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Poisson structures on $\mathbb{C}[X_1, \ldots, X_n]$ associated with rigid Lie algebras

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Abstract

We present the classical Poisson-Lichnerowicz cohomology for the Poisson algebra of polynomials $\mathbb{C}[X_1, \ldots, X_n]$ using exterior calculus. After presenting some non-homogenous Poisson brackets on this algebra, we compute Poisson cohomological spaces when the Poisson structure corresponds to a bracket of a rigid Lie algebra.

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1 Introduction

The first Poisson structures appeared in classical mechanics. In 1809, D. Poisson introduced a bracket of functions, which permits to write Hamilton's equations as differential equations. This leaded to define a Poisson manifold, that is, a manifold M whose algebra of smooth functions $F(M)$ is equipped with a skew-symmetric bilinear map:

$$\{\ ,\ \} : F(M)F(M) \longrightarrow F(M),$$

satisfying the Leibniz rule:

$$\{FG, H\} = F\{G, H\} + \{F, H\}G,$$

and the Jacobi identity. In [7], A. Lichnerowicz has also introduced a cohomology, associated to a Poisson structure, called Poisson cohomology.

In this paper, we study in terms of exterior calculus the Poisson structures on the associative algebra of complex polynomials in n variables. We apply this approach to the determination of non-homogenous quadratic Poisson brackets and to the computation of the Poisson cohomology. The linear Poisson structures are naturally related to the n-dimensional Lie algebras. Recall that a complex Lie algebra $\mathfrak{g}$ is rigid when its orbit in the algebraic variety of n-dimensional complex Lie algebra defined by the Jacobi relations is Zariski open. Such an algebra admits a nontrivial Malcev torus and it is graded by the roots of the torus. We study the Poisson structure on $\mathbb{C}[X_1, \ldots, X_n]$ whose Poisson brackets correspond to a solvable rigid Lie bracket with non-zero roots. In a generic example, we compute the corresponding Poisson cohomology.

2 Poisson structures on $\mathbb{C}[X_1, \ldots, X_n]$ and exterior calculus

2.1 Poisson brackets and differential forms

Let $\mathcal{A}^n$ be the commutative associative algebra $\mathbb{C}[X_1, \ldots, X_n]$ of complex polynomials in $X_1, \ldots, X_n$. We define a Poisson structure on $\mathcal{A}^n$ as a bivector:

$$\mathcal{P} = \sum_{1 \le i < j \le n} P_{ij} \partial_i \wedge \partial_j,$$

where $\partial_i = \frac{\partial}{\partial X_i}$ and $P_{ij} \in \mathcal{A}^n$, satisfying the axiom:

$$[\mathcal{P}, \mathcal{P}]_S = 0,$$

where $[,]_S$ denotes Schouten's bracket. If $\mathcal{P}$ is a Poisson structure on $\mathcal{A}^n$, then

$$\{P, Q\} = \mathcal{P}(P, Q)$$

defines a Lie bracket on $\mathcal{A}^n$ which satisfies the Leibniz identity:

$$\{PQ, R\} = P\{Q, R\} + Q\{P, R\}$$

for any $P, Q, R \in \mathcal{A}^n$.

We denote by $\mathrm{Sh}_{p,q}$ the set of unshuffles, where a (p, q)-shuffle is a permutation σ of the symmetric group Σ_{p+q} of degree $p + q$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p + 1) < \sigma(p + 2) < \cdots < \sigma(p + q)$. For any bivector $\mathcal{P}$ we consider the $(n - 2)$-exterior form:

$$\Omega = \sum_{\sigma \in S_{2,n-2}} (-1)^{\varepsilon(\sigma)} P_{\sigma(1)\sigma(2)} dX_{\sigma(3)} \wedge \cdots \wedge dX_{\sigma(n)},$$

where $(-1)^{\varepsilon(\sigma)}$ is the signature of the permutation σ. If $n > 3$, we consider the Pfaffian form $\alpha_{i_1,\ldots,i_{n-3}}$ given by

$$\alpha_{i_1,\ldots,i_{n-3}}(Y) = \Omega\big(\partial_{i_1}, \partial_{i_2}, \ldots, \partial_{i_{n-3}}, Y\big)$$

with $Y = \sum_{i=1}^n Y_i \partial_i$, $Y_i \in \mathcal{A}^n$.

Theorem 2.1. *A bivector $\mathcal{P}$ on $\mathcal{A}^n$ satisfies $[\mathcal{P}, \mathcal{P}]_S = 0$ if and only if:*

- *for $n > 3$,*

  $$d\alpha_{i_1,\ldots,i_{n-3}} \wedge \Omega = 0$$

  for every $i_1, \ldots, i_{n-3}$ such that $1 \le i_1 < \cdots < i_{n-3} \le n$.

- *for $n = 3$,*

  $$d\Omega \wedge \Omega = 0.$$

Proof. The integrability condition $[\mathcal{P}, \mathcal{P}]_S = 0$ writes

$$\sum_{r=1}^{n} P_{ri} \partial_r P_{jk} + P_{rj} \partial_r P_{ki} + P_{rk} \partial_r P_{ij} = 0$$

for any $1 \leq i, j, k \leq n$. But

$$\alpha_{i_1,\ldots,i_{n-3}} = \sum (-1)^N P_{jk} dX_l$$

summing over all triples (j,k,l), such that $(j,k,i_1,\ldots,l,\ldots i_{n-3})$ is a permutation of $S_{2,n-2}$ and $N = \varepsilon(\sigma) + p - 3$, where $(-1)^{\varepsilon}(\sigma)$ is the signum of σ. Then

$$d\alpha_{i_1,\ldots,i_{n-3}} = \sum (-1)^N dP_{jk} \wedge dX_l,$$

and $d\alpha_{i_1,\ldots,i_{n-3}} \wedge \Omega = 0$ corresponds to $[\mathcal{P},\mathcal{P}]_S = 0$. The proof is similar if $n = 3$. □

2.2 Lichnerowicz-Poisson cohomology

We denote by $\mathcal{A}^n_{\mathcal{P}}$ the algebra $\mathcal{A}^n = \mathbb{C}[X_1,\ldots,X_n]$ provided with the Poisson structure $\mathcal{P}$. For $k \geq 1$, let $\chi^k(\mathcal{A}^n_{\mathcal{P}})$ be the vector space of k-derivations that is of k-skew linear maps on $\mathcal{A}^n_{\mathcal{P}}$ satisfying

$$\varphi\big(P_1Q_1, P_2, \ldots, P_k\big) = P_1\varphi\big(Q_1, P_2, \ldots, P_k\big) + Q_1\varphi\big(P_1, P_2, \ldots, P_k\big)$$

for all $Q_1, P_1, \ldots, P_k \in \mathcal{A}^n_{\mathcal{P}}$. For $k = 0$, we put $\chi^0(\mathcal{A}^n_{\mathcal{P}}) = \mathcal{A}^n_{\mathcal{P}}$. Let δ^k be the linear map:

$$\delta^k : \chi^k\big(\mathcal{A}^n_{\mathcal{P}}\big) \longrightarrow \chi^{k+1}\big(\mathcal{A}^n_{\mathcal{P}}\big)$$

given by

$$\begin{aligned}\delta^k\varphi\big(P_1, P_2, \ldots, P_{k+1}\big) = &\sum_{i=1}^{k+1} (-1)^{i-1}\big\{P_i, \varphi\big(P_1, \ldots, \widehat{P_i}, \ldots P_{k+1}\big)\big\} \\ &+ \sum_{1\leq i<j\leq k+1} (-1)^{i+j}\varphi\big(\big\{P_i, P_j\big\}, P_1, \ldots, \widehat{P_i}, \ldots, \widehat{P_j}, \ldots P_{k+1}\big),\end{aligned}$$

where $\widehat{P_i}$ means that the term P_i does not appear. We have $\delta^{k+1} \circ \delta^k = 0$ and the Lichnerowicz-Poisson cohomology corresponds to the complex $(\chi^k(\mathcal{A}_{\mathcal{P}}), \delta^k)_k$. Let us note that $\chi^k(\mathcal{A}^n_{\mathcal{P}})$ is trivial as soon as $k > n$. A description of the cocycle $\delta^k\varphi$ is presented in [11] for $n = 3$ using the vector calculus. We will describe these formulae using exterior calculus for $n > 3$. Let us begin with some notations.

- To any element $P \in \mathcal{A}^n_{\mathcal{P}} = \chi^0(\mathcal{A}^n_{\mathcal{P}})$, we associate the n-exterior form:

  $$\Phi_n(P) = P dX_1 \wedge \cdots \wedge dX_n.$$

- To any $\varphi \in \chi^k(\mathcal{A}^n_{\mathcal{P}})$ for $1 \leq k < n$, we associate the $(n-k)$-exterior form:

  $$\Phi_{n-k}(\varphi) = \sum_{\sigma \in S_{k,n-k}} (-1)^{\varepsilon(\sigma)}\varphi\big(X_{\sigma(1)}, \ldots, X_{\sigma(k)}\big) dX_{\sigma(k+1)} \wedge \cdots \wedge dX_{\sigma(n)}.$$

- To any $\varphi \in \chi^n(\mathcal{A}^n_{\mathcal{P}})$, we associate the function $\Phi_0(\varphi) = \varphi$.

Finally, if θ is an k-exterior form and $Y = \sum_{i=1}^n Y_i\partial_i$ is a vector field with $Y_i \in \mathcal{A}^n_{\mathcal{P}}$, then the inner product $i(Y)\theta$ is the $(k-1)$-exterior form given by

$$i(Y)\theta\big(Z_1, \ldots, Z_{k-1}\big) = \theta\big(Y, Z_1, \ldots, Z_{k-1}\big)$$

for every vector fields $Z_1, \ldots, Z_{k-1}$.

Theorem 2.2. *Assume that $n = 3$. Then we have*

(1) for all $P \in \mathcal{A}^3_{\mathcal{P}}$,

$$\Phi_2\big(\delta^0 P\big) = -\Omega \wedge dP;$$

(2) for all $f \in \chi^1(\mathcal{A}^3_{\mathcal{P}})$,

$$\begin{aligned}\Phi_1\big(\delta^1 f\big) = &- i\big(\partial_1, \partial_2\big)\big[\Omega \wedge d\big(i\big(\partial_3\big)\Phi_2(f)\big) + d\big(i\big(\partial_3\big)\Omega\big) \wedge \Phi_2(f)\big]\\ &+ i\big(\partial_1, \partial_3\big)\big[\Omega \wedge d\big(i\big(\partial_2\big)\Phi_2(f)\big) + d\big(i\big(\partial_2\big)\Omega\big) \wedge \Phi_2(f)\big]\\ &- i\big(\partial_2, \partial_3\big)\big[\Omega \wedge d\big(i\big(\partial_1\big)\Phi_2(f)\big) + d\big(i\big(\partial_1\big)\Omega\big) \wedge \Phi_2(f)\big],\end{aligned}$$

where $i(X,Y)$ denotes the composition $i(X) \circ i(Y)$;

(3) for all $\varphi \in \chi^2(\mathcal{A}^3_{\mathcal{P}})$,

$$\Phi_0\big(\delta^2\varphi\big) = i\big(\partial_1, \partial_2, \partial_3\big)\big(d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi)\big).$$

Proof. If $n = 3$, we have

$$\Omega = P_{12}dX_3 - P_{13}dX_2 + P_{23}dX_1.$$

Then the integrability of $\mathcal{P}$ is equivalent to $\Omega \wedge d\Omega = 0$. The theorem results of a direct computation and of the following general formula:

$$\forall \varphi \in \chi^k\big(\mathcal{A}^n_{\mathcal{P}}\big), \quad \varphi(P_1, \ldots, P_k) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} \varphi\big(X_{i_1}, \ldots, X_{i_k}\big)\partial_{i_1}P_1 \cdots \partial_{i_k}P_k. \qquad \square$$

Example 2.3. We consider the Poisson algebra $\mathcal{A}^n_{\mathcal{P}_1} = (\mathbb{C}[X_1, X_2, X_3], \mathcal{P}_1)$, where $\mathcal{P}_1$ is given by

$$\begin{cases}\mathcal{P}_1\big(X_1, X_2\big) = X_2,\\ \mathcal{P}_1\big(X_1, X_3\big) = 2X_3,\\ \mathcal{P}_1\big(X_2, X_3\big) = 0.\end{cases}$$

Then

$$\dim H^0\big(\mathcal{A}^n_{\mathcal{P}_1}\big) = 1, \quad \dim H^1\big(\mathcal{A}^n_{\mathcal{P}_1}\big) = 3, \quad \dim H^2\big(\mathcal{A}^n_{\mathcal{P}_1}\big) = 2, \quad H^3\big(\mathcal{A}^n_{\mathcal{P}_1}\big) = \{0\}.$$

In this case, $\Omega = X_2 dX_3 - 2X_3 dX_2$ and $d\Omega = 3dX_2 \wedge dX_3$. Let us compute $\dim H^2(\mathcal{A}^n_{\mathcal{P}_1})$. Let $\varphi \in \chi^2(\mathcal{A}^n_{\mathcal{P}1})$. Then $\Phi_0(\delta^2\varphi) = 0$ implies

$$d\Omega \wedge \Phi_1(\varphi) + \Omega \wedge d\Phi_1(\varphi) = 0,$$

that is

$$\begin{aligned}&X_2\big(\partial_1\varphi\big(X_1, X_3\big) + \partial_2\varphi\big(X_2, X_3\big)\big)\\ &\qquad + 2X_3\big(- \partial_1\varphi\big(X_1, X_2\big) + \partial_3\varphi\big(X_2, X_3\big)\big) + 3\varphi\big(X_2, X_3\big) = 0.\end{aligned}$$

Now, if $f \in \chi^1(\mathcal{A}^n_{\mathcal{P}_1})$, then

$$\Phi_1(\delta f) = \big[X_2\big(- \partial_2 f\big(X_2\big) - \partial_1 f\big(X_1\big)\big) - 2X_3\big(\partial_3 f\big(X_2\big)\big) + f\big(X_2\big)\big]dX_3$$

$$- \big[2X_3\big(\partial_1 f(X_1) + \partial_3 f(X_3)\big) + X_2\big(\partial_2 f(X_3)\big) - 2f(X_3)\big]dX_2$$
$$- \big[X_2\big(- \partial_1 f(X_3)\big) - 2X_3\big(\partial_1 f(X_2)\big)\big]dX_1.$$

Comparing these two relations, we obtain that $H^2(\mathcal{A}^n_{\mathcal{P}_1})$ is generated by the two cocycles:

$$\begin{cases} \Phi_1(\varphi_1) = X_3 dX_2, \\ \Phi_1(\varphi_2) = X_2^2 dX_2. \end{cases}$$

Now consider the general case. Let $\mathcal{A} = \mathbb{C}[X_1, \ldots, X_n]$ be provided with the Poisson structure $\mathcal{P}$.

Theorem 2.4. *Let $\varphi \in \chi^k(\mathcal{A}_{\mathcal{P}})$. Then, we have*

$$\Phi_{n-k-1}(\delta^k\varphi) = \varepsilon \sum i\big(\partial_{\sigma(1)}, \ldots, \partial_{\sigma(k+1)}\big)\big[d\big(i\big(\partial_{\sigma(k+2)}, \ldots, \partial_{\sigma(n)}\big)\Omega\big) \wedge \Phi_{n-k}(\varphi)$$
$$+ \Omega \wedge d\big(i\big(\partial_{\sigma(k+2)}, \ldots, \partial_{\sigma(n)}\big)\Phi_{n-k}(\varphi)\big)\big]$$

for all $\sigma \in S_{k+1,n-k-1}$, where $\varepsilon = \varepsilon(n,k) = (-1)^{\frac{(n-k)(n-k+1)}{2}}$.

Proof. To simplify, we write d_i in place of dX_i. We have seen that for every $P \in \mathcal{A}_{\mathcal{P}}$, we have $\delta^0 P = -\Omega \wedge dP$. But

$$\Phi_{n-1}(\delta P) = \sum_{k=1}^{n} (-1)^{k-1}\{X_k, P\} d_1 \wedge \cdots \wedge \hat{d_k} \wedge \cdots \wedge d_n,$$

where $\hat{d_i}$ means that this factor does not appear with $\{P, X_i\} = \sum_{j=1}^n P_{ji}\partial_j P$ with $P_{ji} = -P_{ij}$ when $j > i$. But

$$\begin{aligned} &i(\partial_1)[\Omega \wedge d\big(i(\partial_2, \ldots, \partial_n)\Phi_n(P)\big) + d\big(i(\partial_2, \ldots, \partial_n)\Omega\big) \wedge \Phi_n(P) \\ &\quad = i(\partial_1)\big[\Omega \wedge d\big(i(\partial_2, \ldots, \partial_n)\Phi_n(P)\big)\big] = (-1)^{\frac{n(n-1)}{2}} i(\partial_1)\big[\Omega \wedge dP \wedge d_1\big] \\ &\quad = -(-1)^{\frac{n(n-1)}{2}} \sum_{i=2}^{n} P_{1i}\partial_i P d_2 \wedge \cdots \wedge d_n = (-1)^{\frac{n(n-1)}{2}} \Phi_{n-1}(P)\big(\partial_2, \ldots, \partial_n\big). \end{aligned}$$

Similarly,

$$\begin{aligned} &i(\partial_j)\big[\Omega \wedge d\big(i(\partial_1, \ldots, \hat{\partial_j}, \ldots, \partial_n)\Phi_n(P)\big) + d\big(i(\partial_1, \ldots, \hat{\partial_j}, \ldots, \partial_n)\Omega\big) \wedge \Phi_n(P)\big] \\ &\quad = i(\partial_j)\big[\Omega \wedge d\big(i(\partial_1, \ldots, \hat{\partial_j}, \ldots, \partial_n)\Phi_n(P)\big)\big] = (-1)^{j-1+\frac{n(n-1)}{2}} i(\partial_j)\big[\Omega \wedge dP \wedge dX_j\big] \\ &\quad = (-1)^{j-1+\frac{n(n-1)}{2}} i(\partial_j)\left(\sum_{l=1}^{l=j-1} P_{1j}\partial_l P - \sum_{l=j+1}^{l=n} P_{jl}\partial_l P\right) d_1 \wedge \cdots \wedge d_n \\ &\quad = (-1)^{\frac{n(n-1)}{2}} \left(\sum_{l=1}^{l=j-1} P_{1j}\partial_l P - \sum_{l=j+1}^{l=n} P_{jl}\partial_l P\right) d_1 \wedge \cdots \wedge \hat{d_j} \cdots \wedge d_n \\ &\quad = (-1)^{\frac{n(n-1)}{2}} \{P, X_i\} d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n. \end{aligned}$$

We deduce

$$\Phi_{n-1}(\delta^0 P) = (-1)^{\frac{n(n-1)}{2}} \sum_{j=1}^{n} (-1)^{j-1} i(\partial_j)\big[\Omega \wedge d\big(i(\partial_1, \ldots, \hat{\partial_j}, \ldots, \partial_n)\Phi_n(P)\big)\big],$$

which proves the theorem for $k = 0$. The proof is similar for any k. $\square$

Application 2.5. We consider the n-dimensional complex Lie algebra defined by the brackets:

$$[X_1, X_i] = (i-1)X_i$$

for $i = 2, \dots, n$. Let $\mathcal{P}_2$ be the corresponding Poisson bracket on $\mathbb{C}[X_1, \dots, X_n]$. Let $\chi_2^k(\mathcal{A}_{\mathcal{P}_2})$ be the subspace of $\chi^k(\mathcal{A}_{\mathcal{P}_2})$ whose elements are homogenous of degree 2. We denote by $H_2^2(\mathcal{A}_{\mathcal{P}_2}) = Z_2^2/B_2^2$ the corresponding subspace of $H^2(\mathcal{A}_{\mathcal{P}_2})$. Define $N := \frac{n(n-1)}{2}$.

- If n is even, then

$$\dim B_2^2 = N + (N-1) + \cdots + N - n/2 + 1 = \frac{n\big(2n^2 - 3n + 2\big)}{8}.$$

- If n is odd,

$$\dim B_2^2 = N + (N-1) + \cdots + \big(N - (n-1)/2\big) = \frac{\big(n^2-1\big)(2n-1)}{8}.$$

In fact, if $f \in \chi_2^1(\mathcal{A}_{\mathcal{P}})$, then $f(X_i) = P_i = \Sigma a^i_{i_1,\dots,i_n} X_1^{i_1} X_2^{i_2} \cdots X_n^{i_n}$ is homogenous of degree 2, then:

(1) in $\delta f(X_1, X_{2l})$, we find $N - l$ independent coefficients of P_{2l}. The coefficients which do not appear are

$$a^{2l}_{1,0,0,\dots,0,1,0,\dots,0},\ a^{2l}_{0,1,0,\dots,0,1,0,\dots,0},\ \dots, a^{2l}_{0,0,\dots,1,1,0,\dots,0},$$

where the second 1 in the sequences of indices is, respectively, in the place $2l, 2l-1, \dots, l+1$;

(2) in $\delta f(X_1, X_{2l+1})$, we find $N - l - 1$ independent coefficients of P_{2l+1}. The coefficients which do not appear are

$$a^{2l+1}_{1,0,0,\dots,0,1,0,\dots,0},\ a^{2l+1}_{0,1,0,\dots,0,1,0,\dots,0},\ \dots, a^{2l+1}_{0,0,\dots,0,2,0,\dots,0},$$

where the second 1 in the sequences of indices is in place $2l+1, 2l, \dots, l+2$ and in the last case the 2 is in place $l+1$;

(3) for $i \geq 2$ and $j > i$, $\delta f(X_i, X_j)$ is defined by the $(n-2)$ coefficients $a^i_{1,0,0,\dots,0,1,0,\dots,0}$.

Now we can find the generators of $H_2^2(\mathcal{A}_{\mathcal{P}})$. We can choose $\phi \in \chi_2^2$ such that

$$\begin{cases} \phi\big(X_1, X_2\big) = 0, \\ \phi\big(X_1, X_3\big) = a^{1,3}_{1,3} X_1 X_3 + a^{2,2}_{1,3} X_2^2, \\ \cdots \\ \phi\big(X_1, X_{2l}\big) = a^{1,2l}_{1,2l} X_1 X_{2l} + a^{2,2l-1}_{1,2l} X_2 X_{2l-1} + \cdots + a^{l,l+1}_{1,2l} X_l X_{l+1}, \\ \phi\big(X_1, X_{2l+1}\big) = a^{1,2l+1}_{1,2l+1} X_1 X_{2l+1} + a^{2,2l}_{1,2l+1} X_2 X_{2l} + \cdots + a^{l,l}_{1,2l+1} X_l^2, \\ \cdots \\ \phi\big(X_1, X_n\big) = a^{1,n}_{1,n} X_1 X_n + a^{2,n-1}_{1,n} X_2 X_{n-1} + \cdots, \\ \phi\big(X_i, X_j\big) = A_{i,j}, \end{cases}$$

where $A_{i,j}$ is a degree 2 homogenous polynomial without monomial of types X_1X_k and X_iX_j. By solving $\Phi_{n-2}(\delta\phi)=0$, we obtain the generators of $H_2^2(\mathcal{A}_{\mathcal{P}})$. They are given by

$$\begin{cases}\phi\big(X_1,X_2\big)=0,\\ \phi\big(X_1,X_3\big)=a_{1,3}^{2,2}X_2^2,\\ \dots\\ \phi\big(X_1,X_{2l}\big)=a_{1,2l}^{2,2l-1}X_2X_{2l-1}+\dots+a_{1,2l}^{l,l+1}X_lX_{l+1},\\ \phi\big(X_1,X_{2l+1}\big)=a_{1,2l+1}^{2,2l}X_2X_{2l}+\dots+a_{1,2l+1}^{l+1,l+1}X_{l+1}^2,\\ \dots\\ \phi\big(X_1,X_n\big)=a_{1,n}^{2,n-1}X_2X_{n-1}+\dots+a_{1,n}^{m,m+1}X_mX_{m+1}, \quad \text{if } n=2m,\\ \phi\big(X_i,X_j\big)=A_{i,j},\end{cases}$$

or $\phi(X_1,X_n)=a_{1,n}^{2,n-1}X_2X_{n-1}+\dots+a_{1,n}^{m+1,m+1}X_mX_{m+1}$, if $n=2m+1$. For example:

- if $n=2$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2},\mathcal{A}_{\mathcal{P}_2})=1$;
- if $n=3$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2},\mathcal{A}_{\mathcal{P}_2})=3$;
- if $n=4$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2},\mathcal{A}_{\mathcal{P}_2})=8$;
- if $n=5$, $\dim H_2^2(\mathcal{A}_{\mathcal{P}_2},\mathcal{A}_{\mathcal{P}_2})=16$.

3 Poisson structures of degree 2 on $\mathbb{C}[X_1,X_2,X_3]$

Let $\mathcal{P}$ be a Poisson structure on $\mathcal{A}^3=\mathbb{C}[X_1,X_2,X_3]$ with P_{ij} of degree 2. Then $\mathcal{P}$ writes

$$\mathcal{P}=\mathcal{P}_0+\mathcal{P}_1+\mathcal{P}_2,$$

where $\mathcal{P}_i$ is homogenous of degree i. The associated form Ω is decomposed in homogenous parts $\Omega=\Omega_0+\Omega_1+\Omega_2$ and, since $d\Omega_0=0$, the condition $\Omega\wedge d\Omega=0$ is equivalent to

$$\begin{cases}\Omega_2\wedge d\Omega_2=0,\\ \Omega_0\wedge d\Omega_1+\Omega_1\wedge d\Omega_0=0,\\ \Omega_0\wedge d\Omega_2+\Omega_2\wedge d\Omega_0+\Omega_1\wedge d\Omega_1=0,\\ \Omega_1\wedge d\Omega_2+\Omega_2\wedge d\Omega_1=0.\end{cases} \tag{3.1}$$

If $\Omega_2=0$, then $\mathcal{P}$ is a linear Poisson structure on $\mathcal{A}^3$ ([1]). If $\Omega_2\neq 0$ and $\Omega_0=\Omega_1=0$, then $\mathcal{P}$ is a quadratic homogenous Poisson structure, and the classification is given in [9]. In this section, we will study the remaining cases $\Omega_0\neq 0$ or $\Omega_1\neq 0$. The associative algebra $\mathcal{A}^3$ admits a natural grading $\mathcal{A}^3=\oplus_{n\geq 0}V_n$, where V_n is the space of degree n homogenous polynomial of $\mathcal{A}^3$.

Definition 3.1. A linear isomorphism:

$$f:\oplus_{n\geq 0}V_n\longrightarrow\oplus_{n\geq 0}V_n$$

is called equivalence of order 2 if it satisfies

- $f(V_1)\subset V_1\oplus V_2$,

- $f(V_0) = V_0$,
- $f\mid_{\oplus_{n\geq 2} V_n} = Id$.

Moreover, if V_1 is provided with a Lie algebra structure, then

- $\pi_1 \circ f$ is a Lie automorphism of V_1,

where π_1 is the projection on V_1.

Such a map writes

$$\begin{cases} f(X_i) = \sum_{j=1}^{n} a_i^j X_j + \sum_{j,k=1}^{n} b_i^{jk} X_j X_k, \\ f(X_i X_j) = X_i X_j. \end{cases}$$

Thus, if $\mathcal{P}$ is a degree 2 Poisson structure on $\mathcal{A}^3$, putting $Y_i = f(X_i)$ and

$$\{Y_i, Y_j\} = f^{-1}\big(\{f(X_i), f(X_j)\}\big),$$

we obtain a new Poisson structure of degree 2. These two Poisson structures are called equivalent. In the following, we classify the non-homogenous Poisson structure of degree 2 up to an equivalence of order 2. Note that the quadratic homogenous Poisson structures are classified in [6]. We assume also that these Poisson structures are not trivial extensions of Poisson structures on $\mathcal{A}^2$, that is, Poisson structures which do not depend only on two variables.

3.1 First case: $\Omega = \Omega_2 + \Omega_1$, $\Omega_1 \neq 0$

The integrability condition of Ω reduces to

$$\begin{cases} \Omega_1 \wedge d\Omega_1 = 0, \\ \Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0, \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases} \tag{3.2}$$

As $\Omega^1 \wedge d\Omega^1 = 0$, Ω_1 defines on $\mathcal{A}^3$ a linear Poisson structure, then this form is isomorphic to one of the following:

$$\begin{cases} \Omega_1^1 = X_3 dX_3, \\ \Omega_1^2 = X_2 dX_3 + X_3 dX_2 + X_1 dX_1, \\ \Omega_1^3 = X_2 dX_3 - \alpha X_3 dX_2, \\ \Omega_1^4 = (X_2 + X_3) dX_3 - X_3 dX_2. \end{cases}$$

Consider $\Omega_2 = A_3 dX_3 - A_2 dX_2 + A_3 dX_1$ with

$$\begin{cases} A_1 = a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + a_4 X_1 X_2 + a_5 X_1 X_3 + a_6 X_2 X_3, \\ A_2 = b_1 X_1^2 + b_2 X_2^2 + b_3 X_3^2 + b_4 X_1 X_2 + b_5 X_1 X_3 + b_6 X_2 X_3, \\ A_3 = c_1 X_1^2 + c_2 X_2^2 + c_3 X_3^2 + c_4 X_1 X_2 + c_5 X_1 X_3 + c_6 X_2 X_3. \end{cases}$$

3.1.1 $d\Omega_2 = 0$

If $\Omega_1 = \Omega_1^1$ or Ω_1^2, then $d\Omega_1 = 0$, and (3.2) is satisfied. An equivalence of order 2 of type $Y_1 = X_1, Y_2 = X_2, Y_3 = X_3 + B$, where B is an homogenous polynomial of degree 2, allows to reduce the form Ω_2 to a form with $A_1 = 0$. We obtain the following Poisson structure associated to

$$\Omega(1) = \Big(aX_1^2 - \frac{b}{2}X_2^2 - 2cX_1X_2\Big)dX_1 - \big(cX_1^2 + eX_2^2 + bX_1X_2\big)dX_2 + X_3dX_3, \qquad (3.3)$$

corresponding to $\Omega_1 = \Omega_1^1$, and

$$\Omega(2) = \Big(X_1 + aX_1^2 - \frac{b}{2}X_2^2 - 2cX_1X_2\Big)dX_1 + \big(X_3 - cX_1^2 - eX_2^2 - bX_1X_2\big)dX_2 + X_3dX_3, \quad (3.4)$$

corresponding to $\Omega_1 = \Omega_1^2$. If $\Omega_1 = \Omega_1^3$ or Ω_1^4, then $d\Omega_1 = kdX_2 \wedge dX_3$ with $k \neq 0$. Then (3.2) implies $\Omega_2 \wedge dX_2 \wedge dX_3 = 0$ that is $A_3 = 0$. Such a structure is a Poisson structure on $\mathcal{A}^2$.

3.1.2 $d\Omega_2 \neq 0$, $\Omega_1 = \Omega_1^1$

As $d\Omega_1 = 0$, then (3.2) is equivalent to

$$\begin{cases} \Omega_1 \wedge d\Omega_2 = 0, \\ \Omega_2 \wedge d\Omega_2 = 0. \end{cases}$$

This implies $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$, where P is an homogenous polynomial of degree 2. The equivalence of order 2 given by $Y_i = X_i$ for $i = 1, 2$ and $Y_3 = X_3 + B$ with $B \in V_2$ enables to consider $A_1 = 0$. In this case, $\Omega_1 \wedge \Omega_2 = Pd\Omega_2$ is equivalent to

$$\begin{cases} \partial_1 A_2 + \partial_2 A_3 = 0, \\ P\partial_3 A_3 = X_3A_3, \\ P\partial_3 A_2 = X_3A_2. \end{cases}$$

If X_3 is not a factor of P, then $\partial_3 A_2 = \alpha X_3$ and $\partial_3 A_3 = \beta X_3$. If $\alpha = \beta = 0$, then $\Omega_2 = 0$. The case $\alpha\beta \neq 0$ reduces by a change of variables to the case $\alpha \neq 0$ and $\beta = 0$, then $A_3 = 0$. Thus, we obtain

$$\Omega = X_3dX_3 - \big(aX_2^2 + bX_3^2\big)dX_2.$$

This structure is a trivial extension of a Poisson structure on $\mathbb{C}[X_2, X_3]$. If $P = X_3Q$ and Q is a degree 1 homogenous polynomial, then Q satisfies

$$\begin{cases} Q\big(\partial_1 A_2 + \partial_2 A_3\big) = 0, \\ Q\partial_3 A_3 = A_3, \\ Q\partial_3 A_2 = A_2. \end{cases}$$

We deduce the following structures:

$$\begin{aligned} \Omega &= \big(aX_1^2 + bX_1X_3\big)dX_1 + X_3dX_3, \\ \Omega &= \big(aX_1 + X_3/2\big)^2 dX_1 + X_3dX_3, \\ \Omega &= \big(aX_1X_3 + bX_2X_3\big)dX_1 + \big(bX_1X_3 + cX_2X_3\big)dX_2 + X_3dX_3, \end{aligned}$$

The first two equations depend only on X_1 and X_3. Then we obtain the following Poisson structure:

$$\Omega(3) = \big(aX_1X_3 + bX_2X_3\big)dX_1 + \big(bX_1X_3 + cX_2X_3\big)dX_2 + X_3dX_3. \qquad (3.5)$$

3.1.3 $d\Omega_2 \neq 0$, $\Omega_1 = \Omega_1^2$

By an equivalence of degree 2, we can consider that $A_3 = 0$. Then $Pd\Omega_2 = \Omega_1 \wedge \Omega_2$ gives

$$\begin{cases} P\partial_1 A_2 = X_1 A_2, \\ P\partial_1 A_1 = X_1 A_1, \\ P\big(\partial_2 A_1 + \partial_3 A_2\big) = \big(A_2 X_2 + A_1 X_3\big). \end{cases}$$

Solving these equations, we obtain

$$\begin{aligned} \Omega(4) &= X_1 dX_1 + \big(X_3 - aX_1X_3\big)dX_2 + \big(X_2 + aX_1X_2\big)dX_3, \\ \Omega(5) &= X_1 dX_1 + \big(X_3 - aX_1^2 - 2aX_2X_3\big)dX_2 + X_2 dX_3. \end{aligned} \tag{3.6}$$

3.1.4 $d\Omega_2 \neq 0$, $\Omega_1 = \Omega_1^3 = X_2 dX_3 - \alpha X_3 dX_2$

Assume that $\alpha \neq 0$ and $\alpha \neq -1$. The equivalence given by $Y_2 = X_2 + B_2$, $Y_i = X_i$ for $i = 1, 3$ and $B_2 \in V_2$ shows that the structure corresponding to $\Omega = \Omega_1$ is equivalent to a structure of degree 2 defined as follow:

$$\begin{cases} A_1 = a_2X_2^2 + a_3X_3^2 + \dfrac{c_6}{\alpha}X_1X_2 + c_3X_1X_3, \\ A_2 = 0, \\ A_3 = c_3X_3^2 + c_5X_1X_3 + c_6X_2X_3. \end{cases}$$

Thus we can assume that in Ω_2, we have $c_3 = c_5 = c_6 = a_2 = a_3 = a_6 = 0$. The new equivalence of degree 2 given by $Y_3 = X_3 + B_3$, $Y_i = X_i$ for $i = 1, 2$ and $B_3 \in V_2$ gives a Poisson structure of degree 2 equivalent to the structure of degree 1 with

$$\begin{cases} A_1 = 0, \\ A_2 = b_2X_2^2 + b_3X_3^2 - c_2X_1X_2 + \dfrac{c_6}{\alpha}X_1X_3, \\ A_3 = c_2X_2^2 + c_4X_1X_2 + c_6X_2X_3. \end{cases}$$

Thus we can assume that

$$\Omega_2 = \big(a_1X_1^2 + a_4X_1X_2 + a_5X_1X_3\big)dX_1 + \big(b_1X_1^2 + b_4X_1X_2 + b_5X_1X_3\big)dX_2 + c_1X_1^2 dX_3.$$

As $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$, we obtain the following Poisson structure:

$$\Omega(6) = aX_1X_3dX_1 - \alpha X_3dX_2 + \left(X_2 - \frac{a}{2\alpha}X_1^2\right)dX_3 \tag{3.7}$$

with $\alpha \neq 0$ and $\alpha \neq -1$.

If $\alpha = -1$, then $d\Omega_1 = 0$, and this case has already been studied. If $\alpha = 0$, by equivalence of degree 2 we can assume that

$$\begin{cases} A_1 = a_1X_1^2 + a_3X_3^2 + a_4X_1X_2 + a_5X_1X_3, \\ A_2 = b_1X_1^2 + b_3X_3^2 + b_5X_1X_3, \\ A_3 = c_1X_1^2 + c_3X_3^2 + c_5X_1X_3. \end{cases}$$

Then we have $A_3 = 0$, and the Poisson structure concerns only two variables.

3.1.5 $d\Omega_2 \neq 0$, $\Omega_1 = \Omega_1^4 = (X_2 + X_3)dX_3 - X_3 dX_2$

By equivalence of degree 2, we can assume that $A_1 = 0$, $c_5 = 0$, and $b_4 = 0$. The equation $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies that $c_1 = c_4 = b_1 = 0$, $c_6 = -b_5 = 2c_2$. The equation $\Omega_2 \wedge d\Omega_2 = 0$ implies that $c_2 = 0$ and $b_2 c_3 = b_6 c_3 = 0$. Then we obtain the following Poisson structure:

$$\Omega(7) = aX_3^2 dX_1 - \big(X_3 + bX_3^2\big)dX_2 + \big(X_2 + X_3\big)dX_3 \tag{3.8}$$

with $a \neq 0$.

3.2 Second case: $\Omega = \Omega_2 + \Omega_1 + \Omega_0$, $\Omega_0 \neq 0$

The form $\Omega_0 \oplus \Omega_1$ provides the vector space $V_0 \oplus V_1$ with a linear Poisson structure. Then $V_0 \oplus V_1$ is a Lie algebra such that V_0 is in the center. This implies $\Omega_1 \wedge d\Omega_1 = 0$. We deduce that $\Omega_0 + \Omega_1$ is equivalent to

$$\begin{cases} dX_3 - X_3 dX_2, \\ X_3 dX_3 - dX_2, \\ X_2 dX_3 + X_3 dX_2 + dX_1. \end{cases} \tag{3.9}$$

3.2.1 $\Omega_0 + \Omega_1 = dX_3 - X_3 dX_2$

By equivalence, we can assume that $a_3 = a_5 = b_3 = c_5 = 0$. The equation $\Omega_0 \wedge d\Omega_2 = 0$ implies $b_4 = -2c_2$, $c_4 = -2b_1$, $c_6 = -b_5$, $\Omega_1 \wedge d\Omega_2 + \Omega_2 \wedge d\Omega_1 = 0$ implies that $c_1 = c_2 = c_3 = c_4 = 0$, $a_1 = a_4 = 0$, and $\Omega_2 \wedge d\Omega_2 = 0$ gives $b_5 b_2 = b_5 a_2 = b_5 a_6 = 0$. Thus we obtain the following Poisson structures given by

$$\Omega(8) = aX_2 X_3 dX_1 - \big(X_3 - aX_1 X_3 + bX_2 X_3\big)dX_2 + dX_3 \tag{3.10}$$

with $a \neq 0$.

3.2.2 $\Omega_0 + \Omega_1 = -dX_2 + X_3 dX_3$

We can assume that $A_2 = b_2 X_2^2 + b_4 X_1 X_2$. As $d\Omega_1 = 0$, the system reduces to $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0$. This gives $c_4 = c_6 = a_4 = 0$ and $b_4 + 2c_2 = a_5 - 2c_3 = 2a_1 - c_5 = 0$. Thus, $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $(2a_2 X_2 + a_6 X_3)A_3 = 0$. We obtain the following Poisson structures:

$$\begin{aligned} \Omega(9) = &-\big(X_2 + aX_2^2 + bX_1 X_2\big)dX_2 + \big(1 + cX_1^2 + eX_3^2 + fX_1 X_3\big)dX_3 \\ &+ \Big(gX_1^2 - \frac{b}{2}X_2^2 + \frac{f}{2}X_3^2 + 2cX_1 X_3\Big)dX_1. \end{aligned} \tag{3.11}$$

3.2.3 $\Omega_0 + \Omega_1 = dX_1 + X_3 dX_2 + X_2 dX_3$

By equivalence, we can assume $b_5 = b_2 = a_3 = a_5 = c_2 = c_5 = 0$. As $d\Omega_1 = 0$, the equation $\Omega_0 \wedge d\Omega_2 = \Omega_1 \wedge d\Omega_2 = 0$ implies that $b_6 + 2a_2 = a_6 + 2b_3 = a_4 = a_1 = b_1 = b_4 = c_3 = 0$. In this case, $\Omega_2 \wedge d\Omega_2 = 0$ is equivalent to $c_6(X_2 A_2 + X_3 A_1) = 0$. We obtain

$$\Omega(10) = \big(1 + aX_1^2\big)dX_1 + X_3 dX_2 + X_2 dX_3, \tag{3.12}$$

and

$$\Omega(11) = \big(1 + aX_1^2\big)dX_1 + \big(X_3 + bX_3^2 + cX_2 X_3\big)dX_2 + \Big(X_2 + \frac{c}{2}X_2^2 + 2bX_2 X_3\Big)dX_3. \tag{3.13}$$

4 Poisson algebras associated to rigid Lie algebras

4.1 Rigid Lie algebras

Let us fix a basis of $\mathbb{C}^n$. With respect to this basis, a multiplication μ of a n-dimensional complex Lie algebra is determined by its structure constants C_{ij}^k. We denote by L_n the algebraic variety $\mathbb{C}[C_{ij}^k]/I$, where I is the ideal generated by the polynomials:

$$\begin{cases} C_{ij}^k + C_{ji}^k = 0, \\ \displaystyle\sum_{l=1}^{n} C_{ij}^l C_{lk}^s + C_{jk}^l C_{li}^s + C_{ki}^l C_{li}^s = 0 \end{cases}$$

for all $1 \leq i, j, k, s \leq n$. Then every multiplication μ of a n-dimensional complex Lie algebra is identified to one point of L_n. We have a natural action of the algebraic group $\mathrm{Gl}(n, \mathbb{C})$ on L_n whose orbits correspond to the classes of isomorphic multiplications:

$$\mathcal{O}(\mu) = \left\{ f^{-1} \circ \mu \circ (f \times f),\ f \in \mathrm{Gl}(n, \mathbb{C}) \right\}.$$

Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a n-dimensional complex Lie algebra. We denote also by μ the corresponding point of L_n.

Definition 4.1. The Lie algebra $\mathfrak{g}$ is rigid if its orbit $\mathcal{O}(\mu)$ is open (for the Zariski topology) in L_n.

Among rigid complex Lie algebras, there are all simple and semi-simple Lie algebras, all Borel algebras and parabolic Lie algebras. Concerning the classification of rigid Lie algebras, we know the classification up the dimension 8 ([2]), the classification in any dimension of solvable rigid Lie algebras whose nilradical is filiform ([2]). Recall two interesting tools to study rigidity of a given Lie algebra.

Theorem 4.2. *Let $\mathfrak{g} = (\mathbb{C}^n, \mu)$ be a n-dimensional complex Lie algebra. Then:*

(1) $\mathfrak{g}$ is rigid if and only if any valued deformation $\mathfrak{g}'$ is (K^*)-isomorphic to $\mathfrak{g}$, where K^* is the fraction field of the valuation ring R containing the structure constants of $\mathfrak{g}'$;

(2) (Nijenhuis-Richardson theorem) if $H^2(\mathfrak{g}, \mathfrak{g}) = 0$, then $\mathfrak{g}$ is rigid.

The notion of valued deformation, which extends in a natural way the classical notion of Gerstenhaber deformations, is developed in [4]. In the Nijenhuis-Richardson theorem, the second cohomological space $H^2(\mathfrak{g}, \mathfrak{g})$ of the Chevalley cohomology of $\mathfrak{g}$ is trivial. Let us recall that the converse of this theorem is not true. There exists solvable rigid Lie algebras with $H^2(\mathfrak{g}, \mathfrak{g}) \neq 0$ (see for example [2]). In this case, there exists a 2-cocycle $\varphi_1 \in H^2(\mathfrak{g}, \mathfrak{g})$ which is not the first term of a valued (or formal) deformation:

$$\mu_t = \mu + \sum_{i \geq 1} t^i \varphi_i$$

of the Lie multiplication μ of $\mathfrak{g}$.

4.2 Finite dimensional Poisson algebras whose Lie bracket is rigid

We recall in this section some results of [5] which precise the structure of a finite dimensional complex Poisson algebra with rigid underlying Lie bracket. Let $\mathcal{P} = (\mathbb{C}^n, \mathcal{P})$ be a finite dimensional complex Poisson algebra. We denote by $\{X, Y\}$ and $X \cdot Y$ the corresponding Lie bracket and associative multiplication, by $\mathfrak{g}_{\mathcal{P}}$ the Lie algebra $(\mathcal{P}, \{,\})$, and by $\mathcal{A}_{\mathcal{P}}$ the associative algebra $(\mathcal{P}, \cdot)$.

Proposition 4.3 ([5])**.** *If the Lie algebra $\mathfrak{g}_{\mathcal{P}}$ is a simple complex Lie algebra, then the associative product is trivial that is $X \cdot Y = 0$ for every X, Y in $\mathcal{P}$.*

Let us assume now that $\mathfrak{g}_{\mathcal{P}}$ is a complex rigid solvable Lie algebra. Then $\mathfrak{g}$ is written as follows:

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n},$$

where $\mathfrak{n}$ is the nilradical of $\mathfrak{g}$ and $\mathfrak{t}$ is a maximal abelian subalgebra such that the adjoint operators adX are diagonalizable for every $X \in \mathfrak{t}$. This subalgebra $\mathfrak{t}$ is usually called a Malcev torus. All these maximal tori are conjugated and their common dimension is called the rank of $\mathfrak{g}$.

Lemma 4.4. *If there is a non-zero vector $X \in \mathfrak{g}_{\mathcal{P}}$, such that adX is diagonalizable with 0 as a simple root, then $\mathcal{A}_{\mathcal{P}} \cdot \mathcal{A}_{\mathcal{P}} = \{0\}$.*

Proof. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}_{\mathcal{P}}$, such that ade_1 is diagonal with respect to this basis. By assumption, $\{e_1, e_i\} = \lambda_i e_i$ with $\lambda_i \neq 0$ for $i \geq 2$. Since $\{e_1^2, e_1\} = 2e_1 \cdot \{e_1, e_1\} = 0$, it follows that $e_1^2 = ae_1$. But for any $i \neq 1$, $\{e_1^2, e_i\} = 2e_1 \cdot \{e_1, e_i\} = 2\lambda_i e_1 \cdot e_i$ and $\{e_1^2, e_i\} = a\lambda_i e_i$, thus $e_1 \cdot e_i = \frac{a}{2} e_i$. The associativity of the product $X \cdot Y$ implies that $(e_1 \cdot e_1) \cdot e_i = ae_1 \cdot e_i = \frac{a^2}{2} e_i = e_1 \cdot (e_1 \cdot e_i) = \frac{a^2}{4} e_i$. Therefore, $a = 0$ and $e_1^2 = 0 = e_1 \cdot e_i$ for any i. Finally, $0 = \{e_1 \cdot e_j, e_i\} = e_1 \cdot \{e_j, e_i\} + e_j \cdot \{e_1, e_i\} = \lambda_i e_j \cdot e_i$, which implies $e_i \cdot e_j = 0$, $\forall i, j \geq 1$. $\square$

Proposition 4.5. *Let $\mathfrak{g}$ be a rigid solvable Lie algebra of rank 1 with non-zero roots. Then, there is only one Poisson algebra $\mathcal{P}$ such that $\mathfrak{g}_{\mathcal{P}} = \mathfrak{g}$. It corresponds to*

$$X \cdot Y = 0$$

for any $X, Y \in \mathcal{P}$.

Proof. By hypothesis, we have $\dim \mathfrak{t} = 1$ and for $X \in \mathfrak{g}_{\mathcal{P}}$, $X \neq 0$, as the roots of $\mathfrak{g}$ are non-zero, the restriction of the operator adX on $\mathfrak{n}$ is invertible (all known solvable rigid Lie algebras satisfy this hypothesis). By the previous lemma, the associated algebra $\mathcal{A}_{\mathcal{P}}$ satisfies $\mathcal{A}_{\mathcal{P}} \cdot \mathcal{A}_{\mathcal{P}} = \{0\}$. $\square$

Theorem 4.6. *Let $\mathcal{P}$ a complex Poisson algebra, such that $\mathfrak{g}_{\mathcal{P}}$ is rigid solvable of rank 1 (i.e $\dim \mathfrak{t} = 1$) with non-zero roots. Then $\mathcal{P}$ is a rigid Poisson algebra.*

Proof. See [5]. $\square$

4.3 Linear Poisson structures on $\mathcal{A}^{n+1} = \mathbb{C}[X_0, X_1, \dots, X_n]$ given by a rigid Lie bracket

In this section, we consider a linear Poisson bracket on $\mathbb{C}[X_0, \dots, X_n]$, such that the brackets $\{X_i, X_j\} = \mathcal{P}(X_i, X_j)$ correspond to a solvable rigid Lie algebra $\mathfrak{g}$ of rank 1. We assume that the roots (see [2]) of this rigid Lie algebras are $1, \dots, n$. In this case, we have

$$\begin{cases} \{X_0, X_i\} = iX_i, \quad i = 1, \dots, n, \\ \{X_1, X_i\} = X_{i+1}, \quad i = 2, \dots, n-1, \\ \{X_2, X_i\} = X_{i+2}, \quad i = 3, \dots, n-2. \end{cases}$$

We denote this $(n+1)$-dimensional Poisson algebra by $\mathcal{P}(\mathfrak{g})$. This algebra is a deformation of the Poisson algebra studied in Section 2.2. The corresponding $(n-1)-$exterior form is

$$\Omega = \sum_{i=1}^{n} (-1)^{i-1} X_i d_1 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n + \sum_{i=2}^{n-1} (-1)^i X_{i+1} d_0 \wedge d_2 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n$$
$$+ \sum_{i=3}^{n-2} (-1)^{i+1} X_{i+2} d_0 \wedge d_1 \wedge d_3 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n,$$

where d_i denotes dX_i, and $\hat{d_i}$ means that this term does not appear. Let φ be a 2-cochain. We denote by $\varphi(i, j)$ the vector $\varphi(X_i, X_j)$. Then φ is a 2 cocycle if and only if

$$\Phi_{n-1}(\varphi) = (-1)^{n-2} \varphi(1, i) d_0 \wedge d_2 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n$$
$$+ \sum_{i=3}^{n} (-1)^{i-1} \varphi(2, i) d_0 \wedge d_1 \wedge d_3 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge d_n$$
$$+ \sum_{3 \le i < j \le n} (-1)^{j-i-1} \varphi(i, j) d_0 \wedge \cdots \wedge \hat{d_i} \wedge \cdots \wedge \hat{d_j} \wedge \cdots \wedge d_n$$

satisfies

$$d\big[i\big(\partial_{\sigma(1)}, \dots, \partial_{\sigma(n-2)}\big)\Omega\big] \wedge \Phi_{n-1}(\varphi) + \Omega \wedge d\big[i\big(\partial_{\sigma(1)}, \dots, \partial_{\sigma(n-2)}\big)\Phi_{n-2}(\varphi)\big] = 0 \qquad (4.1)$$

for any $\sigma \in S_{3,n-2}$. As $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{n}$, we have the decomposition $\mathcal{P}(\mathfrak{g}) = \mathcal{P}(\mathfrak{t}) \oplus \mathcal{P}(\mathfrak{n})$, where $\mathcal{P}(\mathfrak{t})$ and $\mathcal{P}(\mathfrak{n})$ are the Poisson algebras $(\mathbb{C}[X_0], \mathcal{P})$ and $(\mathbb{C}[X_1, \dots, X_n], \mathcal{P})$. From the Hochschild-Serre factorization theorem, we assume that the cocycles are $\mathfrak{t}$-invariant and with values in $\mathcal{P}(\mathfrak{n})$. We denote this space by $\chi^k(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$. If $f \in \chi^1(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$, then

$$\big\{X_0, f\big(X_i\big)\big\} = if\big(X_i\big),$$

and we obtain

$$f\big(X_1\big) = a_1^1 X_1, f\big(X_2\big) = a_1^{11} X_1^2 + a_2^2 X_2, \dots, f\big(X_i\big) = \sum_{l_1 + \cdots + l_k = i} a_i^{l_1 \cdots l_k} X_1^{l_1} \cdots X_k^{l_k}.$$

Thus, $\delta f(X_1, X_i) = a_1^1 \{X_1, X_i\} + \{X_1, f(X_i)\} - f(X_{i+1})$ and we can reduce any element $\varphi \in Z^2(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ to a 2-cocycle satisfying

$$\varphi\big(X_1, X_i\big) = 0 \quad \text{for } i = 2, \dots, n-1.$$

We denote by $Z_k^*(\mathcal{P}(\mathfrak{g}), \mathcal{P}(\mathfrak{g}))^{\mathfrak{t}}$ the subspace of homogenous cocycles of degree k. Let us have look the system at the $\varphi(i, j)$ which is deduced from equation (4.1).

– If $(\sigma(1),\ldots,\sigma(n-2))=(3,4,\ldots,n)$, then condition (4.1) is trivial.

– If $(\sigma(1),\ldots,\sigma(n-2))=(2,3,\ldots,\hat{l},\ldots,n)$, then condition (4.1) is trivial as soon as $l\neq n$. If $l=n$, we obtain

$$n\varphi\big(X_1,X_n\big)+(-1)^{n-1}\sum iX_i\partial_i\varphi\big(X_1,X_n\big)=0,$$

and $\varphi(X_1,X_n)$ is of weight $n+1$.

– If $(\sigma(1),\ldots,\sigma(n-2))=(1,2,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,n)$, we obtain

$$(i+j)\varphi\big(X_i,X_j\big)=\sum kX_k\partial_k\varphi\big(X_i,X_j\big),$$

and $\varphi(X_i,X_j)$ is of weight $i+j$.

Other relations show that the space of cocycles of degree 2 is generated by $\varphi(X_1,X_n)$ and $\varphi(X_2,X_{2k+1})$ with $k=1,\ldots,l$ where $n=2l+1$ or $n=2l$. The relations between these generators lead to study two cases: $k=1$ and $k=2$.

Case $k=1$. As $\varphi(X_1,X_n)$ is of weight $n+1$, then $\varphi(X_1,X_n)=0$. We have also $\varphi(X_i,X_j)=a_{ij}^{i+j}X_{i+j}$ if $i+j\leq n$.

If $(\sigma(1),\ldots,\sigma(n-2))=(1,2,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,n)$, we obtain

$$(i+j)\varphi\big(X_i,X_j\big)=\sum kX_k\partial_k\varphi\big(X_i,X_j\big),$$

and $\varphi(X_i,X_j)$ is of weight $i+1$. Then,

$$\varphi\big(X_i,X_j\big)=a_{ij}^{i+j}X_{i+j},$$

if $i+j\leq n$.

If $(\sigma(1),\ldots,\sigma(n-2))=(0,1,2,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,\hat{k},\ldots,n)$ with $i\geq 3$, then the related conditions are always satisfied.

If $(\sigma(1),\ldots,\sigma(n-2))=(0,3,\ldots,\hat{\imath},\ldots,n)$, $i\geq 3$, we obtain relation between $\varphi(3,l)$ and $\varphi(2,l+1)$. We deduce that

$$a_{3,i}=-a_{2,i+1}+a_{2,i},$$

and $a_{2,3}=a_{2,4}$.

If $(\sigma(1),\ldots,\sigma(n-2))=(0,1,2,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,\hat{k},\ldots,n)$ with $i\geq 3$, then

$$a_{4,i}=a_{2,i+2}-2a_{2,i+1}+a_{2,i},$$

and

$$a_{3,4}=a_{3,5}.$$

If $(\sigma(1),\ldots,\sigma(n-2))=(0,2,3,\ldots,\hat{\imath},\ldots,\hat{\jmath},\ldots,n)$, $i\geq 4$, then we have

$$a_{i+1,j}=-a_{i,j+1}+a_{i,j},$$

and

$$a_{i,i+2}=a_{i,i+1}.$$

If $(\sigma(1),\dots,\sigma(n-2)) = (0,1,3,\dots,\hat{\imath},\dots,\hat{\jmath},\dots,n)$, $i \geq 4$, then we have

$$a_{i+2,j} = -a_{i,j+2} + a_{i,j},$$

and

$$a_{3,j} = a_{2,j} - a2, j+1.$$

If we solve this linear system, we obtain the following.

Proposition 4.7. *If $n \geq 7$, then $H^2_1(A_p, A_p)$ is of dimension 1 and generated by the cocycle given by*

$$\begin{cases} \varphi\big(X_2, X_i\big) = (4-i)X_{2+i} \quad i = 5,\dots,n-2, \\ \varphi\big(X_3, X_i\big) = X_{3+i} \quad i = 4,\dots,n-3, \\ \varphi\big(X_i, X_j\big) = 0, \quad \textit{in other cases.} \end{cases}$$

Case $k = 2$. The set of generators is of dimension $\frac{p^2+5p}{2}$ if $n = 2p+1$ and $\frac{p^2+3p-2}{2}$ if $n = 2p$. The number of independent relations concerning these parameters is greater than the dimension of the set of generators as soon as $n \geq 6$. For $n = 5$, the dimension is equal to 2, and for $n = 6$, this dimension is 0. We deduce that $\dim H^2_2 = 0$ when $n \geq 7$.

Remark 4.8 (deformations of the enveloping algebra of a rigid Lie algebra)**.** Let $\mathfrak{g}$ be a finite dimensional complex Lie algebra. We denote by $\mathcal{U}(\mathfrak{g})$ its enveloping algebra. One of the most important problems in this time is to look at the deformations of the associative algebra $\mathcal{U}(\mathfrak{g})$. The theory of quantum groups comes from the deformation of $\mathcal{U}(\mathrm{sl}(2))$. In this case, $\mathfrak{g} = \mathrm{sl}(2)$ is a rigid Lie algebra, and $\mathcal{U}(\mathrm{sl}(2))$ is a rigid associative algebra. Thus, we have to look upon what happens for any rigid Lie algebra. The aim of this section is to study the deformations of $\mathcal{U}(\mathfrak{g})$ when $\mathfrak{g}$ is the rigid Lie algebras studied in the previous section.

We denote by $S(\mathfrak{g})$ the symmetric algebra on the vector space $\mathfrak{g}$. This associative commutative algebra is interpreted as the algebra of polynomials on the dual vector space $\mathfrak{g}^*$ of $\mathfrak{g}$ that is $\mathbb{C}[\alpha_1,\dots,\alpha_n]$ where $\{\alpha_1,\dots,\alpha_n\}$ is a basis of $\mathfrak{g}^*$. But the Lie structure of $\mathfrak{g}$ induces a linear Poisson structure (or of degree 1), $\mathcal{P}$, on $\mathfrak{g}^*$. In fact, if $\{X_1,\dots,X_n\}$ is the (dual) basis of $\mathfrak{g}$, this Poisson structure corresponds to the Poisson structure of degree 1 on $\mathbb{C}[X_1,\dots,X_n]$ associated to $\mathfrak{g}$. From the formality theorem of Kontsevich, $\mathcal{U}(\mathfrak{g})$ is a deformation of the Poisson algebra $(\mathbb{C}[X_1,\dots,X_n],\mathcal{P})$. In his thesis, Toukaidine Petit ([10]) shows that every nontrivial deformation of the Poisson structure $\mathcal{P}$ on $\mathbb{C}[X_1,\dots,X_n]$ induces a nontrivial deformation of the associative algebra $\mathcal{U}(\mathfrak{g})$. As a consequence, we have that if $\mathfrak{g}$ is a nonrigid Lie algebra, then there is a nontrivial deformation of $\mathcal{U}(\mathfrak{g})$.

If we consider the rigid Lie algebra $\mathfrak{g}_{n+1}$ studied in the previous section, we have determinate a nontrivial cocycle of degree one for the corresponding Poisson algebra which is not integrable. Thus, we cannot define a deformation of its enveloping algebra. But the Lie algebra $\mathfrak{g}_{n+1}$ admit a deformation in the following nonLie algebra which is written as follows:

$$\begin{cases} \mu\big(X_0, X_i\big) = iX_i, \quad i = 1,\dots,n, \\ \mu\big(X_1, X_i\big) = X_{i+1}, \quad i = 2,\dots,n-1, \\ \mu\big(X_2, X_3\big) = X_5, \\ \mu\big(X_2, X_i\big) = (5-i)X_{2+i}, \quad i = 4,\dots,n-2, \\ \mu\big(X_3, X_i\big) = X_{3+i}, \quad i = 4,\dots,n-3. \end{cases}$$

References

[1] J.-P. Dufour, Formes normales de structures de Poisson. In “Symplectic Geometry and Mathematical Physics (Aix-en-Provence, 1990)”, pp. 129–135, Progr. Math. **99**, Birkhäuser Boston, Boston, MA, 1991.

[2] M. Goze and J. M. Ancochea Bermudez, On the classification of rigid Lie algebras. *J. Algebra*, **245** (2001), 68–91.

[3] M. Goze, Algèbres de Lie. Classifications, Déformations et Rigidité, Géométrie différentielle. In “Algèbre, dynamique et analyse pour la géométrie: aspects récents”. Editions Ellipse, 2010, 39–99.

[4] M. Goze and E. Remm, Valued deformations of algebras. *J. Algebra Appl.*, **3** (2004), 345–365.

[5] M. Goze and E. Remm, Poisson algebras in terms of non-associative algebras. *J. Algebra*, **320** (2008), 294–317.

[6] A. Haraki, Quadratisation de certaines structures de Poisson. *J. London Math. Soc. (2)*, **56** (1997), 384–394.

[7] A. Lichnerowicz, Les variétés de Poisson et leurs algèbres de Lie associées, *J. Differential Geometry*, **12** (1977), 253-300.

[8] M. Markl and E. Remm, Algebras with one operation including Poisson and other Lie-admissible algebras. *J. Algebra*, **299** (2006), 171–189.

[9] P. Monnier, Formal Poisson cohomology of quadratic Poisson structures. *Lett. Math. Phys*, **59** (2002), 253–267.

[10] T. Petit, Sur les algèbres enveloppantes des algèbres de Lie rigides. Thèse de doctorat, Université de Haute Alsace, 2001.

[11] A. Pichereau, Poisson (co)homology and isolated singularities. *J. Algebra*, **299** (2006), 747–777.

[12] E. Remm, Opérades Lie-admissibles. *C. R. Math. Acad. Sci. Paris*, **334** (2002), 1047–1050.

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