Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

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Abstract

In this paper, we prove that equation \( E = u_t - u_{x} + u_{x} + f(u) - auu_{x} - buu_{x} = 0 \) is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where \( f \) is smooth function on \( u \) and \( a, b \) are arbitrary constants. We find Three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitham equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

\[ E = u_t - u_{x} + u_{x} + f(u) - auu_{x} - buu_{x} = 0, \]  
and we show that this equation is self-adjoint and quasi self-adjoint. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) \( b = 0, a = \) arbitrary constant, 2) \( b = 0, a = 0 \), and 3) \( b = a = 0 \). Clarkson, Mansfield and Priestly [3] are concerned with the problem of determining symmetries of the non-linear third order partial differential equation given by

\[ u_t - u_{x} + (k + 3)u_{x} = auu_{x} + buu_{x} = 0, \]  
where \( a, k \), and \( \beta \) are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of (1) are:

Camassa-Holm (CH) equation \( u_t - u_{x} + (k + 3)u_{x} = auu_{x} + buu_{x} \), \( k \)-arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let \( k = 3, a = 2, b = 1 \) in (1)).

Degas peris-Procesi (DP) equation \( u_t - u_{x} + (k + 4)u_{x} = auu_{x} + buu_{x} \), \( k \)-arbitrary (real), is another equation of this class (let \( f = k + 4, a = 3, b = 1 \) in (1)).

Fornberg Whitham (FW) equation \( u_t - u_{x} + (1 + u)_{x} = auu_{x} + buu_{x} \), \( a \) and \( b \) are arbitrary constans.

BBM equation \( u_t - u_{x} + u_{x} + (uu_{x}) = 0 \), is another equation of this class (let \( f = 1 + u, a = 0, b = 0 \) in (1)).

Preliminaries

In this section, we recall the procedure to literature of Ibragimov [1]. Let us introduce the formal Lagrangian

\[ L = vE, \]  

where \( v = v(t, x) \) is a new dependent variable.

We define the adjoint equation by \( E^* = \frac{\delta L}{\delta u} = 0 \). Here

\[ \frac{\delta}{\delta u} = \frac{\partial}{\partial u} + D_x \frac{\partial}{\partial u_x} + D_{xx} \frac{\partial}{\partial u_{xx}} - D_{xxx} \frac{\partial}{\partial u_{xxx}} + \cdots i, j, k = 1, 2, \]  

is the variational derivative and \( D_x \) is the operator of total differentiation.

An equation \( E = 0 \) is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution \( v = u \) is identical with the original equation.

An equation \( E = 0 \) is said to be quasi- self-adjoint [5] if there exists a function \( v = \phi(u) \), \( \phi(u) \neq 0 \) such that \( E^* \) has an undetermined coefficient \( \lambda \). Eq.(1) is said to have a nonlocal conservation law if there exits a vector \( C = (C^i, C^j) \) satisfying the equation

\[ D^i(C^i) + D_j(C^j) = 0, \]  

on any solution of the system of differential equations comprising (E) and the adjoint equation (\( E^* \)). We say that orginal equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry \( v = \xi(x, t, u)\partial_x + \eta(x, t, u)\partial_u \) is obtained by the following formula:

\[ C^i = \xi^i + W^i_j \frac{\partial \xi^j}{\partial u_{i}} \frac{\partial}{\partial u} - D_{j} \frac{\partial \psi}{\partial u_{j}} - D_{ij} \frac{\partial \psi}{\partial u_{ij}} \]  

+ \frac{\partial D_j(W)\partial \psi}{\partial u_{ij}} + D_{ij} \frac{\partial \psi}{\partial u_{ij}}, \]  

where \( i, j, k = 2, 1 \) and \( W = \psi - \phi \). (Here \( \partial_x \) means \( \frac{\delta}{\delta x} \).)

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order \( n \), containing \( p \) independent and \( q \) dependent variables is given as follows

\[ \Delta_p(x, u^{(m)}) = 0, \quad v = 1, \ldots, l, \]  

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involving \( x = (x^1, \ldots, x^r) \), \( u = (u^1, \ldots, u^n) \) and the derivatives of \( u \) with respect to \( x \) up to \( n \), where \( u^{(n)} \) represents all the derivatives of \( u \) of all orders from 0 to \( n \). We consider a one-parameter Lie group of transformations acting on the variables of system (5):

\[
\pi^i = x^i + \xi^i(x, u) + O(\varepsilon^2), \quad \pi^a = u^a + \phi^a(x, u) + O(\varepsilon^2),
\]

where \( i = 1, \ldots, r, \) \( a = 1, \ldots, n \), \( \xi^i \) and \( \phi^a \) are the infinitesimals of the transformations for the independent and dependent variables, respectively, and \( \varepsilon \) is the parameter of the transformation. We consider the general vector field \( v \) as the infinitesimal generator associated with the above group

\[
v = \sum_i \xi_i^i(x, u) \partial_i + \sum_a \phi^a(x, u) \partial_a.
\]

where \( v \) is called the \( n \)th order prolongation of the infinitesimal generator given by \( v^\varepsilon = v + \sum_k \sum_i [\phi^i(x, u^\varepsilon)] \partial_{\varepsilon^k} \partial_i \), where \( k = (i_1, \ldots, i_k) \), \( 1 \leq i_k \leq p \), \( 1 \leq a \leq n \), and the sum is over all \( k \)’s of order \( 0 < k \leq n \). If \( k = a \), the coefficient \( \phi_a \) of \( \partial_\varepsilon^a \) will depend only on \( \varepsilon \)th and lower order derivatives of \( u \) and \( \phi^1(x, u^\varepsilon) = D_1(\phi) - \sum_k [\xi^1(x, u^\varepsilon)] + \sum_k [\partial_{\varepsilon^a}], \) where \( \varepsilon \) is the normal coordinate for \( x \)

where \( \varepsilon \leq 1 \) the group parameter and \( \xi^1 = \xi^1, \xi^2 = \varepsilon \) and \( \phi^1 = \phi \) are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form

\[
\phi^1(x, u^\varepsilon) + \phi^2(x, u^\varepsilon) + \phi^3(x, u^\varepsilon) + \phi^4(x, u^\varepsilon) \partial_\varepsilon^2 + \phi^5(x, u^\varepsilon) \partial_\varepsilon^3 + \phi^6(x, u^\varepsilon) \partial_\varepsilon^4 + \phi^7(x, u^\varepsilon) \partial_\varepsilon^5 + \phi^8(x, u^\varepsilon) \partial_\varepsilon^6,
\]

with coefficient

\[
\phi^i = D_1(\phi) - \sum_k [\xi^1(x, u^\varepsilon)] + \sum_k [\partial_{\varepsilon^a}],
\]

(11)

where \( D_1 \) is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

\[
v^\varepsilon([u, -u_{\varepsilon^1} + af - au_{\varepsilon^2} - bu_{\varepsilon^3}]) = 0,
\]

(12)

whenever \( E = 0 \). The condition (12) is equivalent to

\[
\phi_{\varepsilon^1} - bu_{\varepsilon^3} + (f - au_{\varepsilon^2}) - bu_{\varepsilon^3} - bu_{\varepsilon^3} = 0,
\]

(13)

whenever \( E = 0 \). Substituting (11) into (13), yields the determining equations. There are three cases to consider:

\[ a \text{ and } b \neq 0 \text{ are arbitrary constants} \]

In this case, complete set of determining equation is:

\[
q_2 = 0,
\]

(14)

\[ r_1 = 0, \]

(15)

\[ \phi_2 = 0, \]

(16)

\[ \phi_{\varepsilon^2} + \phi_{\varepsilon^3} + 3\phi_{\varepsilon^4} = 0, \]

(17)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(18)

\[ 3\phi_{\varepsilon^4} + 3\phi_{\varepsilon^5} - 2\phi_{\varepsilon^6} = 0, \]

(19)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(20)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(21)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(22)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(23)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(24)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0, \]

(25)

\[ \phi_{\varepsilon^3} + 2\phi_{\varepsilon^4} = 0. \]

(26)

With substituting (26) into (24) – (25) we have

\[ f = -1 + K(bu_1 + 1), \]

(27)

where \( c_1, c_2 \) and \( K \) are arbitrary constants. With substituting (27) into determining system, we have

\[ \phi = \frac{-c_1(bu_1 + 1)}{b}, \]

(28)

\[ \tau = c_1 + c_2, \]

(29)

\[ \tau = c_1 + c_2. \]

(30)

where \( c_1, c_2 \) and \( K \) are arbitrary constants. With substituting (27) into determining system, we have

\[ \phi = -c_1(bu_1 + 1), \]

(31)

\[ \tau = c_1 + c_2, \]

(32)

\[ \tau = c_1 + c_2. \]

(33)

Theorem 3.1.1. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are

\[
v_1 = -t \partial_1 + \xi^{(1)}(t), \quad \xi^{(2)} = \xi^{(1)}(t), \quad \xi^{(3)} = \alpha(t).
\]

(34)

With substituting (34) into (33) we have

\[
v_1 = \frac{-t}{b} \partial_1 + \frac{(bu_1 + 1)}{b} \partial_2.
\]

(35)

We want to construct the conservation law associated with the symmetry

\[
v_1 = -t \partial_1 + t \partial_2.
\]

(36)
We have
\[ W = u - \frac{1}{b} tu + tu'. \]
Thus, the expression for the conserved vector with\( a = 0 \) becomes
\[ \xi_v^0 = 0, \quad r_v^0 = 0, \quad \tau_v^0 = 0, \quad f_u^0 + \beta = 0. \]

Now, by considering Eq. (33) – (42) it is not hard to find that the components \( \xi, \tau, b \) of infinitesimal generators become
\[ \phi = \frac{dF(t)}{dt} = \frac{d^2F(t)}{dt^2} + F(t), \quad \tau = -F(t) + c_2, \quad \xi = c_1. \]

To find complete solution of the above system, we consider Eq. (43) and \( l = \dim \text{Span}_a(f_u^0 + \beta^1) \). Three general cases are possible:

3.2.i) \( l = 1 \), then \( f = \alpha \) constant;
3.2.ii) \( l = 2 \), then \( f = af + \beta \);
3.2.iii) \( l = 3 \), then \( af^b + \beta f + \gamma = 0, \alpha \neq 0 \).

Case 3.2.i). With substituting \( f = \alpha \) constant in determining system (33)–(44), we have \( \varphi = c_1, \tau = c_2, \xi = c_3 \), where \( c_1, c_2, c_3 \) are arbitrary constants.

Theorem 3.2.1. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:
\[ v_1 = \partial_t, \quad v_2 = \partial_u, \quad v_3 = \partial_v. \]

Case 3.2.ii). With integrating from \( f = af + \beta \) with respect to \( u \), we obtain
\[ f = -\frac{\beta}{\alpha} + Ce^{\alpha u}, \]
where \( C \) is an integrating constant. With substituting (46) into Eq. (43)–(44) and Eq. (45), we have
\[ \xi = c_1, \quad \tau = -c_2, \quad \phi = \frac{c_3(Ca - e^{\alpha u} \beta)}{Ca^2}. \]

Theorem 3.2.2. Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:
\[ v = \partial_t - t\partial_u + \frac{Ca - e^{\alpha u} \beta}{Ca^2} \partial_v. \]

Case 3.2.iii). The Eq. (43) leads to \( \varphi = 0, \tau = c_2, \xi = c_3 \).

Theorem 3.2.3. Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:
\[ v_1 = \partial_t, \quad v_2 = \partial_u, \quad v_3 = \partial_v. \]
\[ b = 0, \quad a = 0. \]

Complete set of determining equation is
\[ \xi_v^1 = 0, \quad \tau_v^1 = 0, \quad \xi_v^2 = 0. \]
We find that \( \xi \), \( \tau \) and \( \phi \) are \( \xi = \xi(x), \tau = \tau(x) \) and \( \phi = \phi(x) \). Hence we have two cases possible:

\[
C^2 = -u_x v_x + f u_x v_x + 2u v_x - tu_x v_x - tv_x u_x - 2fu_x v_x + tfu^2 x, \tag{50}
\]

Now, we substitute in (68) and (70) the expression \( v = u \), therefore arrive at the conserved vector with the following components:

\[
C^2 = -cu + cu_x + fu_x v_x + tu_x u_x - tfu_x v_x - tu_x u_x, \tag{69}
\]

\[
C^2 = -cu - u_x u_x + fu_x v_x + 2tu_x u_x - tu_x u_x - 2fu_x v_x + 2tu_x u_x, \tag{70}
\]

where \( f = Le^{ux} \).

**Case 3.3.ii)** By considering Eq. (49) – (54), we find that the components \( \xi, \tau \) and \( \phi \) are \( \xi = \xi(x), \tau = \tau(x) \) and \( \phi = \phi(x) \). By considering Eq. (55) and (56) we have

\[
\xi = c_0 \exp 2x + c_1 \exp -2x + c_1, \tag{71}
\]

where \( c_i, i = 1, 2, 3 \) are arbitrary constants.

From the following identity:

\[
A(x)u + B(x,t) = \int \frac{f}{L}L = Ku, \tag{72}
\]

we find that \( \xi = c_0 = 0 \) and \( \phi = -(fL)_x \). Hence we have two particular cases:

\[
\frac{f}{L} = Ku, \quad \frac{f}{L} = Ku = g(u), \tag{73}
\]

where \( K \) is an arbitrary nonzero constant. For the first case, we have

\[
\xi = c_0, \quad \tau = c_0 + c_1, \quad \phi = -Ku_x, \tag{74}
\]

and for the second case, we have

\[
\xi = c_0, \quad \tau = c_0 + c_1, \quad \phi = 0. \tag{75}
\]

**Theorem 3.2.** Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when \( \frac{f}{L} = Ku \) are

\[
v_1 = \partial_x, \quad v_2 = \partial_y, \quad v_3 = -(u_3 + au_1), \tag{76}
\]

and when \( \frac{f}{L} = Ku = g(u) \) are

\[
v_1 = \partial_x, \quad v_2 = \partial_y, \quad v_3 = -(u_3 + au_1), \tag{77}
\]

where \( K \) is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry \( v = \partial_x, -au_1 \), we find that \( W = -u \). Therefore, we have the conserved vector with the following components:

\[
C^1 = u^2 + au_x - tu_x u_x + tu_x u_x - fu_x u_x + fu_x u_x - fu_x u_x, \tag{78}
\]

\[
C^1 = -u^2 + au_x - tu_x u_x + tu_x u_x - fu_x u_x + fu_x u_x - fu_x u_x, \tag{79}
\]

\[
C^1 = -u^2 - fu_x u_x + fu_x u_x - fu_x u_x - fu_x u_x + fu_x u_x - fu_x u_x, \tag{80}
\]

where \( f = Le^{ux} \).

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