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Research Article

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Regularity of Solutions of Degenerate Parabolic Non-linear Equations and Removability of Solutions

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Abstract

In this paper we prove regularity of solutions of degenerate parabolic nonlinear equations. We also the proof of a removability theorem for solutions to degenerate parabolic nonlinear equations.

Keywords: Degenerate; Nonlinear parabolic equations; Regularity; Removability

Introduction

Let we are considered in cylindrical domains $Q_T = \Omega \times (0,T)$, where $\Omega \subset R^n$, $n \geq 2$ is a bounded Lipschitz domain, T>0, degenerate non-linear parabolic equations

$$u_t - \mathrm{div}(\omega(x)|Du|^{p-2}Du) = 0 \quad (1.1)$$

$$u|_{\Gamma(Q_T)} = h, \quad (1.2)$$

where $\Gamma(Q_T) = (\Omega^- \times \{0\}) \cup (\partial\Omega \times [0,T])$ denote the parabolic boundary of Q_T, $h : Q_T \to R$ continuous function, $\omega(x)$-Makenxhoupt weight function [1].

To regularity of solutions to the degenerate parabolic non-linear operator introduced by DiBenedetto et al. [2,3]. Let $C_w^\alpha(Q_T)$ weighted space, where norm following:

$$\|f\|_{C^\alpha_{w(x)}(Q_T)} = \sup_{z_1, z_2 \in Qr} \frac{|f(z_1)w(x_1) - f(z_2)w(x_2)|}{\|z_1 - z_2\|\alpha} < \infty$$

where the parabolic metric is defined as

$$\|(x_1, t_1) - (x_2 t_2)\|_\alpha = \max\left[|x_1 - x_2|\right], |t_1 - t_2|^{\frac{1}{[p-\alpha(p-2)]}}, 0 < \alpha < 1$$

Main Results

We are now ready to state our result which concerns regularity for solutions to the problem (1.1), (1.2).

Theorem 2.1

Let's consider problem (1.1), (1.2)and let u(x,t) solve this problem. Let $Q^0_T \subset Q_T$ be a bounded space time cylinder such that (interior regularity)

$Q^0_T \cap \Gamma(Q_T) = \emptyset$. Then $u \in C^\alpha_{w(x)}(Q'_T)$ and

$$\|u(x,t)\|_{u \in C^\alpha_{w(x)}(Q'_T)} \leq c(n, p, w(x), Q_T, Q'_T, |h(x) \in C^\alpha_{w(x)}(\Omega), osu(x,t)|) \quad (2.1)$$

Theorem 2.1 concerns optimal interior regularity. We also establish optimal regularity up to initial state. In particular, in this case we prove $C^\alpha_w(Q_T)$ estimates on $Q_r = \Omega \times (0,T)$) for every $\Omega, \subset \Omega$. We doing remark that in this case Q_T is not a compact subset of Q_T.

In this context hold following result [1-12].

Theorem 2.2

Let u(x,t) solve problem (1.1), (1.2) and(Initial time regularity)

$$h(x) \in C^\alpha_{w(x)}(\Omega), \Omega' \subset \Omega\ Q'_T = \Omega'\ X(0,T), u \in C^\alpha_{w(x)}(Q'_T)$$

And

$$\|u(x,t)\|_{u \in C^\alpha_{w(x)}(Q'_T)} \leq c(n, p, w(x), Q_T, Q'_T, |h(x) \in C^\alpha_{w(x)}(\Omega), osu(x,t)|) \quad (2.2)$$

We also can be is considered obstacle problem similarly to problem (1.1), (1.2). In the case of linear uniformly parabolic equations [4]. Optimal regularity problem of the solution is considered [5].

We are study weak solutions from $L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))$ space. In the space

$W^{1,p}_{w(x)}(\Omega)$ the norm denote the space of equivalence classes of functions f with distributional gradient Df, both of which are p[th] power integral on Q_T. Let

$$\|f\|_{W^{1,p}_{w(x)}(\Omega)} = \|w(x) f(x)\|_{LP(\Omega)} + \|w(x)\| |Df|_{LP(\Omega)}$$

be the norm in $W^{1,p}_{w(x)}(\Omega)$.

Given $t_1 < t_2$ we denote by $L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))$ the space of functions such that for almost every t, $t_1 \leq t \leq t_2$ the function

$x \to u(x,t)$ belongs to $W^{1,p}_{w(x)}(\Omega)$ and $\|u\|_{L^P(t_1 t_2, W^{1,p}_{w(x)}(Q_T))}$

$$= \left(\int_{t_1}^{t_2}\int_\Omega (w(x)|u(x,t)|^p + w(x)|Du(x,t)|^p dxdt\right)^{\frac{1}{P}} \leq \infty$$

We say that a function u(x, t) is a weak solution to (1.1), (1.2) in an open set

$Q_T \subset R^{n+1}$ if whenever $Q^0_T = \Omega^0 \times (t_1, t_2) \subset Q_T$ with $\Omega^0 \subset \Omega \subset R^n$ and $t_1 < t_2$ then $u \in L^P(t_1 t_2, W^{1,p}_{w(x)}(\Omega))$ and

$$\int_{Q'_T} (w(x)|Du|^{p-2} Du D\varphi - u\varphi_t) dxdt = 0 \quad (2.3)$$

for all nonnegative $\varphi \in C_0^\infty(Q'_T)$.

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Using Theorem2.1 we are able to establish sharp removability conditions for compact sets. We of cylinders introduced

$$Q_T^{\lambda}(x_0,t_0)=\left\{(x,t)\in R^{n+1}:|x_0-x|<r,|t_0-t|<\lambda^{2-p\tau p}\right\}$$

And a concave modulus of continuity $\psi(\cdot)$. We let ψ: $R_+\to R_+$ be a concave modulus of continuity, i.e., concave non-decreasing function such that $\psi(1)=1$ and $\psi(0)=\lim_{r\to 0}\psi(r)=0$. We also define Hausdorff measure as follows. We let for fixed δ,0< δ<r_0 and $E\subset R^{n+1}, L(\delta,\psi(.);E)=\{Q_{ri}^{\psi(ri)}(x_i,t_i)\}$ be a family of cylinders such that $E\subset\cup Q_{ri}^{\psi(ri)}(x_i,t_i)$ and 0 < r_i < δ for i=1,2,..

Using this notation we let

$$H^{\psi(.)}(E)=\lim_{\delta\downarrow 0}\inf_{L(\delta,\psi(.);E)}\left\{\sum r_i^n\psi(r_i):E\subset\cup Q_{ri}^{\psi(ri)}(x_i,t_i)\right\}$$

where the indium is taken with respect to all possible coverings L(δ,ψ(·); E) of E.

Theorem 2.3

Let Q_T be a cylindrical domain and let E ⊂ Q_T be a closed set. Let u(x,t) is a weak solution to eqn. (1.1) in Q_T\E and that $u(x,t)\in C_{w(x)}^{\alpha}(Q_T)$

Assume also that $H^{\psi(\cdot)}$(E)=0. Then the set E is removable, i.e., u(x,t) can be extended to be a weak solution in Q_T.

Similarly result the fundamental work [6], under assumption Holder continuity of the solution can be found [7-12].

Proof of theorem 2.1

We assume $Q_{,T}\subset Q_T$ such that $Q^0_T\cap\Gamma(Q_T)=\emptyset$.

We define function

$$\underset{\Omega X[0,\infty]}{osc}\ \overline{h} = \underset{\Omega X[0,T]}{osc}\ h$$

Then $\underset{\Omega X[0,\infty]}{osc}\overline{h}=\underset{\Omega X[0,T]}{osc}h$. Let $\bar{u}$ be the unique solution to

$$\overline{u}_t-div(w(x)\left|D\overline{u}\right|^{p-2}D\overline{u})=0 \text{ in } \Omega\times(0,\infty)$$

$\bar{u}(x,t)=\bar{h}(x,t)$ on $\Gamma(\Omega\times(0,\infty))$.

By the uniqueness $\bar{u}=u$ in $\Omega\times[0,T]$ and hence $\bar{u}$ is an extension of u. Let

$R=\max\{1,diam\Omega,T^{1/2}\}$. As clearly

$T\le(\psi(R))^{2-p}R^p\le R^2$.

Whenever $R\ge 1$. By maximum and minimum principle implies that

$$\underset{Qr}{osc}u\le\underset{Qr}{osc}u\le\overline{c}\left(\Omega,T,\underset{Qr}{osc}u\right) \tag{2.4}$$

We may assume that $Q^{'}_{T,\gamma=}\Omega^{'}x(\tau,T)$, where $\Omega^0\subset\Omega$ and τ>0. We let R be

a number subject to the restrictions

$R\le dist(\Omega^0,\partial\Omega),\tau\ge R^p\max\{osch,\psi(R),s\cdot R\}^{2-p}$.

Q_T

As so $\psi(1)=1$, we see that these conditions are satisfied if we take

$$R\le dist(\Omega^{'},\partial(\Omega^{'},\partial\Omega)),\max\left\{T^{\frac{1}{P}}(\Omega,T,\underset{QT}{osch})^{\frac{p-2}{p}},\tau^{\frac{1}{P}},\tau^{\frac{1}{r}},\tau^{\frac{1}{2}},S^{\frac{P-2}{P}}\right\}$$

Taking correspondingly λ it follows that $Q_R^{\lambda\psi(R)}(z)\subset Q_T$ whenever $z\in Q^0_{T,\tau}$.

Now we prove that the following holds whenever $z\in Q^{'}_T$

$$\underset{Q_{T,r}^{c-\lambda\psi(r)}(z_0)\cap Qr}{osc}u\le\underset{QT}{osc}u=\frac{\underset{QT}{osc}u}{\psi(r)}\le\frac{\underset{QT}{osc}u}{\psi(\frac{R}{2})}\le 2\lambda\psi(r)$$

This completes the proof of Theorem 2.1.

Proof of theorem 2.2

After extending $u(x,t)$ as in the above we choose

$R=dist(\Omega^0,\partial\Omega)$ and define

$$\lambda=\max\left(\overline{c}/\psi(R),|b|_{C_{w(x)}(\Omega)},s.R/\psi(R)\right)\quad\text{where}\quad\overline{c}=\overline{c}\left(\Omega,T,\underset{Qr}{osc}u\right)$$

We let $Z=\Omega^{-0}\times(0)$ then

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\ \lambda\psi(r)$ for every $r\in(0,R)$,

$Q\lambda\psi r(r)(z)\cap Q0T$.

Whenever $z\in Z$. Consider $z_1\in\left(Q^{'}_T\cap(\overline{\Omega}^{'}X(0)\right)$ and define $\overline{r}=\overline{r}(z_1)=\sup\left[r\le R:Q_r^{\lambda\psi(r)}(z_1)\cap Z=\theta\right]$ If $r>R/2$, then

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\ \lambda\psi(r)$ for every $r\in(0,R)$.

In the final

$\bar{\lambda}=max\{4\lambda\psi(\bar{r}),s\cdot r/\psi^{-}(\bar{r})\}\ 4max\{\lambda,sR/\psi(R)\}=c\cdot\lambda$, implies that

$\underset{Q_r^{2\lambda\psi(r)}(z_1)}{osc}u\le c\ \lambda\psi(r)$ for every $r\in[0,\bar{r}]$.

Whenever $z_1\in\left(Q^{'}_T\cap(\overline{\Omega}^{'}X(0)\right)$.

This completes the proof of Theorem 2.2.

Proof of theorem 2.3

Let $u(x,t)$ weakly solve of eqn. (1.1) in $Q_T\setminus E$ and assume that $u(x,t)\in C^{\alpha}_{w(x),loc}(Q_T)$ and $H^{\psi(\cdot)}(E)=0$. $Q^2_T\subset Q^1_T\subset Q_T$ be arbitrary space-time smooth cylinders. Our only need to prove the conclusion in Q^1_T since the one of being a weak solution is a local property. By the assumption

$u(x,t)\in C^{\alpha}_{w(x),loc}(Q_T)$ there exists M>0 such that

$$\underset{Q^1_T}{osc}u(x,t)\le M\ \text{ and }\ \underset{Q_r^{\mu-\psi(r)}\cap Q^1_T}{osc}u(x,t)\le M\psi(r) \tag{2.5}$$

If we using the existence result, then see that there exist a unique solution $v(x,t)$ of problem

$$u_t-div(w(x)|Dv|^{p-2}Dv)=0 \tag{2.6}$$

$v|_{\Gamma(}QT^1)=u$

Let μ be the nonnegative Riesz measure associated to $v(x,t)$. Note that from existence μ follows $v(x,t)$ is a supersolution [7]. Let $F=\{(x,t)\in Q^1_T: v(x,t)=u(x,t)\}$. Now prove that the support of μ is contained in $F\cap E$. For these is sufficient to show that $v(x,t)$ is a weak solution to (2.6) in $Q^1_T\setminus(F\cup E)$. We already know that (2.6) satisfy in $Q^1_T\setminus F$ and it therefore remains to show that (2.6) satisfy in $Q^{-}_{T=}Q^1_T\setminus E$. To this aim, we show that if $Q^{-*}_T\subset Q^{-}_T$ is a cylinder and $\alpha\in C^0(Q^{-}_T)$ is a weak solution to $\alpha_t-(w(x)|D\alpha|^{P-2}D\alpha)$ witk $\alpha=u$ on $\Gamma(Q_T)$, then actually v must coincide with α (x,t)in the(Q_T) . Note that such a unique solution $\alpha(x,t)$ exists. We immediately see by the comparison principle that $v\ge\alpha$ in Q^{-*}_T, because $v(x,t)$ is a weak supersolution. To show that $v\le\alpha$ we

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instead argue as follows: since $u(x,t) \le v(x,t)$, we also have $u(x,t) \le \alpha(x,t)$ on ΓQ_T and as $u(x,t)$ solves eqn. (1.1) in Q_T, the comparison principle holdsu(x,t)≤ α (x,t)in. We thus conclude that $v(x,t) \le \alpha(x,t)$ on $\Gamma(Q^{**}_T) \cup F$. Therefore $v(x,t)=\alpha(x,t)$ and consequently also eqn. (2.6) yields in Q^{**}_T. This completes the proof that support of μ is contained in $F \cap E$.

Later using Theorem 2.1 and a covering argument we can conclude that there exists C depending only on $n,p,v,L,M,\psi(\cdot),Q^1_T,Q^2_T$ such that

$$\operatorname*{osc}_{Q_T^{\mu-\psi(t)} \cap Q_T^1} u(x,t) \le M\,\psi(r) \tag{2.7}$$

Whenever $\operatorname*{osc}_{Q_T^1} u(x,t) \le M$. Consider concentric cylinders $\operatorname*{osc}_{Q_T^{\mu-\psi(t)} \cap Q_T^1} u(x,t) \le M\,\psi(r)$. In the following we will use the short notation $\bar{Q}_T = Q^1_T \setminus E$. Let $\bar{Q}_T = Q^1_T \setminus E$ be such 0 and $\phi \equiv 1$ on $Q^{\cdot}_{\tau}$. Let $k=\sup v(x,t)$. Using eqn. (2.6) we have

$Q\tilde{\ }2\tau$

$$0 \le \mu\left(\bar{Q}_r\right) \le \int_{Q2r} \varphi^p d\mu =$$

$$\int_{Q2r}\left[-\left(\varphi^p\right)_{t^v} + \left(w(x)\left|Dv\right|^{p-2} Dv\right) D\varphi^p\right] dxdt \le$$

$$c\int_{Q2r} w(x)\left|Dv\right|^{p-1}\left|D\varphi\right|\varphi^{p-1}dxdt + \int_{Q2r}\left(\varphi^p\right)_t vdxdt \le$$

$$c\left(\int_{Q2r} w(x)\left|Dv\right|^p \varphi^p dxdt\right)^{\frac{p-1}{p}}\left(\int_{Q2r} w(x)\left|D\varphi\right|\varphi^p dxdt\right)^{\frac{1}{p}} +$$

$$\int_{Q2r}\left(\varphi^p\right)_t vdxdt \tag{2.8}$$

For the nonnegative weak sub solution $k-v(x,t)$ we see that

$$c\int_{Q2r}[\mathrm{w}(x)\left(k-v\right)^p\left|D\varphi\right|^{p-1} + \left|k-v\right|^2\left|\left(\varphi^p\right)_t\right| + S^p\varphi^p]dxdt$$

for some const $c=c(n,p,v,L) \ge 1$. By eqn. (2.7)

$$\sup_{Q_{2r}}\left|k-v\right| \le \operatorname*{osc}_{Q_T^1} v(x,t) \le C\psi(r)$$

and putting the estimates (2.8) we obtain that

$$\mu\left(Q_T\right) \le c\left[\left(\psi^2(\tau)\tau^n + s^p\psi^{2-p}(\tau)\tau^{2-p}\right)\right]^{\frac{p-1}{p}} \cdot \left[\left|\psi(\tau)\right|^{2-p}\tau^n\right]^{\frac{1}{p}}$$

$$c\psi(\tau)\tau^n \le c(1+s)^{p-1}\psi(\tau)\tau^n. \tag{2.9}$$

Here we also used the estimate $|\psi(\tau)|^{2-p} \le \tau^{2-p}$ for $\tau \le 1$. Now we consider cylinder $Q^3_T \subset Q^2_T$. We will prove that $\mu(Q^3_T)=0$. We first note using eqn. (2.9) we have

$$\mu\left(Q_\tau^{w(\tau)}\right) \le c\tau^n\varphi(\tau)\cdot \tag{2.10}$$

Whenever $Q_{2\tau}^{w(2\tau)} \subset Q_\tau^2$. Since $H^{\psi(\cdot)}(E)=0$ we obtain for $\varepsilon>0$ and $\delta>0$ given (to be taken smaller that dist$(\Gamma(Q^3_T),Q^2_T)/4$), then there exists a countable family

$$\left\{Q_{\tau_i}^{\psi(\tau_i)}\right\} = \left\{Q_{\tau_i}^{\psi(\tau_i)}\left(x_i,t_i\right)\right\}$$

of cylinders with $0 < \tau_i < \delta, i=1,2,...$, such that $Q_{2\tau_i}^{\psi(2\tau_i)} \subset Q_\tau^2$ and

$$E \cap Q3T \subset [Q\tilde{\ }\psi\tau i(\tau i) \text{ and } X\tau in\psi(\tau i)<\varepsilon. \tag{2.11}$$

Later using eqn. (2.10) we is obtain

$$\mu\left[F \cap \left(E \cap Q_T^3\right)\right] \le \sum_i \mu\left(Q_{\tau_i}^{\psi(\tau_i)}\right) \le \sum_i \tau_i^n\psi\left(\tau_i\right) < C.\in \tag{2.12}$$

proving that $\mu[F \cap (E \cap Q^3_T)]=0$. The fact that both Q^2_T and Q^3_T are arbitrary, we can conclude that $\mu(Q^1_T)=0$. Thus $v(x,t)$ is a solution in Q^1_T. Finally applying the above argument with $u(x,t)$ replaced by $-u(x,t)$ we deduce that there exist two solutions $v_1(x,t)$ and $v_2(x,t)$ i.e., eqn. (2.6) for v_1 equal to eqn. (2.6) for v_2. Such that $v_1(x,t) \le u(x,t) \le v_2(x,t)$ and $v_1(x,t)=v_2(x,t)$ on $\Gamma(Q^1_T)$. It follows that $v_{1=}v_{2=}u$. Theorem is proof.

References

1. Chamillo S, Wheeden RL (1985) Weighted Poincare and Sobolev inequalities. Amer J Math 107: 1191-1226.
2. Di Benedetto E (1993) Degenerate parabolic equations. Springer-Verlag.
3. Di Benedetto E, Gianazza U, Vespri V (2008) Intrinsic Harnack Estimates for Non-Negative Solutions of Quasilinear Degenerate Parabolic Equations. Acta Mathematica 200: 181-209.
4. Friedman A (1995) Parabolic variational inequalities in one-space dimension and smoothness of the free boundary. J Func Anal 18: 151-76.
5. Caffarelli LA (1998) The obstacle problem revisited. J Fourier Anal Appl 4: 383-402.
6. Serrin J (1964) Local behavior of solutions of quasi-linear equations. Acta Mathe-matica 3: 247-302.
7. Heinonen J, Kilpelainen T (1998) A superharmonic functions and supersolutions of degenerate elliptic equations. Arkiv Matematik 26: 87-105.
8. Kilpelainen T, Zhong X (2002) Removable sets for continuous solutions of quasi-linear elliptic equations. Proceedings of the American Mathematical Society 130: 1681-1688.
9. Gadjiev TS, Sadigova NR, Rasulov RA (2011) Removable singularities of solutions of degenerate nonlinear elliptic equations on the boundary of a domain. Nonlinear Analysis: Theory, Methods and Applications 74: 5566-5571.
10. Gadjiev T, Bayramova N (2014) The removability of compact of solutions in classes bounded functions. Ukrayne Mathematics Journal 8: 38-44.
11. Gadjiev T, Aliev O (2013) On Removable Sets of Solutions of Neuman Problem for Quasilinear Elliptic Equations of Divergent Form. Applied Mathematics 4: 290-298
12. Gadjiev T (2013) On Removable Sets for Generated Elliptic Equations. British Journal of Mathematics and Computer Science 3: 290-298.

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