

Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

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Abstract

In this paper, we prove that equation $E \equiv u_t - u_{x^2} + u_x f(u) - au_x u_{x^2} - buu_{x^3} = 0$ is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where f is smooth function on u and a, b are arbitrary constants. We find Three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitam equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

$$E \equiv u_t - u_{x^2} + u_x f - au_x u_{x^2} - buu_{x^3} = 0, \tag{1}$$

and we show that this equation is self-adjoint and quasi self-adjoint. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) $b \neq 0, a =$ arbitrary constant, 2) $b = 0, a \neq 0$, and 3) $b = 0, a = 0$. Clarkson, Mansfield and Priestly [3] are concerned with symmetry reductions of the non-linear third order partial differential equation given by $u_t - \epsilon u_{x^2} + (k - u)u_x - uu_{x^3} - \beta u_x u_{x^2} = 0$, where ϵ, k , and β are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of (1) are:

Camassa-Holm (CH) equation $u_t - u_{x^2} + (k + 3u)u_x = uu_{x^3} + 2u_x u_{x^2}$, k -arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let $f = k + 3u, a = 2, b = 1$ in (1)).

Degas peris-Procesi (DP) equation $u_t - u_{x^2} + (k + 4u)u_x = uu_{x^3} + 3u_x u_{x^2}$, k -arbitrary (real), is another equation of this class (let $f = k + 4u, a = 3, b = 1$ in (1)).

Fornberg Whitham (FW) equation $u_t - u_{x^2} + (1 + u)u_x = uu_{x^3} + 3u_x u_{x^2}$, is another equation of this class (let $f = 1 + u, a = 3, b = 1$ in (1)).

BBM equation $u_t - u_{x^2} + u_x + (uu_x) = 0$, is another equation of this class (let $f = 1 + u, a = 0, b = 0$ in (1)).

Preliminaries

In this section, we recall the procedure in literature of Ibragimov [1]. Let us introduce the formal Lagrangian

$$L \equiv vE, \tag{2}$$

where $v = v(t, x)$ is a new dependent variable.

We define the adjoint equation by $E^* \equiv \frac{\delta L}{\delta u} = 0$. Here

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} + D_x D_j \frac{\partial}{\partial u_{x^j}} - D_x D_j D_k \frac{\partial}{\partial u_{x^j k}} + \dots \quad i, j, k = 1, 2,$$

is the variational derivative and D_i is the operator of total differentiation.

An equation $E = 0$ is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution $v = u$ is identical with the original equation.

An equation $E = 0$ is said to be quasi- self-adjoint [5] if there exists a function $v = \phi(u)$, $\phi'(u) \neq 0$ such that $E^*|_{v=\phi(u)} = \lambda E$ with an undetermined coefficient λ . Eq.(1) is said to have a nonlocal conservation law if there exists a vector $C = (C^1, C^2)$ satisfying the equation

$$D_i(C^1) + D_x(C^2) = 0, \tag{3}$$

on any solution of the system of differential equations comprising (E) and the adjoint equation (E^*). We say that original equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry $v = \xi^i(x, t, u)\partial_x + \xi^2(x, t, u)\partial_t + \phi(x, t, u)\partial_u$ is obtained by the following formula :

$$C^i = \xi^i L + W \left[\frac{\partial L}{\partial u_i} - D_j \left(\frac{\partial L}{\partial u_{ij}} \right) + D_j D_k \left(\frac{\partial L}{\partial u_{ijk}} \right) \right] + D_j(W) \left[\frac{\partial L}{\partial u_{ij}} - D_k \left(\frac{\partial L}{\partial u_{ijk}} \right) \right] + D_j D_k(W) \frac{\partial L}{\partial u_{ijk}}, \tag{4}$$

where $i, j, k = 1, 2$ and $W = \phi - \xi^i u_i$. (Here ∂_x means $\frac{\partial}{\partial x}$).

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order n , containing p independent and q dependent variables is given as follows

$$\Delta_\nu(x, u^{(n)}) = 0, \quad \nu = 1, \dots, l, \tag{5}$$

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involving $x = (x^1, \dots, x^p)$, $u = (u^1, \dots, u^q)$ and the derivatives of u with respect to x up to n , where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n . We consider a one-parameter Lie group of transformations acting on the variables of system (5): $\bar{x}_i = x^i + \epsilon \xi^i(x, u) + O(\epsilon^2)$, $\bar{u}_j = u^j + \epsilon \phi^j(x, u) + O(\epsilon^2)$, where $i=1, \dots, p$, $j = 1, \dots, q$. ξ^i , ϕ^j are the infinitesimal of the transformations for the independent and dependent variables, respectively, and ϵ is the transformation parameter. We consider the general vector field v as the infinitesimal generator associated with the above group $v = \sum_{i=1}^p \xi^i(x, u) \partial_{x^i} + \sum_{j=1}^q \phi^j(x, u) \partial_{u^j}$. A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (5) under the infinitesimal transformation leads to the invariance conditions. (Theorem 2.36 of studies of Olver [6], Theorem 6.5 of literature of Olver [7]).

$$v^n[\Delta_v(x, u^n)] = 0, \quad \Delta_v(x, u^n) = 0, \quad v = 1, \dots, r, \quad (6)$$

where v^n is called the n^{th} order prolongation of the infinitesimal generator given by $v^n = v + \sum_{j=1}^q \sum_{k=1}^n \phi_k^j(x, u^{(n)}) \partial_{u_k^j}$, where $k = (i_1, \dots, i_\alpha)$, $1 \leq i_\alpha \leq p$, $1 \leq \alpha \leq n$, and the sum is over all k 's of order $0 < \#k \leq n$. If $\#k = \alpha$, the coefficient ϕ_k^j of $\partial_{u_k^j}$, will depend only on α^{th} and lower order derivatives of u and $\phi_k^j(x, u^{(n)}) = D_k(\phi_j - \sum_{i=1}^p (\xi^i u_i^j)) + \sum_{i=1}^p \xi^i u_{k,i}^j$, where $u_i^j := \partial u^j / \partial x^i$ and $u_{k,i}^j := \partial u_k^j / \partial x^i$.

Adjoint Equation and Classical Symmetry Method

Formal Lagrangian for Eq. (1) is

$$L = vE = v[u_t - u_{x^2 t} + u_x f - au_x u_{x^2} - buu_{x^3}]. \quad (7)$$

Therefore, the adjoint equation E^* to Eq. (1) is

$$f v_x + v_t + au_{x^2} v_x + au_x v_{xx} = 3bu_{x^2} v_x + 3bu_x v_{xx} + buv_{xxx} + v_{xxx}. \quad (8)$$

Upon setting $v = u$ it becomes

$$u_t = u_{x^2 t} - u_x f - 2au_x u_{x^2} + 6bu_x u_{x^2} + buu_{x^3}.$$

Hence, Eq. (1) is self-adjoint if and only if it has the form

$$a = 2b. \quad (9)$$

Consider again Eq. (1), and substitute

$$v = \phi(u), \quad v_t = \phi' u_t,$$

$$v_x = \phi' u_x, \quad v_{xx} = \phi' u_{x^2} + \phi'' u_x^2,$$

$$v_{xxx} = \phi' u_{x^3} + 3\phi'' u_x u_{x^2} + \phi''' u_x^3,$$

$$v_{xxt} = \phi' u_{x^2 t} + \phi'' u_x u_{x^2} + \phi''' u_x^2 u_t + 2\phi'' u_x u_{xt},$$

in the adjoint equation (8), then

$$\begin{aligned} & -f\phi' u_x - \phi' u_t - 2a\phi' u_x u_{x^2} - a\phi'' u_x^3 + 6b\phi' u_x u_{x^2} \\ & + 3b\phi'' u_x^3 + b\phi' u u_{x^3} + 3b\phi'' u u_x u_{x^2} + b\phi''' u u_x^3 \\ & + \phi' u_{x^2 t} + \phi'' u_x u_{x^2} + \phi''' u_x^2 u_t + 2\phi'' u_x u_{xt} \\ & = \lambda(u_t - u_{x^2 t} + u_x f - au_x u_{x^2} - buu_{x^3}). \end{aligned}$$

Hence, Eq. (1) is quasi self-adjoint if and only if it has the form

$$a = 2b, v = -\lambda u + \epsilon \quad (10)$$

In this section, we will perform Lie group method for Eq. (1) on $(x^1 = x, x^2 = t, u^1 = u)$, $(\bar{x}, \bar{t}, \bar{u}) = (x, t, u) + \epsilon(\xi(x, t, u), \tau(x, t, u), \phi(x, t, u)) + O(\epsilon^2)$,

where $\epsilon \leq 1$ the group parameter and $\xi^1 = \xi$, $\xi^2 = \tau$ and $\phi^1 = \phi$ are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form $v = \xi(x, t, u) \partial_x + \tau(x, t, u) \partial_t + \phi(x, t, u) \partial_u$ and the third prolongation of v is the vector field

$$v^{(3)} = v + \phi^x \partial_{u_x} + \phi^t \partial_{u_t} + \phi^{x^2} \partial_{u_{x^2}} + \phi^{xt} \partial_{u_{xt}} + \dots + \phi^{(n)} \partial_{u^{(n)}},$$

with coefficient

$$\phi^k = D_k(\phi - \sum_{i=1}^2 \xi^i u_i^j) + \sum_{i=1}^2 \xi^i u_{k,i}^j, \quad (11)$$

where D_k is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

$$v^{(3)}[u_t - u_{x^2 t} + u_x f - au_x u_{x^2} - buu_{x^3}] = 0, \quad (12)$$

whenever $E = 0$. The condition (12) is equivalent to

$$\phi^t - bu_{x^2} \phi + (f - au_{x^2}) \phi^x - au_x \phi^{x^2} - bu \phi^{x^3} - \phi^{x^2 t} = 0, \quad (13)$$

whenever $E = 0$. Substituting (11) into (13), yields the determining equations. There are three cases to consider:

a and $b \neq 0$ are arbitrary constants

In this case, complete set of determining equation is:

$$\xi_u = 0, \quad (14)$$

$$\tau_u = 0, \quad (15)$$

$$\tau_x = 0, \quad (16)$$

$$\phi_{u^2} = 0, \quad (17)$$

$$a\phi_{ux^2} + a\phi_u + a\tau_t - 3a\xi_x = 0, \quad (18)$$

$$\xi_{x^2} - 2\phi_{ux} = 0, \quad (19)$$

$$3bu\phi_{ux} + a\phi_x - 3b\xi_{x^2} u + 2\xi_{xt} + \phi_{ut} = 0, \quad (20)$$

$$2\xi_x - \phi_{ux^2} = 0, \quad (21)$$

$$\xi_t - b\phi_x - \phi_{ux^2} - bu\tau_t + 3bu\xi_x = 0, \quad (22)$$

$$a\xi_{x^2} u - 2a\phi_{ux} = 0, \quad (23)$$

$$\xi_{x^2 t} + bu\xi_x + \phi_{f_u} + \phi_{ux^2} f + f\xi_x + \tau_t f = a\phi_{x^2} + \xi_t + 2\phi_{xut} + 3bu\phi_{ux^2}, \quad (24)$$

$$-bu\phi_{x^3} + \phi_t + \phi_x f - \phi_{x^2 t} = 0. \quad (25)$$

With substituting (14) – (17) into (18) – (23) we have

$$\phi = c_1 + \frac{1}{b} \alpha'(t), \quad \tau = -c_1 t + c_2, \quad \xi = \alpha(t). \quad (26)$$

With substituting (26) into (24) – (25) we have

$$f = -1 + K(bu + 1), \quad (27)$$

where c_1 , c_2 and K are arbitrary constants. With substituting (27) into determining system, we have

$$\phi = \frac{-c_1(bu + 1)}{b}, \quad \tau = c_1 t + c_2, \quad \xi = -c_1 t + c_3,$$

where c_i , $i = 1, 2, 3$ are arbitrary constants.

Theorem 3.1.1. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu + 1)}{b} \partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu + 1)}{b} \partial_u.$$

We have

$$W = -u - \frac{1}{b} - tu_t + tu_x. \quad (28)$$

The right-hand side of (4) is written

$$\begin{aligned} C^1 &= W(v - D_x^2(v)) + (D_x W)(D_x v) - D_x^2(W)v, \\ C^2 &= W[vf - avu_{x^2} + D_x(avu_x) - D_x^2(buv) - 2D_x D_x(v)] \\ &+ D_x(W)[-avu_x + D_x(buv) + D_t(v)] + D_t(W)[D_x(v)] \\ &- 2D_x D_t(W)[v] - D_x^2(W)[buv]. \end{aligned} \quad (29)$$

We eliminate the term $\xi^i L$ since the Lagrangian L is equal to zero on solution of Eq.(1). Substituting in (29), the expression (7) for L and (28) for W , we obtain

$$\begin{aligned} C^1 &= -uv - \frac{1}{b}v - tu_t v + tu_x v + uv_{xx} + \frac{1}{b}v_{xx} + tu_t v_{xx} - tu_x v_{xx} \\ &- u_x v_x - tu_{xt} v_x + tv_x u_{x^2} + u_{x^2} v + tu_{x^2} v - tu_{x^3} v, \end{aligned} \quad (30)$$

and

$$\begin{aligned} C^2 &= -u(vf - bvu_{x^2} - buv_{xx} - 2v_{xt}) \\ &- (b^{-1}(vf - bvu_{x^2} - buv_{xx} - 2v_{xt})) \\ &- tu_t(vf - bvu_{x^2} - buv_{xx} - 2v_{xt}) \\ &+ tu_x(vf - bvu_{x^2} - buv_{xx} - 2v_{xt}) - u_x(buv_x - bvu_x + v_t) \\ &- (tu_{xt})(buv_x - bvu_x + v_t) + (tu_{x^2})(buv_x - bvu_x + v_t) - 2u_t v_x \\ &- tu_{x^2} v_x + u_x v_x + tu_{xt} v_x + buv_{xx} + tbvu_{x^2} - tbvu_{x^3} \\ &+ 4u_{xt} v + 2tu_{x^2} v - 2u_{x^2} v - 2u_{x^2} v. \end{aligned} \quad (31)$$

We can eliminate u_t by using Eq.(1) and then substitute in (30) and (31) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$\begin{aligned} C^1 &= \frac{-1}{b}(t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2})ub \\ &- t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2})u_{x^2} b - tu_x ub \\ &+ tu_{xt} u_x b + tu_{x^3} ub - tu_{x^2} ub + u^2 b - 2uu_{x^2} b + u_x^2 b + u - u_{x^2}), \end{aligned} \quad (32)$$

$$\begin{aligned} C^2 &= -u(uf - 2buu_{xx} - 2u_{xt}) - (b^{-1}(uf - 2buu_{xx} - 2u_{xt})) \\ &- t(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2})(uf - 2buu_{xx} - 2u_{xt}) \\ &+ tu_x(uf - 2buu_{xx} - 2u_{xt}) - u_x(u_t) - tu_{xt}(u_t) \\ &+ (tu_{x^2})(u_t) - 2(2bu_x u_{x^2} + buu_{x^3} - u_x f + u_{x^2})u_x \\ &- tu_{x^2} u_x + u_{xt} + tu_{xt} u_x + bu^2 u_{x^2} + tbu^2 u_{x^2} - tbu^2 u_{x^3} + 4u_{xt} u \\ &+ 2tu_{x^2} u - 2uu_{x^2} - 2u_{x^2} u. \end{aligned}$$

Where $f = -1 + K(bu + 1)$.

a is an arbitrary nonzero constant and $b = 0$.

In this case Eq.(1) is not self adjoint because $a \neq 2b$. Complete set of determining equation is:

$$\phi_{uu} = 0, \quad (33)$$

$$\xi_u = 0, \quad (34)$$

$$\xi_t = 0, \quad (35)$$

$$\tau_u = 0, \quad (36)$$

$$\tau_x = 0, \quad (37)$$

$$3a\xi_x = a\phi_{ux^2} + a\tau_t - a\phi_u, \quad (38)$$

$$\phi_{ut} + a\phi_x = 0, \quad (39)$$

$$-2\phi_{ux} + \xi_{x^2} = 0, \quad (40)$$

$$2\xi_x - \phi_{ux^2} = 0, \quad (41)$$

$$a\xi_{x^2} - 2a\phi_{ux} = 0, \quad (42)$$

$$\tau_t f + \phi f_u + \phi_{ux^2} f = 2\phi_{ux} + f\xi_x + a\phi_{x^2}, \quad (43)$$

$$\phi_t + \phi_x f = \phi_{x^2}. \quad (44)$$

Now, by considering Eq. (33) – (42) it is not hard to find that the components ξ , τ and ϕ of infinitesimal generators become

$$\phi = u \frac{dF_1(t)}{dt} - \frac{x}{a} \frac{d^2 F_1(t)}{dt^2} + F_2(t), \quad \tau = -F_1(t) + c_2, \quad \xi = c_1. \quad (45)$$

To find complete solution of the above system, we consider Eq. (43) and $l = \dim \text{Spam}_{\mathbb{R}}\{f_u, f, 1\}$. Three general cases are possible:

3.2.i) $l = 1$, then $f = \text{constant}$;

3.2.ii) $l = 2$, then $f_u = \alpha f + \beta$;

3.3.iii) $l = 3$, then $\alpha f_u + \beta f + \gamma \neq 0$, $\alpha \neq 0$.

Case 3.2.i). With substituting $f = \text{constant}$ in determining system (33)-(44), we have $\phi = c_1$, $\tau = c_2$, $\xi = c_3$, where c_i , $i = 1, 2, 3$ are arbitrary constants.

Theorem 3.2.1. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = \partial_u.$$

Case 3.2.ii). With integrating from $f_u = \alpha f + \beta$ with respect to u , we obtain

$$f = \frac{-\beta}{\alpha} + C e^{\alpha u}, \quad (46)$$

where C is an integrating constant. With substituting (46) into Eq. (43)-(44) and Eq. (45), we have

$$\xi = c_1, \quad \tau = -c_1 t, \quad \phi = \frac{c_1(C\alpha - e^{-\alpha u}\beta)}{C\alpha^2}. \quad (47)$$

Theorem 3.2.2. Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:

$$v = \partial_x - t\partial_t + \frac{C\alpha - e^{-\alpha u}\beta}{C\alpha^2} \partial_u. \quad (48)$$

Case 3.2.iii). The Eq. (43) leads to $\phi = 0$, $\tau = c_1$, $\xi = c_2$.

Theorem 3.2.3. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = \partial_t, \quad v_2 = \partial_x.$$

$b = 0$, $a = 0$.

Complete set of determining equation is

$$\xi_u = 0, \quad (49)$$

$$\xi_t = 0, \tag{50}$$

$$\tau_u = 0, \tag{51}$$

$$\tau_x = 0, \tag{52}$$

$$\phi_{uu} = 0, \tag{53}$$

$$\phi_{uu} = 0, \tag{54}$$

$$\phi_{ux^2} = 2\xi_x, \tag{55}$$

$$2\phi_{ux} = \xi_{x^2}, \tag{56}$$

$$\phi_t + f\phi_x = \phi_{x^2}, \tag{57}$$

$$f\tau_t + f\xi_x + \phi f_u = 0. \tag{58}$$

To find a complete solution of the above system we consider Eq. (58) and with assumption $f/f_u \neq 0$ we rewrite:

$$\phi = \frac{-f}{f_u}(\tau_t + \xi_x). \tag{59}$$

Two general cases are possible:

$$3.3.i) \frac{f}{f_u} = c, \quad 3.3.ii) \frac{f}{f_u} = h(u),$$

where c is constant.

Case 3.3.i).

With integrating from $f/f_u \neq c$ with respect to u , we have

$$f = Le^{uc}, \tag{60}$$

where L is an integrating constant. Then the Eq. (58) reduce to

$$\phi = -c(\tau_t + \xi_x). \tag{61}$$

With substituting Eq. (61) into determining equation, we have

$$\xi = c_1, \quad \tau = c_2t + c_3, \quad \phi = -cc_2, \tag{62}$$

where $c_i, i = 1,2,3$ are arbitrary constants.

Theorem 3.3.1. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:

$$v_1 = t\partial_t - c\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = t\partial_t - c\partial_u.$$

We have

$$W = -c - tu_t. \tag{63}$$

The right-hand side of (4) is written

$$C^1 = W(v - v_{xx}) + (D_x(W))[v_x] - D_x^2(W)[v], \tag{64}$$

$$C^2 = W[vf - 2v_{xt}] + D_x(W)[v_x] + D_t(W)[v_x] - 2D_x D_t(W)[v]. \tag{65}$$

Substituting in (64) and (65), the expression (7) for L and (63) for W , we obtain

$$C^1 = -cv + cv_{xx} - tvu_t + tv_{xx}u_t - tv_xu_{tx} + tvu_{txx}, \tag{66}$$

$$C^2 = -cvf - u_t v_x + 2cv_{xt} - tu_{xt}v_t - tv_xu_{x^2} - tvfu_t + 2vu_{tx} + 2tu_{xt^2}v + 2tu_xv_{xt}. \tag{67}$$

We can eliminate u_t by using Eq. (1) and obtain

$$C^1 = -cv + cv_{xx} + tvfu_x + tv_{xx}u_{x^2} - tv_{xx}u_x - tv_xu_{tx}, \tag{68}$$

$$C^2 = -u_{x^2}v_x + fu_xv_x + 2cv_{xt} - tu_{xt}v_t - tv_xu_{x^2} - tvfu_{x^2} + tvf^2u_x + 2vu_{tx} + 2tu_{xt^2}v - cvf + 2tv_{xt}u_{xxt} - 2tvfu_xv_{xt}. \tag{69}$$

Now, we substitute in (68) and (69) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$C^1 = -cu + cu_{x^2} + tvfu_x + tu_{x^2}u_{x^2} - tvfu_{x^2}u_x - tv_xu_{tx}, \tag{70}$$

$$C^2 = -cuf - u_{x^2}u_x + fu_x^2 + 2cu_{xt} - tu_{xt}u_t - tu_xu_{x^2} + 2tu_{xt}u_{xxt} - 2tvu_xu_{xt} - tvfu_{x^2} + tvf^2u_x + 2vu_{tx} + 2tu_{xt^2}u, \tag{71}$$

where $f = Le^{u/c}$.

Case 3.3.ii). By considering Eq. (49) – (54), we find that the components ξ , τ and ϕ are $\xi = \xi(x)$, $\tau = \tau(t)$ and $\phi = A(x)u + B(x,t)$. By considering Eq. (55) and (56) we have

$$\xi = c_1 \exp 2x + c_2 \exp -2x + c_3,$$

$$A(x) = c_1 \exp 2x - c_2 \exp -2x + c_4.$$

By considering Eq. (57) we have

$$\tau = ft^2(2c_1 \exp 2x + 2c_2 \exp -2x) + c_5t + c_6,$$

where $c_i, i = 1..6$ are arbitrary constants.

From the following identity:

$$A(x)u + B(x,t) = \frac{-f}{f_u}(\tau_t + \xi_x),$$

we find that $c_1 = c_2 = 0$ and $\phi = -(f/f_u)c_5$. Hence we have two particular cases:

$$\frac{f}{f_u} = Ku, \quad \frac{f}{f_u} \neq Ku = g(u),$$

where K is an arbitrary nonzero constant. For the first case, we have

$$\xi = c_3, \quad \tau = c_5t + c_6, \quad \phi = -Kuc_5,$$

and for the second case, we have

$$\xi = c_3, \quad \tau = c_6, \quad \phi = 0.$$

Theorem 3.2. Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when $f/f_u = Ku$ are

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_t - u\partial_u,$$

and when $f/f_u \neq Ku = g(u)$ are

$$v_1 = \partial_x, \quad v_2 = \partial_t,$$

where K is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry $v = t\partial_t - u\partial_u$, we find that $W = -u - tu_x$. Therefore, we have the conserved vector with the following components:

$$C^1 = -u^2 + uu_{xx} - tuu_{xxt} + tvuu_x + tu_{xx}u_{xxt}$$

$$-tvfu_xu_{xx} - u_{x^2} - tu_xu_{xt} + uu_{xx} + tuu_{xxt},$$

$$C^2 = -u^2f - tvfu_{xxt} + tvf^2u_x + 2uu_{xt} + 2tu_{xxt}u_{xt} - 2tvu_{xt}u_x$$

$$-u_xu_t - tu_{xt}u_t + 4u_{xt}u + 2tu_{tx}u - 2u_{xt}u_x + 2fu_x^2 - 2u_{tt}u_x,$$

where $f/f_u \neq Ku = g(u)$.

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