Partial Interior Stabilization of a Coupled Wave Equations on an Exterior Bounded Obstacle

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Abstract

We consider a stabilization problem for a coupled wave equations on an exterior of bounded domain $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ with interior stabilization. Under a geometrical control condition (BLR condition), for any initial data in the energy space, we show a result of exponential stability in odd dimensional case and polynomial stability in the case of even dimension.

Keywords: Defect measure; stabilization; Energy; Resolvent; Low and High frequencies

Introduction

In this paper we study the stabilization of a coupled wave equations. More precisely, we consider the following initial and boundary value problem:

\begin{align}
\partial_t^2 u_1 - \Delta u_1 + a(x)\partial_t u_2 + \partial_t^2 u_2 &= 0 \text{ in } \Omega \times (0, +\infty), \\
\partial_t^2 u_2 - \alpha \Delta u_2 - \partial_t u_1 &= 0 \text{ in } \Omega \times (0, +\infty), \\
u_1 &= 0 \text{ on } \partial \Omega \times (0, +\infty), \\
u_2 &= 0 \text{ on } \partial \Omega \times (0, +\infty), \\
u_1(x,0) &= u_1^0(x), \quad \partial_t u_1(x,0) = u_1^1(x) \text{ in } \Omega, \\
u_2(x,0) &= u_2^0(x), \quad \partial_t u_2(x,0) = u_2^1(x) \text{ in } \Omega,
\end{align}

where $\Omega = \mathbb{R}^d \setminus \overline{\mathcal{O}}$ and $\mathcal{O}$ an open bounded set of $\mathbb{R}^d$ with smooth boundary $\partial \Omega \setminus \partial \mathcal{O}$, $a(x) \in C^\infty(\overline{\Omega})$ is a positive functions and $\alpha$ is a positive constant.

The study of systems like (1)-(6) (and more generally coupled PDEs systems) is motivated by several physical considerations. In fact, There are many applied problems that can be modeled using coupled partial differential equations, for instance in heating processes, magnetohydrodynamics, quantum mechanics, optics, fluid dynamics...

Among the nowadays many contributions, using different methods and techniques, are given, and relevant reference therein [1,2].

One of the earliest tools in the stabilization analysis of partial differential equation is the micro-local default action of Gérard [3], Tartar [4].

Such techniques have been used firstly to study and to explicit the value of the best decay rate of damped waves equation [5], reduce the boundary and the regularity of the initial data or to show that the geometric condition for control by the board is required [6,7].

Similar works, based on the use of microlocal defect measures in the spirit of the article, have been achieved [5]. In the large time behavior of solutions of the wave equation were studied. The microlocal defect measures have been used to provide estimates of energy was shown in particular how these demonstrate the results of exact controllability, observation and stabilization [8], without any assumption on the dynamics, the logarithmic decay of the local energy with respect to any Sobolev norm larger than the initial energy is proved [6]. In the two and three-dimensional system of linear thermoelasticity in a bounded smooth domain with Dirichlet boundary conditions were studied. In two space dimensions they proved a sufficient (and almost necessary) condition for the uniform decay under an assumption on the boundary of the domain and in three space dimensions sufficient conditions for the uniform decay are given [9].

Also, These techniques are used to study the stabilization of the wave equation in a domain with exterior Dirichlet condition [1], for the equation of damped waves equation in an outside field and under an "Exterior Geometric Control" condition inspired from the so-called microlocal condition of Bardos et al. [10] then for the stabilization of electromagnetic waves on an exterior bounded obstacle in 2D and 3D is treated, and under an exterior geometric control condition the behavior of the solution for large time is studied [11-13].

Later, in three dimension space and under a microlocal geometric condition, the rate of decay of the local energy for solutions of the Lamé system on exterior domain, with localized nonlinear damping was given in ref. [14].

Recently these techniques are also used to study the stabilization of different coupled equations and different results have been established in this domain, some results are given, by Duyckaerts [15] the exponential and the polynomial stabilization of a coupled hyperbolic-parabolic system of thermoelasticity are addressed with microlocal techniques, explained by Atallah-Baraket and Kammerer [16] the energy decay of thermoelasticity system with a degenerated second order operator in the Heat equation was studied, a stabilization problem for a coupled wave equations on a compact Riemannian manifold under a geometrical control condition was examined and a logarithmic decay result of the energy is given [13]. And finally in the exact controllability problem on a compact manifold for two coupled wave equations, with a control function acting on one of them only was treated [17].

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Our aim in this work is to establish the energy decay and to give the best rate of convergence of a coupled damped wave equation on an exterior bounded obstacle. We prove this result in a geometric hypothesis and by using the arguments of the analysis microlocal.

The organization of this paper is as follows. In section 2, we give the main result and recalled some preliminary results. In section 3, we study the poles of the resolvent, in the first, by means of conventional techniques is given a location on the low frequencies and by the defect measures theory we study the high frequencies. In section 4, the main results concerning the stability of systems are established. startsection

Main result and recalled some preliminary results. In section 3, we will hypothesize and by using the arguments of the analysis microlocal.

Let $u=(u_1,u_2)$ then the system of equations (1)-(6) is equivalent to the following system

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\ddot{u}_1 - D_u K \ddot{u}_1 + K \ddot{u}_2 = 0 \quad \text{in } \Omega \times (0, +\infty), \\
u_1 = 0 \quad \text{on } \partial \Omega \times (0, +\infty), \\
u_2(0) = u_0 \quad \text{in } \Omega, \\
\end{array}
\right.
\end{align*}
\]

(7)

where

\[
D_u = \begin{pmatrix}
\Delta & 0 \\
0 & \alpha \Delta
\end{pmatrix}, \quad K = \begin{pmatrix}
2\alpha \nu_1 & 1 \\
1 & 0
\end{pmatrix}.
\]

We define

\[
A_u = \begin{pmatrix}
0 & \text{id} \\
D_u & -K_u
\end{pmatrix}
\]

and

\[
D(A_u) = \left\{ (u_1, u_2) \in (u_1, u_2) \in H \quad \text{and} \quad (u_1, D_u K \nu_2 - K \nu_2) \in H \right\}
\]

(8)

(9)

We can write the problem (7) as the following form

\[
\begin{align*}
\ddot{u}_2(t) f &= \text{A} u_2(t) f \\
\dot{u}_2(t) f &= (u, \dddot{u}_2) .
\end{align*}
\]

(10)

where $u_2(t) f$ is a solution of (7). The problem (7) and (10) are equivalents if and only if the domain $D(A_u)$.

The problem (7) has an unique solution, obtained by using the Hille-Yosida theorem for an unbounded operator.

Let $u(x, t) = (u_1(x, t), u_2(x, t))$ solution of (7) and we set $V_{\alpha, \nu_2} = (V_{\nu_1}, \sqrt{\alpha} V_{\nu_2})$. We define the energy functional at the time $t$ by

\[
E(u, t) = \frac{1}{2} \int_{\Omega} \left( \left| \ddot{u}_1 \right|^2 + |V_{\alpha, \nu_2} u_2 |^2 \right) dx
\]

(11)

that satisfy the following estimation

\[
E(u, 0) - E(u, t) = \int_0^t \int_{\Omega} \left( \left| \ddot{u}_1 \right|^2 + |V_{\alpha, \nu_2} u_2 |^2 \right) dx dt.
\]

(12)

Let $R > 0$ such that $\Omega \subset \Omega \subset (x \in R^2, \|x\| < R)$, we set $\Omega = \Omega \cap B_1$.

For $u=(u_1,u_2)$ solution of (7), we denote $E_\alpha(u)(t)$ the local energy of at instant $t > 0$ define by

\[
E_\alpha(u)(t) = \int_{\Omega} \left( \left| \ddot{u}_1 \right|^2 + |V_{\alpha, \nu_2} u_2 |^2 \right) dx (13)
\]

Now, according to the research of Moulahi [19], we recall that at the boundary point $(t, x) \in \partial \Omega$, it is a tangential direction to $(t, x) \in \partial \Omega$; that is $\eta_1 \nu_1(x) = 0, \nu_1(x)$ being the exterior normal to $\partial \Omega$ at $x$. With the assumption $\alpha=1$ we can consider $\alpha \eta = \eta$ as an element of $T'_{\nu_1}(\partial \Omega)$, and for its inverse image is the both characteristic sets means to look for $\lambda \in R$ such that

\[
p_1(t, x, \tau, \eta + \lambda \nu(x)) = 0, \quad p_1(t, x, \tau, \eta + \lambda \nu(x)) = 0.
\]

That is

\[
p_1(t, x, \tau, \eta + \lambda \nu(x)) = \eta^2 + \lambda^2 - \tau^2, \quad p_1(t, x, \tau, \eta + \lambda \nu(x)) = \alpha \eta_1 \nu_1(x) + \lambda \nu(x) - \tau
\]

and we write

\[
\lambda = \pm \sqrt{\tau^2 - \eta_1 \nu_1^2}, \quad \lambda = \pm \sqrt{\eta_1^2 - \eta^2}.
\]

Hence, for the existence of such real $\lambda$, one of the two relations $r_1 = \tau^2 - \eta_1 \nu_1^2 \geq 0$ or $r_2 = \alpha \eta^2 - \eta_1 \nu_1^2 \geq 0$ must be fulfilled. From the geometrical point of view there are some possibilities for a tangential direction $\zeta = (\tau, \eta) = (0,0)$ with different number of inverse image with respect to the projection $T'_{\nu_1}(\partial \Omega) \to T(\partial \Omega)$. We introduce the characteristic transversal manifold:

\[
\text{Char} T = \text{Char} T_{\nu_1} \cup \text{Char} T_{\nu_1},
\]

where

\[
\text{Char} T_{\nu_1} = \left\{ (t, x, \tau, \eta), \tau^2 - \alpha \eta_1 \nu_1^2 = 0, t > 0 \right\}
\]

\[
\text{Char} T_{\nu_1} = \left\{ (t, x, \tau, \eta), \eta \in \partial \Omega, t > 0, r_1 \geq 0 \right\}
\]

(12)

and the characteristic longitudinal manifold of the wave coupled system is

\[
\text{Char} L = \text{Char} T_{\nu_1} \cup \text{Char} T_{\nu_1},
\]

where

\[
\text{Char} L_{\nu_1} = \left\{ (t, y, \tau, \eta), y \in \partial \Omega, \quad t > 0, \eta \geq 0 \right\}
\]

\[
\text{Char} L_{\nu_1} = \left\{ (t, y, \tau, \eta), y \in \partial \Omega, \quad t > 0, r_1 \geq 0 \right\}
\]

the characteristic manifold of the system is

\[
\text{char} P_{\alpha} = \text{char} P_{\alpha} \cup \text{char} P_{\alpha},
\]

and the assumption on the coupled wave ($\alpha=1$) one obtains

\[
\text{char} P_{\alpha} = \text{Char} T_{\nu_1} \cup \text{Char} L_{\nu_1}
\]

(12)

and

\[
\text{Char} P_{\alpha} = \text{Char} T_{\alpha} \cup \text{Char} L_{\alpha}
\]

(13)

either

\[
\text{Char} P_{\alpha} = \text{Char} L_{\alpha} \quad \text{if } \alpha < 1.
\]

According, we recall the following definition [12,14]
Definition 0.1

Let \( \eta \in \mathcal{T} \partial \Omega \), we say that

1. \( \eta \) is a elliptic (or \( \eta \in \mathcal{E} \)) if and only if \( \forall \rho \neq \text{Char} \mathcal{P}^{\alpha}, \)

2. \( \eta \) is a hyperbolic for the longitudinal wave (or \( \eta \in \mathcal{H} \)) if and only if \( r_{\eta} = 0 \)

3. \( \eta \) is a glancing for the longitudinal wave (or \( \eta \in \mathcal{G} \)) if and only if \( r_{\eta} > 0 \)

4. \( \eta \) is a hyperbolic for the transversal wave (or \( \eta \in \mathcal{G} \)) if and only if \( r_{\eta} < 0 \)

5. \( \eta \) is a glancing for the transversal wave (or \( \eta \in \mathcal{G} \)) if and only if \( r_{\eta} = 0 \)

Now, we are going to make a description of a generalized bicharacteristic path and refer to the research of Lebeau \([5]\) for more details. The generalized bicharacteristic flow lives in \( \text{Char} \mathcal{P}^{\alpha} \subset \mathcal{T} \Omega \) and for \( \rho \in \text{Char} \mathcal{P}^{\alpha} \), we denote by \( G(s, \rho) \) the generalized bicharacteristic path starting from \( \rho \). Since \( \text{char} \mathcal{P}^{\alpha} \) is the disjoint union of \( \text{char} \mathcal{P}_{\alpha}^{\alpha} \) and \( \mathcal{G} \), if \( \alpha > \alpha \) or \( \alpha < \alpha \), we shall consider separately the case where \( \rho \) belongs to each one of these sets. Moreover, all the description below holds for \( |s| \) small, in the following we assume \( |s| > 1 \).

Case 1. \( \rho \in \text{Char} \mathcal{P}^{\alpha} \). Here \( \rho = (x, t, \xi, \tau) \) where \( x \in \Omega, t \in (0, T) \) and \( t_0 = (x, t, \xi, \tau) \). Then for \( |s| \) small, we have

\[
G(s, \rho) = (x(s), t(s)), \text{Char} \mathcal{P}^{\alpha} \cap \mathcal{T} \Omega.
\]

Where \((x(s), t(s))\) is the characteristic starting from the point \((x, t)\) of

1. \( p_\rho = -r_\rho |\xi|^2 \), if \( \rho \in \text{Char} \mathcal{P}^{\alpha} \),

2. \( p_\rho = -r_\rho + \alpha |\xi|^2 \), if \( \rho \in \text{Char} \mathcal{P}^{\alpha} \).

Case 2. \( \rho \in \text{Char} \mathcal{P}^{\alpha} \). Here \( \rho = (x, t, \eta, \omega, \tau) \) where \( \omega = \xi/\alpha \) and \( \eta \neq 0 \) in (0, T) and the equation \( p_{\omega}(x, t, \eta, +\omega, -\omega, \tau) = 0 \) has roots \( \xi_\omega^\pm = \lambda(t) \) described in ref. [12] and we have one of the two relation

\[
r_\omega = r_\omega = 0, \quad r_\omega = a \eta_\omega.
\]

For \( s > 0 \) (resp. \( s < 0 \)), let \( G(s, \rho, \omega) = (x(s), t(s), \xi(s), \tau(s)) \) (resp. \( G^-(s, \rho, \omega) = (x(s), t(s), \xi(s), \tau(s)) \)) be the outgoing (resp. incoming) bicharacteristic of \( \mathcal{P}^{\alpha} \). The generalized bicharacteristic path is such that \( G(0, \rho, \omega) = \rho \) and

\[
G(s, \rho, \omega) = \begin{cases}
G^+(s, \rho, \omega) & 0 < s < \varepsilon \\
G^-(s, \rho, \omega) & -\varepsilon < s < 0.
\end{cases}
\]

Four possibilities may occur

1. \( x(s) = x + 2\alpha s^2 \xi^+, \quad 0 < s < \varepsilon, \)

\( x(s) = x + 2\alpha s^2 \xi^-, \quad -\varepsilon < s < 0, \)

where \( \xi^+ = \eta_\omega - \sqrt{\frac{\alpha}{\sqrt{\alpha}}} v(x) \) and \( \xi^- = \eta_\omega + \sqrt{\frac{\alpha}{\sqrt{\alpha}}} v(x) \).

In particular, if \( 0 < \varepsilon \) one has \( x(s) \in \Omega \) for small \( |s| = 0 \).

2. If \( 0 \leq p_{\rho} \) (i.e., \( \eta \in \mathcal{G} \)), \( G^-(s, \rho, \omega) = (x(s), t(s), \xi(s), \tau(s)) \);\n
\[
(x(s) = x + 2\alpha s^2 \xi^+, \quad 0 < s < \varepsilon, \)

\( x(s) = x + 2\alpha s^2 \xi^-, \quad -\varepsilon < s < 0, \)

where \( \xi^+ = \eta_\omega - \sqrt{\frac{\alpha}{\sqrt{\alpha}}} v(x) \) and \( \xi^- = \eta_\omega + \sqrt{\frac{\alpha}{\sqrt{\alpha}}} v(x) \).
Location of the outgoing resonant poles

We consider the operator $\mathcal{R}_\alpha^* (\lambda)$ define by the following expression

$$
\mathcal{R}_\alpha^* (\lambda) f = \int_\Omega e^{-i\lambda \partial_\alpha} (t) f(t) dt \quad \text{for} \quad \text{Im} \lambda < 0
$$

$\mathcal{A}_\alpha^*$ is dissipative operator, by the Hille-Yosida theorem, generate a contraction semigroup $\{ u(t) \}_{t\geq 0}$.

Then, it is clear that the relation (16) define a bounded family of operators from $(\tilde{L}^2(\Omega))^2$ onto $H^2(\Omega)$ and it is holomorphic in $\{ \text{Im} \lambda < 0 \}$.

Moreover, we have the following characterization of the resonant $\mathcal{R}_\alpha^*"$:

**Lemma 0.5** For all $f \in (\tilde{L}^2(\Omega))^2$ with support in $B_\alpha$ and for all $\lambda \neq 0$ and $\text{Im} \lambda \leq 0$ we have $\mathcal{R}_\alpha^* (\lambda) f$ is the unique solution satisfies the outgoing radiation condition (ORRC) of the following problem:

$$
\begin{align*}
-D_\alpha \psi - \alpha \psi + i\lambda K_\alpha \psi &= \sin \Omega, \\
\psi &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
$$

(17)

Firstly, we recall that $w(u_1, u_2)$ satisfy the outgoing radiation condition if the identity follows

$$
\begin{align*}
\lim_{x \to \infty} \left| \frac{\partial}{\partial x} u_1 + i\lambda u_2 \right|^2 + \left| \int_{\partial \Omega} \frac{\partial}{\partial n} u_1 + i\lambda u_2 d\sigma \right|^2 &= 0.
\end{align*}
$$

(18)

Now, let $\psi$ the difference between two solution of (17). Then, $\psi$ satisfy the homogenous problem with Diriclet boundary. By integration on $\Omega$, for $R$ large enough, we have

$$
\begin{align*}
\ii \lambda \mathcal{L}(\psi, D_\alpha \psi) + i\lambda \mathcal{L}(\psi, \psi) - i\lambda \mathcal{L}(\psi, K_\alpha \psi) &= 0.
\end{align*}
$$

(19)

This implies that

$$
\begin{align*}
\ii \lambda \mathcal{L}(\psi, D_\alpha \psi) + i\lambda \mathcal{L}(\psi, \psi) &= 0.
\end{align*}
$$

In particular, given that the real part of (19) is zero, gives

$$
\begin{align*}
-2\ii \lambda \mathcal{L}(\psi, D_\alpha \psi) &= 0.
\end{align*}
$$

Since

$$
\begin{align*}
\ii \lambda \mathcal{L}(\psi, D_\alpha \psi) &= 2|\alpha| |\psi_1| |\psi_2| + \alpha |\partial_\alpha \psi_1| - \alpha |\partial_\alpha \psi_2| + |\partial_\alpha \psi_1| |\psi_2| - |\partial_\alpha \psi_1| |\psi_2|,
\end{align*}
$$

Using the outgoing radiation condition, we get

$$
\begin{align*}
\lim_{x \to \infty} \left( -2|\alpha| |\psi_1| |\psi_2| + \alpha |\partial_\alpha \psi_1| - \alpha |\partial_\alpha \psi_2| + |\partial_\alpha \psi_1| |\psi_2| - |\partial_\alpha \psi_1| |\psi_2| 
\right)
\end{align*}
$$

(20)

Therefore, if $\text{Im} \lambda < 0$ then we have $\int_{\partial \Omega} |\psi|^2 d\sigma = 0$ that implies $\psi = 0$ in $\Omega$. Assuming that $\text{Im} \lambda > 0$ and $\lambda \neq 0$ the equation (19) and

$$
\begin{align*}
\lim_{x \to \infty} \left( -2|\alpha| |\psi_1| |\psi_2| + \alpha |\partial_\alpha \psi_1| - \alpha |\partial_\alpha \psi_2| + |\partial_\alpha \psi_1| |\psi_2| - |\partial_\alpha \psi_1| |\psi_2| d\sigma(x)
\right)
\end{align*}
$$

(21)

and combining the radiation condition, we conclude that $\psi = 0$. Moreover, if $\text{meas}(\Omega \cap \{ a > 0 \}) > 0$, it is easily to see that

$$
\Psi = 0
$$

Which proves the lemma.

In the following, we study the outgoing resolvent $\mathcal{R}_\alpha^* (\lambda)$ on the real axis. We show firstly that it has no real pole and secondly it is bounded in the neighborhood of 0 in any angular sector does not meet the imaginary axis $iR$.

**Boundedness of the Resolvent Near Zero**

$$
\begin{align*}
(-D_\alpha - \lambda^2 id + i\lambda K_\alpha) w &= f \quad \text{in} \quad \Omega, \\
u &= g \quad \text{on} \quad \partial \Omega.
\end{align*}
$$

(22)

Before beginning the study of holomorphic of the resolvent $\mathcal{R}_\alpha^* (\lambda)$, let us note that we can see (17) as a perturbation of the following problem in a free space

$$
\begin{align*}
(-D_\alpha - \lambda^2 id + i\lambda J) w &= g \quad \text{in} \quad \Omega, \\
u &= 0 \quad \text{on} \quad \partial \Omega.
\end{align*}
$$

(23)

where $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

The solution of the eqn. (23) is given by $w = \mathcal{R}_\alpha^* (\lambda) g$ with $\mathcal{R}_\alpha^* (\lambda)$ is the free outgoing resolvent given by

$$
\mathcal{R}_\alpha^* (\lambda) g = \int_\Omega \Gamma_\alpha^* (x - y, \lambda) g(y) dx.
$$

(24)

$$
\Gamma_\alpha^* (x - y, \lambda) = \frac{1}{(2\pi r)^{\frac{\alpha}{2}}} e^{-\frac{r}{2}} H^2(\lambda r)
$$

where $\gamma_\alpha^*(r, \lambda) \simeq r^{\frac{\alpha}{2} - 1} e^{-\frac{r}{2}}$ for $r$ large [6].

Now, let

$$
\begin{align*}
u &= w - \Theta v
\end{align*}
$$

where

$$
\begin{align*}
D_\alpha v + \xi K_\alpha v &= 0 \quad \text{in} \quad \Omega, \\
v &= g \quad \text{on} \quad \partial \Omega \\
v &= 0 \quad \text{on} \quad \{ x \in R^d \mid |x| = R \}
\end{align*}
$$

and $\Theta \in C^\infty$ equal to 1 on a neighborhood of $\partial \Omega$ with support in $B_\alpha$. The parameter $\xi$ being chosen and subsequently fixed the following discussion. And $w$ is completely determined by $g$ and $v$ is completely determined by $w$. The problem then is to determine the function $g$ for which the function $u$ verifies (22).

$$
\begin{align*}
\int_\Omega (-D_\alpha - \lambda^2 id + i\lambda K_\alpha) w u &= f \quad \text{in} \quad \Omega, \\
\int_{\partial \Omega} v u &= 0.
\end{align*}
$$

where

$$
\begin{align*}
T \mathcal{R} g &= -\Theta(D_\alpha) v - \Psi^\infty \Theta v - \left( \lambda^2 - \xi^2 + i(\lambda - \xi) K_\alpha \right) \psi^{\infty} - i\lambda(K_\alpha - K_\alpha^*) w
\end{align*}
$$

(25)

**Lemma 0.6:** We have

1. $T$ is a bounded operator on $(\tilde{L}^2(\Omega))^2$ for any $\lambda \in C \{ 0 \}$

2. $T$ is a holomorphic function at $\lambda$ in $C$ on the Riemannian Logarithmic surface.

Proof. Let $(H^1(\Omega))^2$ the Sobolev space functions with the following norm
\[ \| g \|_\infty = \left\{ \sum_{n=0}^{N} |e^{ig}\partial_r g_r|^2 + |e^{ig}\partial_\theta g_\theta|^2 \right\}^{1/2}. \]

By (24) and the oscillatory integral theory we can see that
\[ \| \Theta w \|_\infty \leq C_\delta \| g \|_\infty \]  
(25)

Where \( C_\delta \) is bounded uniformly on any compact Riemannian Logarithmic surface [20]. Now we set \( \phi = v - \Theta w \) satisfy the following problem
\[ D_\delta \phi + \zeta^2 \phi - i \zeta K_\delta \phi = -D_\delta (\Theta w) - \zeta^2 \Theta w \text{ in } \Omega, \]
\[ \phi = 0 \text{ on } \partial \Omega, \]
\[ \phi = 0 \text{ on } \{ |x| = R \}, \]

by the ellipticity argument we deduce that
\[ \| \phi \|_\infty \leq C_\delta \| \Theta w \|_\infty \]  
(27)

we obtain by eqn. (25)
\[ \| \phi \|_\infty \leq \tilde{C}_\delta C_\delta \| g \|_\infty \]  
(28)

Moreover \( T_\delta \) contains only derivations of order less than or equal to 1 of \( \phi \)
\[ \| T_\delta g \|_\infty \leq \| \phi \|_\infty \| g \|_\infty \]  
(29)

by the Rillich identity, we deduce that \( T_\delta \) is compact operator on \((L^2(\Omega))\) and this implies that \( R_{\lambda_\delta}^\epsilon(\lambda) \) is meromorphic on \( C \) (resp. Riemannian logarithmic surface \( C \)) if \( d \) odd (resp. \( d \) is even).

Low Frequencies

First we prove that the resolvent \( R_{\lambda_\delta}^\epsilon(\lambda) \) have not poles in the real axis and it is bounded in an angular sector contain the real axis at a neighborhood of zero. For this we begin by the following result

Now, we prove that in a neighborhood of the zero, the resolvent \( R_{\lambda_\delta}^\epsilon(\lambda) \) is bounded in an angular sector contain a real axis. The same result has proved by Morawetz CS [21] in the standard Laplacian case where the dimensional space \( d=2 \) with Neumann or Dirichlet boundary condition and generalized by Burq N in the Dirichlet case [6].

Proposition 0.7: Let \( \gamma = \epsilon^\theta \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \) an \( dL_\delta \), the angular sector opening \( \epsilon = \left\{ \lambda \in C^+; \ Re(\gamma \lambda) \geq \| Im(\gamma \lambda) \| \right\} \).

Then \( R_{\lambda_\delta}^\epsilon(\lambda) \) uniformly bounded in \( \lambda \).

Proof. Let \( f \in L^2(\Omega) \) compact support in \( B_\rho \). By the previous lemma, the function \( \Phi = R_{\lambda_\delta}^\epsilon(\lambda) f \) is the unique solution satisfying the (OGR)of the problem
\[ \left[-D_\delta - \lambda^2 + i \zeta K_\delta \right] \Phi = f \text{ in } \Omega \]
\[ \Phi = 0 \text{ on } \partial \Omega \]  
(30)

Let \( \lambda \in \Lambda_\delta \) and \( u \) solution satisfy the outgoing radiation condition of the following system
\[ \left(D_\delta + \lambda^2 - i \zeta K_\delta \right) u = g \text{ in } \Omega, \]
\[ u = 0 \text{ on } \partial \Omega, \]  
(31)

where \( \text{supp} \subset \left\{ |x| < R \right\} \). We choose a function \( \tilde{g} \in C^0(\Omega) \) equal to 0 for \( |x| < R \) and to 1 for \( |x| > 2R \). We follow the proof of ref. [6], we obtain for \( \lambda \in \Lambda_\delta \)
\[ \text{Re} \left\{ e^{i\lambda \gamma} \tilde{g} \right\} g - \text{Re} \int_{\Omega} e^{i\lambda \gamma} \tilde{g} \left( D_\delta + \lambda^2 - i \zeta K_\delta \right) u \]
\[ - \text{Re} \int_{\Omega} e^{i\lambda \gamma} \left( -[\nabla u_1]^2 + \zeta [\nabla u_1]^2 + \lambda [\nabla u_1]^2 \right) dx \]
\[ + i \lambda \text{Re} \int_{\Omega} \left( \partial x_1 + \partial \zeta x_1 \right) u \left( \partial x_1 + \partial \zeta x_1 \right) dx \]  
(32)

This implies that
\[ \int_{\Omega} e^{i\lambda \gamma} \left( [\nabla u_1]^2 + \alpha |\nabla u_1|^2 \right) dx \leq C \lambda \int_{\Omega} \int_{\Omega} e^{i\lambda \gamma} \left( |u_1|^2 + |u|^2 \right) dx \]
\[ + c \int_{\Omega} e^{i\lambda \gamma} \left( |\nabla u_1|^2 + |u|^2 \right) dx \]  
(33)

Then for \( |\lambda| \leq 1 \),
\[ \left| \int_{\Omega} e^{i\lambda \gamma} \left( [\nabla u_1]^2 + \alpha |\nabla u_1|^2 \right) dx \right| \leq c \left| \int_{\Omega} e^{i\lambda \gamma} \left( |u_1|^2 + |u|^2 \right) dx \right| \]  
(34)

and \( \lambda \in \Lambda_\delta \) we get
\[ \left| \int_{\Omega} e^{i\lambda \gamma} \left( [\nabla u_1]^2 + \alpha |\nabla u_1|^2 \right) dx \right| \leq c \left| \int_{\Omega} e^{i\lambda \gamma} \left( |u_1|^2 + |u|^2 \right) dx \right| \]  
(35)

Using the Hardy- Poincaré inequality for \( d \geq 2 \), we obtain
\[ \left| \int_{\Omega} e^{i\lambda \gamma} \left( [\nabla u_1]^2 + \alpha |\nabla u_1|^2 \right) dx \right| \leq c \left| \int_{\Omega} e^{i\lambda \gamma} \left( |u_1|^2 + |u|^2 \right) dx \right| \]  
(36)

For \( d=2 \) is used [6]. So in both cases we give a uniform bound of norm of the resolvent from \((L^2(\Omega)) \) into \( H^{1,\text{loc}} \) for \( \lambda \) close to zero and in the \( \Lambda \). By choosing a finite number of real \( \gamma \) it covers a neighborhood of upper half-plane (which is excluded 0)\( \Lambda_\delta \), which leads to the conclusion that the resolvent is bounded near zero and we have the assumption (1.1) in ref. [6].

which implies that one have to \( \lambda \) goes to zero and \( |\arg(\lambda) + \pi/2| \leq \pi \), the following behavior:

Proposition 0.8: \( R_{\lambda_\delta}^\epsilon \) does not allow the accumulation point, has no zero on the real axis and admits the following behavior

\[ R_{\lambda_\delta}^\epsilon(\lambda) \approx \left\{ \begin{array}{ll} R_{\lambda_\delta}(\lambda) - \frac{\gamma}{1 - \frac{\gamma}{\sqrt{d}}} & \text{if } d \text{ is odd} \\ 1 - \frac{\gamma}{1 - \frac{\gamma}{\sqrt{d}}} & \text{if } d \text{ is even} \end{array} \right. \]

and

\[ R_{\lambda_\delta}^\epsilon(\lambda) = \left\{ \begin{array}{ll} \lambda \text{Re} \int_{\Omega} e^{i\lambda \gamma} \left( \partial x_1 + \partial \zeta x_1 \right) u \left( \partial x_1 + \partial \zeta x_1 \right) dx \]  
(37)

where \( \text{rank}(\mathcal{L}_\delta) \leq 1 \) and \( \mathcal{F}_\delta \) is analytic at \( \lambda=0 \).
We begin by considering the following lemma inspired from [2] which will be useful to the proof of our proposition.

which gives a good uniform bound on the norm of the resolvent from $(L^2(\Omega))^2$ onto $(H^1(\Omega))^2$ for a close to zero and in the sector.

**Proposition 0.9:** $R_\alpha^+(\lambda)$ and $R_\alpha^-(\lambda)$ have the same behavior near zero.

**Proof.** Let $R_\alpha^+(\lambda)(L^2(\Omega))^2 \rightarrow (L^2(\Omega))^2$ the operator defined by $R_\alpha^+(\lambda)f = \frac{1}{i\lambda} f - i\lambda(K_\alpha - J)R_\alpha^-(\lambda)f$ such that $u = R_\alpha^+(\lambda)f$.

Then we obtain

$$-D_\alpha - \lambda id + i\lambda K_\alpha u = f + i\lambda(K_\alpha - J)u$$

and $u$ satisfy the (OGRC). Let $f \in (L^2(\Omega))^2$ supported in $B_r$ a(x) is supported in $B_k$ and $u = R_\alpha^+(\lambda)f$. Then we obtain

$$u = R_\alpha^+(\lambda)f = \frac{1}{i\lambda} f + i\lambda(K_\alpha - J)u$$

so for any $f \in (L^2(\Omega))^2$ supported in $B_r$, we have

$$R_\alpha^+(\lambda)f = \frac{1}{i\lambda} f + i\lambda(K_\alpha - J)R_\alpha^-(\lambda)f$$

where $R_\alpha^+ = \frac{1}{i\lambda} R_\alpha^-$ is the truncated free resolvent.

**Lemma 0.10:** $\lambda R_\alpha^+(\lambda)$ is analytic at $\lambda = 0$ and $i\lambda(K_\alpha - J)R_\alpha^-(\lambda) \rightarrow 0$ when $\lambda \rightarrow 0$.

Taking into account the Lemma 0.10 we deduce that $R_\alpha^+(\lambda)$ and $R_\alpha^-(\lambda)$ have the same behavior near zero.

**Studies of High Frequencies**

This section is devoted to the proof of Theorem 0.11.

**Theorem 0.11:** There exist $\delta_0 > 0$ and $\lambda_0 > 0$ such that the truncated outgoing resolvent $R_\alpha^+$ extends so as holomorphic in the region $G_\alpha = \{ \lambda \in \mathbb{C}, |Im\lambda| < \delta_0, |Re\lambda| > \lambda_0 \}$.

More precisely, there exists $c > 0$ such that for $f \in (L^2(\Omega))^2$, supp $\subset B_k$ and for all $\alpha \in \mathbb{C}$ we have

$$\| \nabla R_\alpha^+(\lambda)f \|_{L^2(\Omega)^2} + \| R_\alpha^-(\lambda)f \|_{L^2(\Omega)^2} \leq c \| f \|_{L^2(\Omega)^2}.$$  

(37)

Firstly, we denote that the operator $R_\alpha^+(\lambda)$ defined by $(L^2(\Omega))^2$ in $H$ is meromorphic on $C$ (resp. the Riemann surface of the logarithm) if $\alpha$ is even (resp. odd), holomorphic on $|Im\lambda| < 0$. Moreover, $c$, $\delta_0$, and $\lambda_0$ don’t depend of and we can check that $R_\alpha^+(\lambda) = R_\alpha^+(\tilde{\lambda})$. This allows us to limit our study to $|Re\lambda| > 0$. The proof of (37) is based on a reduction ad absurdum argument. We assume that for any $\varepsilon > 0$ such that

$$\| \nabla R_\alpha^+(\lambda)f \|_{L^2(\Omega)^2} + \| R_\alpha^-(\lambda)f \|_{L^2(\Omega)^2} \geq \varepsilon \| f \|_{L^2(\Omega)^2}.$$  

(38)

We note that $u = (u, u^\dagger) = R_\alpha^-(\lambda)(f, f^\dagger)$ is normalized by $\| \nabla u \|_{L^2(\Omega)^2} + \| u \|_{L^2(\Omega)^2} < \infty$. We obtain

$$-D_\alpha u - \lambda u + i\lambda K_\alpha u = f \text{ in } \Omega$$

and $u$ satisfy the outgoing radiation condition

$$u \text{ on } \partial \Omega\times \mathbb{R}.$$  

$$\frac{\partial u}{\partial \nu} + i\lambda K_\alpha u = 0 \text{ on } \partial \Omega \times \mathbb{R}.$$  

(39)

$$\| u \|_{L^2(\Omega)^2} \rightarrow 0, \frac{1}{Re\lambda} \rightarrow 0 \text{ and } \text{Im}(\lambda) \rightarrow 0.$$  

**Lemma 0.12:** We have $u_{\alpha} \rightarrow 0$ in $(H_0^2(\Omega))^2$, $\lambda u_{\alpha} \rightarrow 0$ in $(L^2_{\text{loc}}(\Omega))^2$.

**Proof.** By (39), we obtain that $u_{\alpha} \rightarrow 0$ in $(H_0^2(\Omega))^2$, where $\Omega = \Omega \times (\mathbb{R}^2 \times 0)$.

Moreover

$$\lambda u_{\alpha} = \frac{1}{\lambda} D_\alpha u_{\alpha} - \frac{1}{\lambda} f_{\alpha} - iK_\alpha u_{\alpha}.$$  

By effecting the scalar product with $\Phi \in \left(C_0^\infty(\Omega)^2\right)^\perp$, we get

$$\langle \lambda u_{\alpha}, \Phi \rangle = \langle \frac{1}{\lambda} D_\alpha u_{\alpha} - \frac{1}{\lambda} f_{\alpha}, \Phi \rangle - i\langle K_\alpha u_{\alpha}, \Phi \rangle$$

we get

$$\| \lambda u_{\alpha} \|_{L^2(\Omega)^2} \leq \frac{1}{\lambda} \| D_\alpha u_{\alpha} \|_{L^2(\Omega)^2} + C \| f_{\alpha} \|_{L^2(\Omega)^2}$$

(39)

this implies $\lambda u_{\alpha} \rightarrow 0$ in $(L^2(\Omega))^2$.

Let $\chi \in C_0(\Omega, \mathbb{R})$ equal to the $id$ near the boundary and supported in $B_k$. We set $w_{\alpha} = (\chi - \chi_{\partial})u_{\alpha}$.

We can see that

$$-D_{\alpha} w_{\alpha} - \lambda^2 w_{\alpha} + i\lambda K_\alpha w_{\alpha} = [D_{\alpha}, \chi] u_{\alpha} + (\chi - \chi_{\partial}) f_{\alpha}$$

(40)

$\| w_{\alpha} \|_{L^2(\Omega)^2} \rightarrow 0$, $\| w_{\alpha} \|_{H_0^2(\Omega)^2} \rightarrow 0$.

By $\| w_{\alpha} \|_{H^1(\Omega)^2} \rightarrow 0$, we obtain $\| w_{\alpha} \|_{H^2(\Omega)^2} \rightarrow 0$, and $\| w_{\alpha} \|_{H^0(\Omega)^2} \rightarrow 0$.

$\| w_{\alpha} \|_{H^1(\Omega)^2} \rightarrow 0$ for all $\alpha$. We can see that

$$\langle \lambda u_{\alpha}, \Phi \rangle = \langle \frac{1}{\lambda} D_\alpha u_{\alpha} - \frac{1}{\lambda} f_{\alpha}, \Phi \rangle - i\langle K_\alpha u_{\alpha}, \Phi \rangle$$

we get

$$\langle \lambda u_{\alpha}, \Phi \rangle \rightarrow 0 \text{ in } (L^2(\Omega))^2$$

(39)

this implies $\lambda u_{\alpha} \rightarrow 0$ in $(L^2(\Omega))^2$.

Let $\chi \in C_0(\Omega, \mathbb{R})$ equal to the $id$ near the boundary and supported in $B_k$. We set $w_{\alpha} = (\chi - \chi_{\partial})u_{\alpha}$.

We can see that

$$-D_{\alpha} w_{\alpha} - \lambda^2 w_{\alpha} + i\lambda K_\alpha w_{\alpha} = [D_{\alpha}, \chi] u_{\alpha} + (\chi - \chi_{\partial}) f_{\alpha}$$

(40)

$\| w_{\alpha} \|_{L^2(\Omega)^2} \rightarrow 0$, $\| w_{\alpha} \|_{H_0^2(\Omega)^2} \rightarrow 0$.

By $\| w_{\alpha} \|_{H^1(\Omega)^2} \rightarrow 0$, we obtain $\| w_{\alpha} \|_{H^2(\Omega)^2} \rightarrow 0$, and $\| w_{\alpha} \|_{H^0(\Omega)^2} \rightarrow 0$.

$\| w_{\alpha} \|_{H^1(\Omega)^2} \rightarrow 0$ for all $\alpha$. We can see that
\[ \hat{f}_s \to 0 \text{ in } L^2_w(\Omega) \text{ in fact we have:} \]
\[ \| f_s \|_{L^2_w(\Omega)} \to 0 \]
\[ \| (\text{Im}\lambda_s)^{\frac{1}{2}} u_s \|_{L^2_w(\Omega)} \leq (\text{Im}\lambda_s)^{\frac{1}{2}} \| u_s \|_{L^2_w(\Omega)} \to 0 \]
\[ \| \tilde{R}_s \lambda_s (\text{Im}\lambda_s - \delta_s)^{\frac{1}{2}} u_s \|_{L^2_w(\Omega)} \leq \left( \frac{\text{Re}\lambda_s}{\lambda_s} \right) \| \text{Im}\lambda_s - \delta_s \| \tilde{R}_s \lambda_s u_s \|_{L^2_w(\Omega)} \to 0 \]
\[ \| \tilde{R}_s \lambda_s u_s \|_{L^2_w(\Omega)} \leq C \| \lambda_s u_s \|_{L^2_w(\Omega)} \]
\[ \| \lambda_s \|_{L^2_w(\Omega)} \leq M \| \lambda_s u_s \|_{L^2_w(\Omega)} \to 0 \]

We can associate a microlocal defect measure \( \mu \) in \( (H^s_w(\Omega \setminus \Omega'))^\prime \), the support of which is a subset of the characteristic of \( \Omega \). On the other hand, \( \text{supp}\mu \cap B_{\delta} (0, +\infty) \neq \emptyset \) because if \( \mu = 0 \) on \( B_{\delta} \), which contradicts the fact that \( \| v \|_{L^2} = 1 \).

\textbf{Lemma 0.13:} For all \( x \in B_{\delta} \), we have:
\[ \text{Supp}\mu \subset \{(x, t, \tau, \xi); (|\xi|^2 = r^2; \text{or} |\langle \xi \rangle| = r^2) \land x \cdot \xi > 0\} \]

\textbf{Proof:} Indeed, let \( \omega \) a borel set of \( T \left(B_{\delta}\right) \times \{0, 1\} \) such that \( \mu(\omega) = 0 \) on \( \omega \cup U_\alpha \) where \( \omega_1 \) and \( \omega_2 \) are two defined by:
\[ \omega_1 = \{ \rho \in \omega : \exists \text{G} \in B_{\delta}, \rho \in T \left(B_{\delta}\right) \}, \quad \omega_2 = \{ \rho \in \omega : \forall \text{G} \in B_{\delta}, \rho \in T \left(B_{\delta}\right) \} \]
We have \( \mu(\omega) = \mu(\omega_1) + \mu(\omega_2) \) or from Lemma 0.13. Indeed, if \( \text{if } \rho \in \omega \) there exists such that \( G(\rho) \notin B_{\delta} \), then \( G(\rho) \) is outgoing and by lemma 0.13 we obtain \( \mu(\omega) = 0 \). And it follows that \( \mu(\omega) = \mu(\omega_1) \).

We note that if \( \Omega \) is non-captive, \( \omega = \omega_1 \) implies \( \mu(\omega) = 0 \), which is absurd. So it remains the case where \( \Omega \) is captive with the assumption of CGE above \( B_{\delta} \).

On the one hand, we have \( \forall s \geq 0, G(s)w_\alpha \in T \left(B_{\delta}\right) \times \{s, s + 1\} \), then
\[ \mu(G(s)w_\alpha) \leq \mu(B_{\delta} \times \{s, s + 1\}) \leq \| \nu \|_{L^2_w(\Omega)} \leq 1 \]

On the other hand,
\[ \mu(G(s)w_\alpha) = \int_{\mathbb{R}^5} \exp \left( \int_{\mathbb{R}^5} \left( \frac{2G(s)(G(\sigma) - \delta_\alpha)}{2s} - 2\delta_\alpha \right) d\sigma d\nu(t, x, \xi, \tau) \right) d\alpha d\sigma d\nu(t, x, \xi, \tau) \]
\[ = \int_{\mathbb{R}^5} \exp \left( \int_{\mathbb{R}^5} \left( \frac{2G(s)(G(\sigma) - \delta_\alpha)}{2s} - 2\delta_\alpha \right) d\sigma d\nu(t, x, \xi, \tau) \right) d\alpha d\sigma d\nu(t, x, \xi, \tau) \]
\[ \geq \int_{\mathbb{R}^5} \exp \left( -\frac{2G(s)(G(\sigma) - \delta_\alpha)}{2s} - 2\delta_\alpha \right) d\nu(t, x, \xi, \tau) \]

And by using the fact that
\[ C(t) = \min \{C_1(t), C_2(t)\} = \min \left\{ \int_{\mathbb{R}^5} |\nabla_x \phi(x, s, \xi)| dx, \int_{\mathbb{R}^5} |\nabla_x \phi(x, s, \xi)| dx \right\} \]
\[ C(\omega_\delta) \]
there exists \( \varepsilon > 0 \) such that:
\[ \forall s \geq s_0, C(\omega_\delta) - C(s_0) \geq \frac{1}{2} \varepsilon \] we obtain
\[ \mu(G(s)w_\alpha) \geq e^{\frac{2\varepsilon}{2s}(\varepsilon - 2s_0)\|G(s)\|_{L^2_w(\Omega)}} \mu(\omega_\alpha) \]
\[ \geq e^{\frac{2\varepsilon}{2s}(\varepsilon - 2s_0)} \|G(s)\|_{L^2_w(\Omega)} \mu(\omega_\alpha) \]
\[ \geq e^{\varepsilon} \mu(\omega_\alpha) \]
\[ = e^{\varepsilon} \mu(\omega_\alpha) \]
And as \( \mu(\omega) = 0 \), it follows that for sufficiently large \( n \) we get that \( \mu(G(s)w_\alpha) > 1 \), which contradicts (41) start section1@-3.5ex plus -1ex minus -.2ex 3.5ex plus .2ex Stabilization

Using the Theorem 15 and the bound of resolvent in a neighborhood of zero we deduce the decreasing exponential (resp. polynomial) of energy in odd dimensional (resp. even dimensional). The Theorem 15 give a stabilization result by the boundary for the local energy for a coupled wave equation, on the exterior domain \( \Omega = R^n \setminus \Omega' \). Some results of decreasing exponential has proved in ref. [1]. The proof is based on a method of the resolvent (Location of poles) in which we use a lemma recovery and a theorem of propagation for microlocal defect measures

\textbf{Proof:}

We will proceed in similar way to that one in ref. [21]. Let consider the function \( \phi \epsilon C^0 \)
\[ \phi(t) = \begin{cases} 0 & t \leq 1 \\ 1 & t \geq 2 \end{cases} \]
and \( V(t) = \phi(t) e^{\omega t} \), where \( G_\omega = -iA_\omega \).

Note that by a simple calculation, one can find that for \( Im \omega = 0 \)
\[ (G_\omega - I)^{-1} = -i \lambda R_\omega - i \lambda R_\omega \]

Hence, \( (G_\omega - I)^{-1} : H_{weak} \rightarrow H_{weak} \) can be extended to an meromorphic operator on \( C \) if \( d \) is even, and on the Riemann logarithmic surface if \( d \) is even. Moreover, in view of Remark 3.2, \( (G_\omega - I)^{-1} \) is analytic at \( \lambda = 0 \) if \( n \) is odd and it has the following form, modulo an analytic function at \( \lambda = 0 \),
\[ (G_\omega - I)^{-1} = M_\omega e^{\omega \lambda} \text{ln} \lambda + O(|\lambda|^{n-1}), \lambda \rightarrow 0, \]
If \( n \) is even.

Furthermore, it is easy to see that under the assumptions of Theorem, \( (G_\omega - I)^{-1} \) can be extended by an analytical function on the set \( \lambda \in C : 0 \leq \text{Im} \lambda \leq C, \pm \text{Re} \lambda > 0 \) and it satisfies the estimate
\[ \| (G_\omega - I)^{-1} f \|_{C_{\lambda}} \leq C_1(f) \| f \|_{C_{\lambda}}, \forall \lambda \in C_{\lambda}, \forall f \in C \text{ for compactly supported } f \in H \]
compact independent of \( t \). Therefore, \( \varphi(t)\Upsilon(t) : H_{\text{comp}} \to H_{\text{comp}} \) extends to an entire function on \( \mathbb{C} \).

\[
V(\lambda) = \frac{1}{2\pi} \int_{\mathbb{R}^+} e^{\lambda t} (A_\nu^* - \lambda)^{-1} \varphi(t)\ Mon(\nu(t)) d\lambda
\]

In other words, \( (44) \) extends to \( 22 \). Let

\[
(45) \quad W(t) - dy + O(t^{-\gamma}) = O(t^{-\gamma}).
\]

An easy computation gives

\[
(46) \quad \text{we obtain:}
\]

\[
(47) \quad \beta \eta \leq \lambda \beta
\]

This completes the proof of Theorem.

\[
\text{Theorem 0.14}
\]

\[
\beta = 2 \min(D(0), C(\infty))
\]

It results from theorem 0.11 that

\[
\beta \geq \alpha = 2 \min(D(0), C(\infty))
\]

It remains to prove that \( \beta \leq \alpha \)

Assume that \( \beta > 2D(0) \), then there exists \( \lambda_1 \) pole of \( R(\lambda) \) such that

\[
2 \Im \lambda_1 < \beta.
\]

From \( R(\lambda_1) = \int_0^\infty e^{-\lambda_1 t} u(t) dt \) we obtain:

\[
\| \lambda(\lambda_1) f \|_2 \leq \| \int_0^\infty e^{-\lambda_1 t} u(t) dt \|_2 dt
\]

\[
\leq \| \int_0^\infty e^{-\lambda_1 t} e^{-\alpha t} dt \|_2 dt
\]

\[
\leq \| \int_0^\infty e^{-\lambda_1 t} e^{-\alpha t} dt \|_2 dt
\]

\[
\leq - \| f \| \frac{1}{\Im \lambda_1} < 0
\]

which contradicts the fact that \( \lambda_1 \) is a pole of \( R(\lambda) \). And therefore it follows that \( \beta < 2D(0) \)

Assume that \( \beta > 2C(\infty) \), then there exists \( \eta > 0 \) such that

\[
\beta = 2C(\infty) + 4\eta
\]

and there exists \( \eta > 0 \) such that for all \( \text{f} \in H_{\text{g}} \) and \( t > 0 \)

\[
E_g(u(t)) \leq c e^{-\beta t - 4\eta t} E(0).
\]

Let \( t > 0 \) such that \( c e^{-\beta t - 4\eta t} < c e^{-\beta - 4\eta t} \) and then

\[
E_g(u(t)) \leq c e^{-\beta t - 4\eta t} E(0), \quad \text{if} \quad u(t) \in H_{\text{g}}
\]

and as \( C(t_0) \leq C(\infty) < 0 \) (note that if \( C(\infty) = 0 \) the inequality is trivially satisfied, there exists such that:

\[
C(t_0) \leq \frac{\beta - \eta}{2}
\]

and

\[
E_g(u(t)) \leq c e^{-C(t_0) t - \eta} E(0)
\]

Indeed, let \( f \) such that \( L = 1 \). Was noted the \( u(t) \) solution of (1.3) with \( u(0) = 0 \) and \( f \) is a measure of the defect to microlocal association (\( u_0 \) in \( H \) The function \( f \) is chosen such that \( \mu \) is carried by (bicharacteristic ray from BR).

\[
\int_0^t E_g(u(t_0)) ds \leq c e^{-C(t_0) t - \eta} \int_0^t E_g(u(t_0)) ds
\]

\[
\mu(0, \rho) \leq c e^{-C(t_0) - \eta} \mu(0, \rho)
\]

\[
eq e^{-C(t_0) - \eta} \mu(0, \rho)
\]

Or \( \mu(0, \rho) \neq 0 \) (otherwise \( u(t) \) tends to zero in \( H^1(0, \rho < B) \), which contradicts that \( \| f \|_2 = 1 \). This completes the proof of Theorem.

**Conclusion**

A stabilization problem for a coupled wave equations on an exterior bounded domain is derived through the research.

**References**


