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Physics Illustrations of Converging Infinite Series

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Abstract

The elementary problem of a bird oscillating between two approaching trains is a simple one- dimensional mechanics exercise that can be utilized to visualize the fact that some infinite series can add up to a finite number. In addition, the exercise can be used to recalculate some series sums or even to find new ones.

Keywords: Infinite series; Pochhammer symbol

Introduction

Infinite series appear in many areas of science as an investigated matter or as a tool of calculation. The idea of adding numbers infinite times and ending up with a finite sum in some cases puzzles first-year college students. A more profound concept related to such a problem is the philosophical question of the possibility of a convergence if every bird's trip is followed by a next trip. This is known as Zeno's paradox of Achilles and the Tortoise [1]. This issue will not be considered here because the current interest is to illustrate the convergence of infinite sums by means of physical examples In Sec. II, the original exercise is detailed and discussed as an illustration of a converging geometrical series, while in Sec. III, other linear tracks for the bird's motion are analyzed. The motions along spiral trajectories are studied in Sec. IV to illustrate the same idea for integrals instead of sums.

Original Problem

The problem statement is as follows (Figure 1): Two trains, each having a speed of v, are headed at each other on the same straight track. A bird that can fly with speed q(q>v) flies the front of one train when they are separated by a distance 2D and heads directly for the other train. On reaching the other train, it flies direct back to first train and so forth. What is the total distance the bird travels?

The two trains will meet at the middle of the distance between them after time t=D/v, by which the bird would have travelled a distance of d_{bird}=qD/v. Another way of calculating the distance travelled by the bird is to sum up the distances travelled by the bird in every trip d_i. If the bird starts its journey at one of the trains, [1] it will reach the other train after time t_1, by which the two trains would have travelled towards each other a distance vt_1. The distance travelled by the bird during its first trip d_1 can be written in terms of q, v, and D as:

$$d_1 = \frac{2qD}{q+v} \tag{1}$$

And generally during the individual trip, the distance travelled by the bird will be

$$d_i = \left(\frac{q-v}{q+v}\right)^{i-1} d_1 \tag{2}$$

where i=1,2,....

The bird flies between the two trains an infinite number of times since it always precedes the train traveling in the same direction. The total distance travelled by the bird is the sum of an infinite series of which terms are the trip distances,

$$d_{bird} = \sum_{i=1}^{\infty} d_i = d_1 \sum_{i=0}^{\infty}\left(\frac{q-v}{q+v}\right)^{i} = \frac{qD}{v} \tag{3}$$

Since the bird travels with a constant speed, the total trip time is T=d_{bird}/q.

Other Linear Variation

When the two trains travel with different speeds, the bird travels a distance d_1=2qD=(q+ v_1) in its first trip, where v_1 is the speed of the facing train. The distance travelled by the bird during its second trip is d_2=(q-v_2)/(q+v_2)d_1, and for the subsequent trips the distances are:

$$d_i = \left[\left(\frac{q-v_1}{q+v_1}\right)\left(\frac{q-v_2}{q+v_2}\right)\right] d_{i-2} \tag{4}$$

In term of odd and even indexes, they can be written as:

$$d_{2i} = \left[\left(\frac{q-v_1}{q+v_1}\right)\left(\frac{q-v_2}{q+v_2}\right)\right]^{i-1} d_2 \tag{5}$$

$$d_{2i-1} = \left[\left(\frac{q-v_1}{q+v_1}\right)\left(\frac{q-v_2}{q+v_2}\right)\right]^{i-1} d_1 \tag{6}$$

The total trip distance is:

$$d_{bird} = (d_1+d_2)\sum_{i=0}^{\infty}\left[\left(\frac{q-v_1}{q+v_1}\right)\left(\frac{q-v_2}{q+v_2}\right)\right] = \frac{2qD}{v_1+v_2} \tag{7}$$

Setting v_1=v_2 gives back the special case in the previous section. Another generalization of the problem is when the two trains travel along linear paths that makes an angle θ with each other as in Figure 2. If the trains are to meet at the vertex, their travel time to reach it must be the same. We consider the case when the two tracks are of the same length, as the other cases can be derived simply from that [2]. The distance travelled by the bird during any individual trip is given by:

$$d_i = \left(\frac{\sin(\beta-\gamma)}{\sin(\beta+\gamma)}\right)^{i-1} \frac{\sin(\theta)}{\sin(\beta+\gamma)} D \tag{8}$$

Angles β and γ are related to angle θ by $\beta = {(\pi-\theta)}/{2}$ and $\sin(\gamma) = \frac{v}{q}\cos\left({\theta}/{2}\right)$. The total distance travelled by the bird is then:

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Figure 1: A bird travelling between to approaching trains.

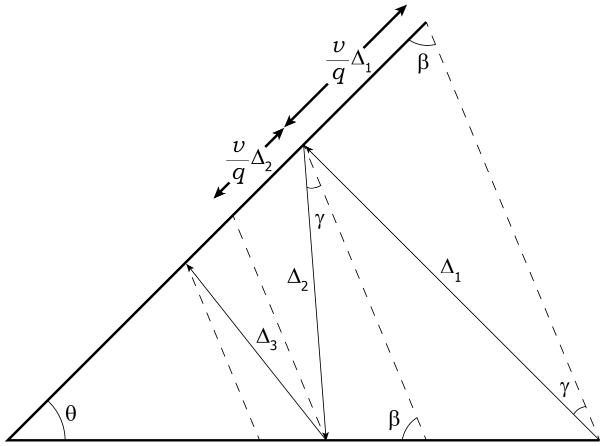

Figure 2: Two trains travelling along paths that make an angle θ with each other.

$$d_{bird} = d_1 \sum_{i=1}^{\infty} \left(\frac{\sin(\beta-\gamma)}{\sin(\beta+\gamma)}\right)^i \tag{9}$$

$$= \frac{qD}{v} \tag{10}$$

An interesting and rich case is when the bird changes its speed between trips. Physically, this happens when the bird collides elastically or inelastically with the trains or when the bird gains energy (thermal, mechanical, etc.) from its contact with the trains. During the first trip, the bird travels a distance of:

$$d_1 = \frac{2qD}{q+v} \tag{11}$$

where q is the bird's speed during its first trip, which is its starting speed. During the subsequent trips, the distances are:

$$d_i = \frac{2q_i D}{q+v}\prod_{j=1}^{i-1}\left(\frac{q_j - v}{q_{j+1}+v}\right) \tag{12}$$

where i=1,2,..., and q_1=q. Similarly, the time taken by the bird to make its i^{th} trip is:

$$d_i = \frac{2D}{q+v}\prod_{j=1}^{i-1}\left(\frac{q_j - v}{q_{j+1}+v}\right) \tag{13}$$

If the bird is accelerating, its speed must stay finite. On the other hand, if it is slowing down, its limiting speed must be greater than or equal to the speed of the trains. Otherwise, one of the trains will drag the bird to the meeting point after some time, and the series will become:

$$q_i = f(i)(q-v)+v \tag{14}$$

The function f(i) is a bound function with f(1)=1 and $\lim_{i\to\infty}$=0.

First, the case when f(i)=1/i is analyzed, for which eqn. (13) becomes:

$$t_i = \frac{2D}{q+v}(q-v)^{i-1}\prod_{j=1}^{i=1}\frac{j+1}{j}\left(\frac{1}{q+(2j+1)v}\right)$$

$$= \frac{D}{v} i \left(\frac{q-v}{2v}\right)^{i-1}\frac{1}{\left(\frac{q+v}{2v}\right)_i} \tag{15}$$

The symbol in the denominator is the shifted factorial or Pochhammer symbol defined by[3]

$$(a)_n \equiv a(a+1)(a+2)...(a+n-1) \tag{16}$$

and it can be written in terms of the Gamma function Γ (x) as:

$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} \tag{17}$$

The total trip duration is the sum of individual trips' durations,

$$T = \frac{D}{v}\sum_{i=0}^{\infty} i \left(\frac{q-v}{2v}\right)^{i-1}\frac{1}{\left(\frac{q+v}{2v}\right)_i}$$

$$= \frac{D}{v}\frac{q-v}{2v}\frac{\Gamma\left(\frac{q-v}{2v}\right)}{\Gamma\left(\frac{q-v}{2v}+1\right)} = \frac{D}{v} \tag{18}$$

where the identity Γ (x + 1)=xΓ(x) has been used in the last step. Second, f(i) is set to be an exponentially decreasing function f(i)=$e^{-(i-1)}$. Eqn. (13) becomes:

$$t_i = \frac{2D}{q+v}\left(\frac{e}{q-v}\right)\frac{1}{\left(\frac{2ev}{q-v};e\right)_i} \tag{19}$$

where the symbol in the denominator this time is the q-shifted factorial or q-Pochhammer symbol defined by[3]

$$(a;q)_n \equiv (1-a)(1-aq)(1-aq^2)....(1-aq^{n-1}) \tag{20}$$

The total trip duration is then

$$T = \frac{2D}{q+v}\left(\frac{e}{q-v}\right)\sum_{i=0}^{\infty}\frac{1}{\left(\frac{2ev}{q-v};e\right)_i} = \frac{D}{v} \tag{21}$$

The first sum in eqn. (18) is a known sum for the Pochhammer symbol, but the sum in eqn. (21) is either an unknown or a very uncommon identity. In either cases, the exercise proves useful in illustrating an important method in finding sums [3-6].

Spiral Paths

The discrete nature of the bird's trajectory produces a series of individual paths. When the bird travels along a spiral path, the distance travelled by the bird can be calculated as an integral instead of a sum.[7,8] A

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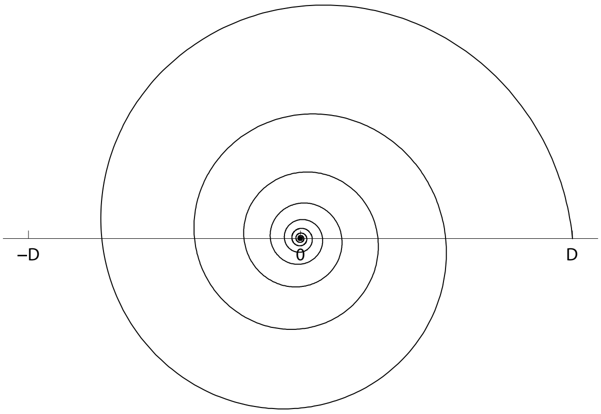

Figure 3: A logarithmeic spiral path.

suitable spiral curve can be taken as a logarithmic function (represented in polar coordinates and shown in Figure 3.

$$r(\theta) = Ae^{-b\theta} \tag{22}$$

The origin of the coordinate system is taken to be the meeting point of the trains, and θ represents the angle made by a line connecting the bird's position to the origin at any point in time. The parameter A and b can be determined from the boundary conditions. At θ=0, the bird is at a distance D from the origin, which sets A=D; at θ=π, the bird is at distance r(π) from the origin.

$$r(\pi) = D - vt_1 = r(0) - \frac{v}{q}d_1$$

$$= r(0) - \frac{v}{q}\int_0^{\pi}\sqrt{r^2(\theta) + \left(\frac{dr(\theta)}{d\theta}\right)^2}\,d\theta \tag{23}$$

which gives b in terms of v and q as

$$b = \frac{v}{\sqrt{q^2 - v^2}} \tag{24}$$

The total distance traveled by the bird is the length of the spiral curve,

$$d_{bird} = \int_0^{\infty} r(\theta)\sqrt{1 + b^2 d\theta} = \frac{Dq}{v} \tag{25}$$

When the path is linear, the change of course can be realized as a result of collisions between the bird and the trains (a ball is more suitable than a bird)[9-12]. In this case the speed changes between individual trips. However, if the bird moves along a curved path, its motion is a continuum and it becomes less complicated to show the effect of a continuous change speed but these calculations will not be presented here because our aim is illustrated in the integral of eqn. (25)[13,14].

Conclusion

These examples clarify several aspects about infinite series. First, as long as every term in a series is less than its predecessor by a multiplicative factor that is smaller than one, the series converges. The competition between the addition process in the series and the terms reduction controls the nature of their sums. Second, the infinite \nature" of the number of terms does not hinder the infinite series convergence. Third, working in the other direction, every number can be represented as an infinite series, as stated by Taylor's theorem. Last, several techniques can be used to find the sum of an infinite series.

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