega; \exists s, G(s)\rho \notin T^*(B_R)\},\ \omega_2 = \{\rho \in \omega; \forall s \geq 0, G(s)\rho \in T^*(B_R)\}$$

We have $\mu(\omega)=\mu(\omega_1)+\mu(\omega_2)$ Or from Lemma 0.13$\mu(_1)$=0, in fact:

if $\rho \in \omega_1$there exists s such that $G(s)\rho \notin B_R$ then $G(s)\rho$ is outgoing and by lemma 0.13 we obtain $\mu(\omega_1)$=0, And it follows that $\mu(\omega)=\mu(\omega_1)$

We note that if Ω is non-captive, $\omega=\omega_1$ implies $\mu(\omega)$=0, which is absurd. So it remains the case where Ω is captive with the assumption of CGE above B_R

On the one hand,we have $\forall s \geq 0, G(s)\omega_2 \subset T^*(B_R) \times [s,s+1]$, then

$$\mu(G(s)\omega_2) \leq \mu\left(B_R \times [s,s+1]\right) \leq \| v_n \|^2_{H^1_{B_R \times [0,1]}} \leq 1 \tag{41}$$

On the other hand,

$$\begin{aligned} \mu(G(s)\omega_2) &= \int_{\omega_2} \exp\left(\int_0^s \begin{pmatrix} 2a(G(\sigma)-\delta_0) & 0 \\ 0 & -2\delta_0 \end{pmatrix}\right) d\sigma d\mu(x,t,\xi,\tau) \\ &= \int_{\omega_2} \exp(\int_0^s 2a(G(\sigma)-\delta_0)d\sigma) d\mu_1 + \int_{\omega_2} \exp -2\delta_0 s d\mu_2 \\ &\geq \int_{\omega_2} \exp \int_0^s {}^{2a(G(\sigma)-\delta_0)d\sigma} d\mu_1 \end{aligned}$$

And by using the fact that

$$C(t) = \min\left(C_1(t), C_2(t)\right) = \min\left(\inf_{\rho_0} \frac{1}{t}\int_0^t a(x_1(s,\rho_0))ds, \inf_{\rho_0} \frac{1}{t}\int_0^t a(x_2(s,\rho_0))ds\right)$$

And $C(\infty)\delta_0$

there exists ε>0 such that: $\forall s \geq s_0; /C(\infty) - C(s_0) \geq \frac{1}{2}\varepsilon$ we obtain

$$\begin{aligned} \mu(G(s)\omega_2) &\geq e^{[2C(s)-2\delta_0]s}\mu(\omega_2) \\ &\geq e^{[2C(s)-2C(\infty)+2C(\infty)-2\delta_0]s}\mu(\omega_2) \\ &\geq e^{[-\varepsilon+2\varepsilon]s}\mu(\omega_2) \\ &\geq e^{\varepsilon s}\mu(\omega_2) \end{aligned}$$

And as $\mu(\omega) \neq 0$, it follows that for sufficiently large n we get that $\mu(G(s)\omega_2)>1$, which contradicts(41) startsection section1@-3.5ex plus -1ex minus -.2ex2.3ex plus .2ex Stabilization

Using the Theorem 15 and the bound of resolvent in a neighborhood of zero we deduce the decreasing exponential (resp. polynomial) of energy in odd dimensional (resp. even dimensional). The Theorem 15 give a stabilization result by the boundary for the local energy for a coupled wave equation, on the exterior domain $\Omega = R^d \setminus \overline{O}$. Some results of decreasing exponential has proved in ref. [1]. The proof is based on a method of the resolvent (Location of poles) in which we use a lemma recovery and a theorem of propagation for microlocal defect measures

Proof

We will proceed in similar way to that one in ref. [21]. Let consider the function $\phi \in \mathcal{C}^\infty$ such that:

$$\varphi(t) = \begin{cases} 0 & t \leq 1 \\ 1 & t \geq 2 \end{cases}$$

and $V(t) = \varphi(t)e^{itG_a^\alpha}$, where $G_a^\alpha = -iA_a^\alpha$.

Note that by a simple calculation, one can find that for Im<0

$$(G_a^\alpha - I)^{-1} = -i\begin{pmatrix} \mathcal{R}_a^\alpha(\lambda)(K_a + i\lambda) & \mathcal{R}_a^\alpha(\lambda) \\ i\lambda\mathcal{R}_a^\alpha(\lambda)(K_a + i\lambda) - I & i\lambda\mathcal{R}_a^\alpha(\lambda) \end{pmatrix}$$

Hence, $(G_a^\alpha - \lambda)^{-1} : H_{comp} \to H_{loc}$ can be extended to an meromorphic operator on $\mathcal{C}$ if d is even, and on the Riemann logarithmic surface if d is even. Moreover, in view of Remark 3.2, $(G_a^\alpha - \lambda)^{-1}$ is analytic at λ=0 if n is odd and it has the following form, modulo an analytic function at λ=0,

$$(G_a^\alpha - \lambda)^{-1} = M'_n \lambda^{n-1} \ln(\lambda) + O(|\lambda|^{n-1}), \lambda \to 0,$$

If n is even.

Furthermore, it is easy to see that under the assumptions of Theorem, $(G_a^\alpha - \lambda)^{-1}$ can be extended by an analytical function on the set $\Lambda_\pm = \{\lambda \in \mathbb{C} : 0 \leq \mathrm{Im}\lambda \leq C,\ \pm\mathrm{Re}\lambda > 0\}$ and it satisfies the estimate

$$\| (G_a^\alpha - \lambda)^{-1} f \| \leq C_1 \| f \| \ \ for |Im\lambda| \leq C, |Re\lambda| \geq C_2,$$

for every compactly supported $f \in H$

Now, the Fourier transform of the function V is given by the integral:

$$\widehat{V}(\lambda) = \int_{-\infty}^{+\infty} e^{-it\lambda} V(t) dt$$

is well defined for $Im\lambda$<0 ,as a bounded operator on H. Furthermore, the inverse Fourier Transform of v is given by:

$$V(t) = \text{——}\int_{-\infty}^{+\infty} e^{it}\ \widehat{V}(\lambda)d\lambda$$

$$= \frac{1}{2\pi}\int_{Im\lambda=-\varepsilon} e^{it\lambda}\widehat{V}(\lambda)d\lambda, \quad \forall \varepsilon > 0$$

and satisfy

$$(\partial_t - A_a^\alpha)V(t) = \varphi'(t)e^{tA_a^\alpha}.$$

Then it follows that for $Im\lambda$<0:

$$\widehat{V}(\lambda) = i(G_a^\alpha - \lambda)^{-1}\widehat{\varphi'(t)U(\lambda)}$$

By the finite speed of the wave propagation, we have that for every compactly supported $f \in H, \forall t \in \mathbb{R}, \varphi'(t)U(t)f$ is supported in some

Citation: Moulahi A, Dlala M (2017) Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle. J Phys Math 8: 248. doi: 10.4172/2090-0902.1000248

compact independent of t. Therefore, $\varphi'(t)U(t): H_{comp} \to H_{comp}$ extends to an entire function on $\mathcal{C}$.

$$\begin{aligned} V(\lambda) &= \frac{1}{2\pi}\int_{Im\lambda=-\varepsilon} e^{it\lambda}(A_a^\alpha-\lambda)^{-1}\widehat{\varphi'(t)U(\lambda)}fd\lambda \\ &= \frac{1}{2\pi}e^{-\delta t}\int_{-\infty}^{+\infty} e^{itz}(A_a^\alpha-(z+i\delta))^{-1}\widehat{\varphi'(t)U(z+i\delta)}fdz \\ &+ \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left\{\int_{\{Re\lambda=-\varepsilon,0\le Im\lambda\le\delta\}} e^{it\lambda}(A_a^\alpha-\lambda)^{-1}\widehat{\varphi'(t)U(\lambda)}fd\lambda\right. \\ &- \left.\int_{\{Re\lambda=\varepsilon,0\le Im\lambda\le\delta\}} e^{it\lambda}(A_a^\alpha-\lambda)^{-1}\widehat{\varphi'(t)U(\lambda)}fd\lambda\right\} \\ &= \frac{1}{2\pi}e^{-\delta t}W_1(t)f+\frac{1}{2\pi}W_2(t)f \end{aligned}$$

Clearly, $W_2(t)f\equiv 0$ if n is odd, while for n even, we have in view of (3.13), $W_2(t)=dy+O(t^{-n})=O(t^{-n})$. In other words,

$$\|W_2(t)f\|_R\le \tilde{C}t^{-d}\|f\|$$

for every compactly supported $f\in H$

To estimate $\|W_1(t)f\|_R$ we will use Plancherel identity together with (3.12). We have

$$\begin{aligned} \int_{-\infty}^{+\infty}\|W_1(t)f\|_R^2\,dt &= \int_{-\infty}^{+\infty}\|(G_a^\alpha-(z+i\delta))^{-1}\widehat{\varphi'(t)U(z+i\delta)}f\|^2\,dz \\ &\le C_1\int_{-\infty}^{+\infty}\|\widehat{\varphi'(t)U(z+i\delta)}f\|^2\,dz \\ &= C_1\int_{-\infty}^{+\infty}e^{2\delta t}\|\widehat{\varphi'(t)U(t)}f\|^2\,dt \\ &\le C_2\|f\|^2. \end{aligned}$$

Let $\chi\in\mathcal{C}^\infty(\mathbb{R}^d), \chi=1$ for $|x|\le R$. An easy computation gives

$$\begin{aligned} (\partial_t-iG_a^\alpha)\chi W_1(t)f &= \delta\chi W_1(t)f-i\chi\int_{-\infty}^{+\infty}e^{-itz}\widehat{\varphi'(t)U(z+i\delta)}fdz \\ &= \tilde{W}_1(t)f \end{aligned}$$

and hence

$$\chi W_1(t)f=U(t)\chi W_1(t_0)f+\int_{t_0}^{t}U(t-t_0-s)\widehat{W_1(s)}fds.$$

This implies

$$\begin{aligned} \|W_1(t)f\| &\le \|\chi W_1(t)f\|\le C_3\|f\|+\int_{t_0}^{t}\|\widehat{W_1(s)}f\|\,ds \\ &\le C_3\|f\|+(t-t_0)^{1/2}\left(\int_{t_0}^{t}\|\widehat{W_1(s)}f\|^2\,ds\right)^{1/2}. \end{aligned}$$

It is easy to see that (3.18) holds with $\|W_1(t)f\|_R$ replaced by $\|\tilde{W}_1(t)f\|$. Hence, for $t\ge 1$,

$$\|W_1(t)f\|_R\le C_4t^{1/2}\|f\|.$$

Thus, (3.10) follows from (3.16), (3.17) and (3.19).

The Best Rate of Decay in Odd Dimension

Let β be the best exponential decay rate defined by:

$$\beta=sup\{\delta>0;\exists c>0,\forall f\in H,\mathrm{supp}f\subset B_R,E_R(u)(t)\le Ce^{-\gamma t}E(u)(0),\ \forall t\ge 0\}$$

Then we have the following result

Theorem 0.14

$$\beta=2\min(D(0),C(\infty))$$

It results from theorem 0.11 that

$$\beta\ge\alpha=2\min(D(0),C(\infty))$$

It remains to prove that $\beta\le\alpha$

Assume that β>2D(0), then there exists λ_1 pole of R(λ) such that 2Im $\lambda_1<\beta$. From $R(\lambda_1)f=\int_0^{+\infty}e^{-i\lambda_1 t}u(t)fdt$ we obtain:

$$\begin{aligned} \|R(\lambda_1)f\|_R &\le \int_0^{+\infty}e^{\mathrm{Im}\,\lambda_1 t}\|u(t)f\|_R\,dt \\ &\le \int_0^{+\infty}e^{\mathrm{Im}\,\lambda_1 t}\|u(t)f\|_R\,dt \\ &\le \int_0^{+\infty}e^{\mathrm{Im}\,\lambda_1 t}e^{-1/2\beta t}\|f\|\,dt \\ &\le \int_0^{+\infty}e^{1/2t(2\mathrm{Im}\,\lambda_1-\beta}\|f\|\,dt \\ &\le -\frac{\|f\|}{\mathrm{Im}\,\lambda_1-1/2\beta}<\infty \end{aligned} \tag{42}$$

which contradicts the fact that λ_1 is a pole of $R(\lambda)$. And therefore it follows that β <2D(0)

Assume that $\beta>2C(\infty)$, then there exists η>0 such that $\beta=2C(\infty)+4\eta$ and there exists c>0 such as for all $f\in H_R$ and t>0

$$E_R(u(t))\le ce^{-(\beta-\eta)t}E(0).$$

Let t_0>0 such that $ce^{-(\beta-\eta)t_0}<e^{-(\beta-2\eta)t_0}$ and then

$$\begin{aligned} E_R(u(t_0)) &\le c\,e^{-(\beta-\eta)t_0}E(0),\quad\forall f\in H_R \\ &\le c\,e^{-2(C(\infty)+\eta)t_0}E(0) \end{aligned} \tag{43}$$

and as $C(t_0)\le C(\infty)<\infty$,(note that if C(∞)=∞ the inequality is trivially satisfied, there exists such that:

$$C(t_0)\le\frac{\beta}{2}-\eta$$

and

$$E_R(u(t_0))\ \le\ c\,e^{-2(C(t_0)+\eta)t_0}E(0) \tag{44}$$

Indeed, let f such that L2 = 1. Was noted the u_k solution of (1.3) with $u_k(0)=0, tu_k(0)=fk$ and μ is a measure of the defect to microlocal assoociation (u_k) in H The function f is chosen such that μ is carried by (bicharacteristic ray from BR).

$$\int_0^\rho E_R(u_s(t_0))ds\ \le\ e^{-2(C(t_0)+\eta)t_0}\int_0^\rho E(u_s(0))ds \tag{45}$$

$$\mu(]t_0,t_0+\rho[)\ \le\ c\,e^{-2(C(t_0)+\eta)t_0}\mu(]0,\rho[) \tag{46}$$

$$e^{-\frac{2}{t_0}\int_0^{t_0}a(x(s,\rho_0))ds}\mu(]0,\rho[)\ \le\ e^{-\frac{2}{t_0}\int_0^{t_0}a(x(s,\rho_0))ds-2\eta t_0}\mu(]0,\rho[). \tag{47}$$

Or $\mu(]0,\rho[)\ne 0$ (otherwise u_k tends to zero in $H^1(]0,\rho[\times B_R)$, which contradicts that $\|f\|_{L^2}=1$). This completes the proof of Theorem.

Conclusion

A stabilization problem for a coupled wave equations on an exterior of bounded domain is deveried through the research.

References

1. Aloui. L , Khenissi M (2002) Stabilisation for the Equation of Waves in an Exterior Domain. Rev Mast Iberoamericana 18: 1-16.

2. Vodev G (1999) On the uniform decay of the local energy. Serdica Math J 25: 191-206.

3. Gerard P (1991) Microlocal Defect Measures. Comm P. D. E. 16: 1761-1794.

4. Tartar L (1990) H-measures, a new approach for studying homogenization, oscillations and concentration effects in partial differential equations. Proc Roy Soc Edinburgh A 115: 193-230.

5. Lebeau G (1996) Equation des ondes amorties. Algebric and Geometric Methods in Mathematical Physics. Kluwer Academic Publishers 73-109.
6. Burq N (1998) Decay of the Local Energy of the Wave Equation for the External Problem and Absence of Resonance in the Vicinity of The Real. Acta Math 180: 1-29.
7. Burq N (1998) Control of the Equation of Waves in Open Spaces with Corners Bull. Soc Math La France 126: 601-637.
8. Lebeau G (1996) Equation des ondes amorties. Algebric and Geometric Methods in Mathematical Physics. Kluwer Academic Publishers 73-109.
9. Lebeau G and Zuazua E (1999) Decay rates for the three-dimensional linear system of thermoelasticity. Arch Rational Mech Anal 148:179-231.
10. Bardos C, Lebeau G, Rauch G (2006) Sharp Sufficient Conditions for the Observation, Control and Stabilization of Waves from the boundary. SIAM J Control Optim 30: 1024-1065.
11. Khenissi M (2003) Equation of Damped Waves in an External domain. Bull Soc Math France 131: 211-228
12. Moulahi A (2006) Stabilization of electromagnetic waves in a 2D external domain. Mathematical Rendering 342: 853-858.
13. Moulahi A (2013) Partial stabilization of a coupled wave equations. Palestine J Mathematics 2: 265-286.
14. Daoulatli M, Dehman B, Khenissi M (2010) Local Energy Decay for the Elastic System with Nonlinear Damping in an Exterior Domain. SIAM J Control Optim 48: 5254-5275.
15. Duyckaerts T (2004) Stabilization of the Linear System of Magnetoelasticity.
16. Atallah-Baraket A, Kammerer CF (2012) Analysis of the Energy Decay of a Degenerated Thermoelasticity System
17. Dehman BJ, Rousseau J, Leautaud M (2014) Controllability of Two Coupled Wave Equations on a Compact Manifold. Arch Rational Mech Anal 211: 113-187.
18. Pazy A (1983) Semigroups of linear Operators and applications to partial. Springer-Verlag, New York.
19. Moulahi A (2016) Boundary partial stabilization of a coupled wave equations on an exterior bounded obstacle. J Dynamical and Control Systems 22: 325-34.
20. Vainberg BR (1988) Asymptotique methods in equations of mathematical physics. Gordon and Breach, New York.
21. Morawetz CS (1975) Decay of solutions of the exterior problem for the wave equation. Comm Pure Appl Math 28: 229-264.