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Self-adjointness, Group Classification and Conservation Laws of an Extended Camassa-Holm Equation

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Abstract

In this paper, we prove that equation $E \equiv u_t - u_{x^2 t} + u_x f(u) - a u_x u_{x^2} - b u u_{x^3} = 0$ is self-adjoint and quasi self-adjoint, then we construct conservation laws for this equation using its symmetries. We investigate a symmetry classification of this nonlinear third order partial differential equation, where *f* is smooth function on *u* and *a*, *b* are arbitrary constans. We find Three special cases of this equation, using the Lie group method.

Keywords: Lie symmetry analysis; Self-adjoint; Quasi self-adjoint; Conservation laws; Camassa-Holm equation; Degas peris-Procesi equation; Fornberg whitham equation; BBM equation

Introduction

A new procedure for constructing conservation laws was developed by Ibragimov [1]. For Camassa-Holm equation are calculated in studies of Ibragimov, Khamitova and Valenti [2]. In this paper, we study the following third-order nonlinear equation

$$E \equiv u_t - u_{x^2 t} + u_x f - a u_x u_{x^2} - b u u_{x^3} = 0, \qquad (1)$$

and we show that this equation is self-adjoint and quasi self-adjont. Therefore we find Lie symmetries and conservation laws. There are three cases to consider: 1) $b \neq 0$, a = arbitrary constant, 2) $b = 0$, $a \neq 0$, and 3) $b = 0$, $a = 0$. Clarkson, Mansfield and Priestly [3] are concerned with symmetry reductions of the non-linear third order partial differential equation given by $u_t - \epsilon u_{x^2 t} + (k-u)u_x - u u_{x^3} - \beta u_x u_{x^2} = 0$, where ϵ, k, and β are arbitrary constants. Symmetry classification and conservation laws for higher order Camassa-Holm equation are calculated in framework of Nadjafikhah and Shirvani-Sh [4].

The special cases of (1) are:

Camassa-Holm (CH) equation $u_t - u_{x^2 t} + (k+3u)u_x = u u_{x^3} + 2u_x u_{x^2}$, k-arbitrary (real), describing the unidirectional propagation of shallow water waves over a flat bottom (let $f = k + 3u$, $a = 2$, $b = 1$ in (1).

Degas peris-Procesi (DP) equation $u_t - u_{x^2 t} + (k+4u)u_x = u u_{x^3} + 3u_x u_{x^2}$, k-arbitrary (real), is another equation of this class (let $f = k + 4u$, $a = 3$, $b = 1$ in (1).

Fornberg Whitham (FW) equation $u_t - u_{x^2 t} + (1+u)u_x = u u_{x^3} + 3u_x u_{x^2}$, is another equation of this class (let $f = 1 + u$, $a = 3$, $b = 1$ in (1)).

BBM equation $u_t - u_{x^2 t} + u_x + (u u_x) = 0$, is another equation of this class (let $f = 1 + u$, $a = 0$, $b = 0$ in (1)).

Preliminaries

In this section, we recall the procedure in literature of Ibragimov [1]. Let us introduce the formal Lagrangian

$$L \equiv vE, \qquad (2)$$

where $v = v(t, x)$ is a new dependent variable.

We define the adjoint equation by $E^* \equiv \dfrac{\delta L}{\delta u} = 0$. Here

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} - D_i \frac{\partial}{\partial u_i} + D_i D_j \frac{\partial}{\partial u_{ij}} - D_i D_j D_k \frac{\partial}{\partial u_{ijk}} + \cdots \qquad i, j, k = 1, 2,$$

is the variational derivative and D_i is the operator of total diferentiation.

An equation $E = 0$ is said to be self-adjoint [5] if the equation obtained from the adjoint equation by substitution $v = u$ is identical with the original equation.

An equation $E = 0$ is said to be quasi- self-adjoint [5] if there exists a function $v = \varphi(u)$, $\varphi'(u) \neq 0$ such that $E^*|_{v=\varphi(u)} = \lambda E$ with an undetermined coefficient λ. Eq.(1) is said to have a nonlocal conservation law if there exits a vector $C = (C^1, C^2)$ satisfying the equation

$$D_t(C^1) + D_x(C^2) = 0, \qquad (3)$$

on any solution of the system of differential equations comprising (E) and the adjoint equation (E^*). We say that orginal equation has a local conservation law if (3) is satisfied on any solution of Eq.(1). In studies of Ibragimov [1], the conserved vector associated with the Lie point symmetry $v = \xi^1(x,t,u)\partial_x + \xi^2(x,t,u)\partial_t + \phi(x,t,u)\partial_u$ is obtained by the following formula :

$$C^i = \xi^i L + W\Big[\frac{\partial L}{\partial u_i} - D_j\Big(\frac{\partial L}{\partial u_{ij}}\Big) + D_j D_k\Big(\frac{\partial L}{\partial u_{ijk}}\Big)\Big]$$

$$+ D_j(W)\Big[\frac{\partial L}{\partial u_{ij}} - D_k\Big(\frac{\partial L}{\partial u_{ijk}}\Big)\Big] + D_j D_k(W)\frac{\partial L}{\partial u_{ijk}}, \qquad (4)$$

where $i, j, k = 1{,}2$ and $W = \phi - \xi^i u_i$. (Here ∂_x means $\dfrac{\partial}{\partial x}$).

We recall the general procedure for determining symmetries for an arbitrary system of partial differential equations [6]. Let us consider the general system of a nonlinear system of partial differential equations of order n, containing p independent and q dependent variables is given as follows

$$\Delta_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, l, \qquad (5)$$

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involving $x=(x^1,\cdots,x^p)$, $u=(u^1,\cdots,u^q)$ and the derivatives of u with respect to x up to n, where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n. We consider a one-parameter Lie group of transformations acting on the variables of system (5): $\overline{x}_i = x^i + \in \xi^i(x,u) + O(\in^2)$, $\overline{u}_j = u^j + \in \phi^j(x,u) + O(\in^2)$, where $i=1,\cdots, p$, $j = 1,\cdots, q$. ξ^i, φ^j are the infinitesimal of the transformations for the independent and dependent variables, respectively, and $\in$ is the transformation parameter. We consider the general vector field v as the infinitesimal generator associated with the above group $v=\sum_{i=1}^p \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^q \phi^j(x,u)\partial_{u^j}$. A symmetry of a differential equation is a transformation, which maps solutions of the equation to other solutions. The invariance of the system (5) under the infinitesimal transformation leads to the invariance conditions. (Theorem 2.36 of studies of Olver [6], Theorem 6.5 of literature of Olver [7]).

$$v^n[\Delta_\nu(x,u^n)]=0, \quad \Delta_\nu(x,u^n)=0, \quad \nu=1,\cdots,r, \tag{6}$$

where v^n is called the n^{th} order prolongation of the infinitesimal generator given by $v^n = v + \sum_{j=1}^q \sum_k \phi_k^j(x,u^{(n)})\partial_{u_k^j}$, where $k = (i_1,\cdots i_\alpha)$, $1 \le i_\alpha \le p$, $1 \le \alpha \le n$, and the sum is over all k's of order $0 < \#k \le n$. If $\#k = \alpha$, the coeficent ϕ_k^j of $\partial_{u_k^j}$, will depend only on α'th and lower order derivatives of u and $\phi_j^k(x,u^n) = D_k(\phi_j - \sum_{i=1}^p(\xi^i u_i^j)) + \sum_{i=1}^p \xi^i u_{k,i}^j$, where $u_i^j := \partial u^j/\partial x^i$ and $u_{k,i}^j := \partial u_k^j/\partial x^i$.

Adjoint Equation and Classical Symmetry Method

Formal Lagrangian for Eq. (1) is

$$L = vE = v[u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - b u u_{x^3}]. \tag{7}$$

Therefore, the adjoint equation E^* to Eq. (1) is

$$f v_x + v_t + a u_{x^2} v_x + a u_x v_{xx} = 3b u_{x^2} v_x + 3b u_x v_{xx} + b u v_{xxx} + v_{xxt}. \tag{8}$$

Upon setting $v = u$ it becomes

$$u_t = u_{x^2t} - u_x f - 2a u_x u_{x^2} + 6b u_x u_{x^2} + b u u_{x^3}.$$

Hence, Eq. (1) is self-adjoint if and only if it has the form

$$a = 2b. \tag{9}$$

Consider again Eq. (1), and substitute

$$v=\varphi(u), \qquad v_t = \varphi' u_t,$$

$$v_x = \varphi' u_x, \qquad v_{xx} = \varphi' u_{x^2} + \varphi'' u_x^2,$$

$$v_{xxx} = \varphi' u_{x^3} + 3\varphi'' u_x u_{x^2} + \varphi''' u_x^3,$$

$$v_{xxt} = \varphi' u_{x^2t} + \varphi'' u_t u_{x^2} + \varphi''' u_x^2 u_t + 2\varphi'' u_x u_{xt},$$

in the adjoint equation (8), then

$$\begin{aligned}&-f\varphi' u_x - \varphi' u_t - 2a\varphi' u_x u_{x^2} - a\varphi'' u_x^3 + 6b\varphi' u_x u_{x^2}\\ &+3b\varphi'' u_x^3 + b\varphi' u u_{x^3} + 3b\varphi'' u u_x u_{x^2} + b\varphi''' u u_x^3\\ &+\varphi' u_{x^2t} + \varphi'' u_{x^2} u_t + \varphi''' u_x^2 u_t + 2\varphi'' u_x u_{xt}\\ &= \lambda(u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - b u u_{x^3}).\end{aligned}$$

Hence, Eq. (1) is quasi self-adjoint if and only if it has the form

$$a = 2b,\ v = -\lambda u + \varepsilon \tag{10}$$

In this section, we will perfom Lie group method for Eq. (1) on ($x^1 = x$, $x^2 = t$, $u^1 = u$), $(\tilde{x},\tilde{t},\tilde{u}) = (x,t,u) + \in(\xi(x,t,u),\tau(x,t,u),\phi(x,t,u)) + O(\in^2)$, where $\varepsilon \le 1$ the group parameter and $\xi^1 = \xi$, $\xi^2 = \tau$ and $\phi^1 = \phi$ are the infinitesimals of the transformations for the independent and dependent variables respectively. The associated vector fields is of the form $v = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \phi(x,t,u)\partial_u$ and the third porolongation of v is the vector field

$$v^{(3)} = v + \phi^x \partial_{u_x} + \phi^t \partial_{u_t} + \phi^{x^2}\partial_{u_{x^2}} + \phi^{xt}\partial u_{xt} + \cdots + \phi^{ttt}\partial u_{ttt},$$

with coefficent

$$\phi^k = D_k(\phi - \sum_{i=1}^2 \xi^i u_i^j) + \sum_{i=1}^2 \xi^i u_{k,i}, \tag{11}$$

where D_k is the total derivative with respect to independent variables. The invariance condition (6) for Eq. (1) is given by,

$$v^{(3)}[u_t - u_{x^2t} + u_x f - a u_x u_{x^2} - a u u_{x^3}] = 0, \tag{12}$$

whenever $E = 0$. The condition (12) is equvalent to

$$\phi^t - b u_{x^3}\phi + (f - a u_{x^2})\phi^x - a u_x \phi^{x^2} - b u \phi^{x^3} - \phi^{x^2t} = 0, \tag{13}$$

whenever $E = 0$. Substituting (11) into (13), yields the determining equations. There are three cases to consider:

a and $b \neq 0$ are arbitrary constants

In this case, complete set of determining equation is:

$$\xi_u = 0, \tag{14}$$

$$\tau_u = 0, \tag{15}$$

$$\tau_x = 0, \tag{16}$$

$$\phi_{u^2} = 0, \tag{17}$$

$$a\phi_{ux^2} + a\phi_u + a\tau_t - 3a\xi_x = 0, \tag{18}$$

$$\xi_{x^2} - 2\phi_{ux} = 0, \tag{19}$$

$$3bu\phi_{ux} + a\phi_x - 3b\xi_{x^2}u + 2\xi_{xt} + \phi_{ut} = 0, \tag{20}$$

$$2\xi_x - \phi_{ux^2} = 0, \tag{21}$$

$$\xi_t - b\phi_x - \phi_{ux^2} - bu\tau_t + 3bu\xi_x = 0, \tag{22}$$

$$a\xi_{x^2}u - 2a\phi_{ux} = 0, \tag{23}$$

$$\xi_{x^2t} + bu\xi_{x^3} + \phi f_u + \phi_{ux^2} f + f\xi_x + \tau_t f = a\phi_{x^2} + \xi_t + 2\phi_{xtu} + 3bu\phi_{ux^2}, \tag{24}$$

$$-bu\phi_{x^3} + \phi_t + \phi_x f - \phi_{x^2t} = 0. \tag{25}$$

With substituting (14) – (17) into (18) – (23) we have

$$\phi = c_1 + \frac{1}{b}\alpha'(t), \quad \tau = -c_1 t + c_2, \quad \xi = \alpha(t). \tag{26}$$

With substituting (26) into (24) – (25) we have

$$f = -1 + K(bu+1), \tag{27}$$

where c_1, c_2 and K are arbitrary constants. With substituting (27) into determining system, we have

$$\phi = \frac{-c_1(bu+1)}{b}, \quad \tau = c_1 t + c_2, \quad \xi = -c_1 t + c_3,$$

where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.1.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu+1)}{b}\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = -t\partial_x + t\partial_t - \frac{(bu+1)}{b}\partial_u.$$

We have

$$W=-u-\frac{1}{b}-tu_t+tu_x. \quad (28)$$

The right-hand side of (4) is written

$$C^1=W(v-D_x^2(v))+(D_xW)(D_xv)-D_x^2(W)v,$$
$$C^2=W[vf-avu_{x^2}+D_x(avu_x)-D_x^2(buv)-2D_tD_x(v)] \quad (29)$$
$$+D_x(W)[-avu_x+D_x(buv)+D_t(v)]+D_t(W)[D_x(v)]$$
$$-2D_xD_t(W)[v]-D_x^2(W)[buv].$$

We eliminate the term $\xi^i L$ since the Lagrangian L is equal to zero on solution of Eq.(1). Substituting in (29), the expression (7) for L and (28) for W, we obtain

$$C^1=-uv-\frac{1}{b}v-tu_tv+tu_xv+uv_{xx}+\frac{1}{b}v_{xx}+tu_tv_{xx}-tu_xv_{xx}$$
$$-u_xv_x-tu_{xt}v_x+tv_xu_{x^2}+u_{x^2}v+tu_{x^2t}v-tu_{x^3}v, \quad (30)$$

and

$$C^2=-u(vf-bvu_{x^2}-buv_{xx}-2v_{xt})$$
$$-(b^{-1}(vf-bvu_{x^2}-buv_{xx}-2v_{xt}))$$
$$-tu_t(vf-bvu_{x^2}-buv_{xx}-2v_{xt})$$
$$+tu_x(vf-bvu_{x^2}-buv_{xx}-2v_{xt})-u_x(buv_x-bvu_x+v_t)$$
$$-(tu_{xt})(buv_x-bvu_x+v_t)+(tu_{x^2})(buv_x-bvu_x+v_t)-2u_tv_x$$
$$-tu_{t^2}v_x+u_xv_x+tu_{xt}v_x+buvu_{x^2}+tbuvu_{x^2t}-tbuvu_{x^3}$$
$$+4u_{xt}v+2tu_{xt^2}v-2u_{x^2}v-2u_{x^2t}v. \quad (31)$$

We can eliminate u_t by using Eq.(1) and then substitute in (30) and (31) the expression $v=u$, therefore arrive at the conserved vector with the following components:

$$C^1=\frac{-1}{b}(t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})ub$$
$$-t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})u_{x^2}b-tu_xub \quad (32)$$
$$+tu_{xt}u_xb+tu_{x^3}ub-tu_{x^2t}ub+u^2b-2uu_{x^2}b+u_x{}^2b+u-u_{x^2}),$$
$$C^2=-u(uf-2buu_{xx}-2u_{xt})-(b^{-1}(uf-2buu_{xx}-2u_{xt}))$$
$$-t(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})(uf-2buu_{xx}-2u_{xt})$$
$$+tu_x(uf-2buu_{xx}-2u_{xt})-u_x(u_t)-tu_{xt}(u_t)$$
$$+(tu_{x^2})(u_t)-2(2bu_xu_{x^2}+buu_{x^3}-u_xf+u_{x^2t})u_x$$
$$-tu_{t^2}u_x+u_{xt}+tu_{xt}u_x+bu^2u_{x^2}+tbu^2u_{x^2t}-tbu^2u_{x^3}+4u_{xt}u$$
$$+2tu_{xt^2}u-2uu_{x^2}-2u_{x^2t}u.$$

Where $f=-1+K(bu+1)$.

a is an arbitrary nonzero constant and *b* = 0.

In this case Eq.(1) is not self adjoint because $a\neq 2b$. Complete set of determining equation is:

$$\phi_{uu}=0, \quad (33)$$
$$\xi_u=0, \quad (34)$$
$$\xi_t=0, \quad (35)$$
$$\tau_u=0, \quad (36)$$
$$\tau_x=0, \quad (37)$$
$$3a\xi_x=a\phi_{ux^2}+a\tau_t-a\phi_u, \quad (38)$$
$$\phi_{ut}+a\phi_x=0, \quad (39)$$
$$-2\phi_{ux}+\xi_{x^2}=0, \quad (40)$$
$$2\xi_x-\phi_{ux^2}=0, \quad (41)$$
$$a\xi_{x^2}-2a\phi_{ux}=0, \quad (42)$$
$$\tau_tf+\phi f_u+\phi_{ux^2}f=2\phi_{uxt}+f\xi_x+a\phi_{x^2}, \quad (43)$$
$$\phi_t+\phi_xf=\phi_{x^2t}. \quad (44)$$

Now, by considering Eq. (33) – (42) it is not to hard to find that the components ξ, τ and φ of infinitesimal generators become

$$\phi=u\frac{dF_1(t)}{dt}-\frac{x}{a}\frac{d^2F_1(t)}{dt^2}+F_2(t),\quad \tau=-F_1(t)+c_2,\quad \xi=c_1. \quad (45)$$

To find complete solution of the above system, we consider Eq. (43) and $l=\dim \mathrm{Spam}_{\mathbb{R}}\{f_u, f, 1\}$. Three general cases are possible:

3.2.i) $l=1$, then $f=$ constant;

3.2.ii) $l=2$, then $f_u=\alpha f+\beta$;

3.3.iii) $l=3$, then $\alpha f_u+\beta f+\gamma\neq 0$, $\alpha\neq 0$.

Case 3.2.i). With substituting $f=$ constant in determining system (33)-(44), we have $\varphi=c_1$, $\tau=c_2$, $\xi=c_3$, where c_i, $i=1,2,3$ are arbitrary constants.

Theorem 3.2.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1=\partial_x,\quad v_2=\partial_t,\quad v_3=\partial_u.$$

Case 3.2.ii). With integrating from $f_u=\alpha f+\beta$ with respect to u, we obtain

$$f=\frac{-\beta}{\alpha}+Ce^{\alpha u}, \quad (46)$$

where C is an integrating constant. With substituting (46) into Eq. (43)-(44) and Eq. (45), we have

$$\xi=c_1,\qquad \tau=-c_1t,\qquad \phi=\frac{c_1(C\alpha-e^{-\alpha u}\beta)}{C\alpha^2}. \quad (47)$$

Theorem 3.2.2. *Infinitesimal generator of every one parameter Lie group of point symmetries in this case is:*

$$v=\partial_x-t\partial_t+\frac{C\alpha-e^{-\alpha u}\beta}{C\alpha^2}\partial_u. \quad (48)$$

Case 3.2.iii). The Eq. (43) leads to $\varphi=0$, $\tau=c_1$, $\xi=c_2$.

Theorem 3.2.3. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1=\partial_t,\quad v_2=\partial_x.$$

b = 0, *a* = 0.

Complete set of determining equation is

$$\xi_u=0, \quad (49)$$

$$\xi_t = 0, \quad (50)$$

$$\tau_u = 0, \quad (51)$$

$$\tau_x = 0, \quad (52)$$

$$\phi_{ut} = 0, \quad (53)$$

$$\phi_{uu} = 0, \quad (54)$$

$$\phi_{ux^2} = 2\xi_x, \quad (55)$$

$$2\phi_{ux} = \xi_{x^2}, \quad (56)$$

$$\phi_t + f\phi_x = \phi_{x^2 t}, \quad (57)$$

$$f\tau_t + f\xi_x + \phi f_u = 0. \quad (58)$$

To find a complete solution of the above system we consider Eq. (58) and with assumption $f/f_u \neq 0$ we rewrite:

$$\phi = \frac{-f}{f_u}(\tau_t + \xi_x). \quad (59)$$

Two general cases are possible:

3.3.i) $\dfrac{f}{f_u} = c$, 3.3.ii) $\dfrac{f}{f_u} = h(u)$,

where c is constant.

Case 3.3.i).

With integrating from $f/f_u \neq c$ with respect to u, we have

$$f = Le^{u/c}, \quad (60)$$

where L is an integrating constant. Then the Eq. (58) reduce to

$$\phi = -c(\tau_t + \xi_x). \quad (61)$$

With substituting Eq. (61) into determining equation, we have

$$\xi = c_1, \quad \tau = c_2 t + c_3, \quad \phi = -cc_2, \quad (62)$$

where c_i, $i = 1,2,3$ are arbitrary constants.

Theorem 3.3.1. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case are:*

$$v_1 = t\partial_t - c\partial_u, \quad v_2 = \partial_t, \quad v_3 = \partial_x.$$

We want to construct the conservation law associated with the symmetry

$$v_1 = t\partial_t - c\partial_u.$$

We have

$$W = -c - tu_t. \quad (63)$$

The right-hand side of (4) is written

$$C^1 = W(v - v_{xx}) + (D_x(W))[v_x] - D_x^2(W)[v], \quad (64)$$

$$C^2 = W[vf - 2v_{xt}] + D_x(W)[v_t] + D_t(W)[v_x] - 2D_xD_t(W)[v]. \quad (65)$$

Substituting in (64) and (65), the expression (7) for L and (63) for W, we obtain

$$C^1 = -cv + cv_{xx} - tvu_t + tv_{xx}u_t - tv_xu_{tx} + tvu_{txx}, \quad (66)$$

$$C^2 = -cvf - u_tv_x + 2cv_{xt} - tu_{xt}v_t$$
$$-tv_xu_{t^2} - tvfu_t + 2vu_{tx} + 2tu_{xt^2}v + 2tu_tv_{xt}. \quad (67)$$

We can eliminate u_t by using Eq. (1) and obtain

$$C^1 = -cv + cv_{xx} + tvfu_x + tv_{xx}u_{x^2t} - tfv_{xx}u_x - tv_xu_{tx}, \quad (68)$$

$$C^2 = -u_{x^2t}v_x + fu_xv_x + 2cv_{xt} - tu_{xt}v_t - tv_xu_{t^2} - tvfu_{x^2t} + tvf^2u_x$$
$$+2vu_{tx} + 2tu_{xt^2}v - cvf + 2tv_{xt}u_{xxt} - 2tfu_xv_{xt}. \quad (69)$$

Now, we substitute in (68) and (??) the expression $v = u$, therefore arrive at the conserved vector with the following components:

$$C^1 = -cu + cu_{x^2} + tufu_x + tu_{x^2}u_{x^2t} - tfu_{x^2}u_x - tu_xu_{tx}, \quad (70)$$

$$C^2 = -cuf - u_{x^2t}u_x + fu_x^2 + 2cu_{xt} - tu_{xt}u_t - tu_xu_{t^2} + 2tu_{xt}u_{xxt}$$
$$-2ftu_xu_{xt} - tufu_{x^2t} + tuf^2u_x + 2uu_{tx} + 2tu_{xt^2}u, \quad (71)$$

where $f = Le^{u/c}$.

Case 3.3.ii). By considering Eq. (49) – (54), we find that the components ξ, τ and φ are $\xi = \xi(x)$, $\tau = \tau(t)$ and $\phi = A(x)u + B(x,t)$. By considering Eq. (55) and (56) we have

$$\xi = c_1 \exp 2x + c_2 \exp -2x + c_3,$$

$$A(x) = c_1 \exp 2x - c_2 \exp -2x + c_4.$$

By considering Eq. (57) we have

$$\tau = ft^2(2c_1 \exp 2x + 2c_2 \exp -2x) + c_5t + c_6,$$

where c_i, $i = 1..6$ are arbitrary constants.

From the following identity:

$$A(x)u + B(x,t) = \frac{-f}{f_u}(\tau_t + \xi_x),$$

we find that $c_1 = c_2 = 0$ and $\phi = -(f/f_u)c_5$. Hence we have two particular cases:

$$\frac{f}{f_u} = Ku, \qquad \frac{f}{f_u} \neq Ku = g(u),$$

where K is an arbitrary nonzero constant. For the first case, we have

$$\xi = c_3, \quad \tau = c_5t + c_6, \quad \phi = -Kuc_5,$$

and for the second case, we have

$$\xi = c_3, \quad \tau = c_6, \quad \phi = 0.$$

Theorem 3.2. *Infinitesimal generators of every one parameter Lie group of point symmetries in this case, when* $f/f_u = Ku$ *are*

$$v_1 = \partial_x, \quad v_2 = \partial_t, \quad v_3 = t\partial_t - u\partial_u,$$

and when $f/f_u \neq Ku = g(u)$ are

$$v_1 = \partial_x, \quad v_2 = \partial_t,$$

where K is an arbitrary nonzero constant.

To construct the conservation law associated with the symmetry $v = t\partial_t - u\partial_u$, we find that $W = -u - tu_t$. Therefore, we have the conserved vector with the following components:

$$C^1 = -u^2 + uu_{xx} - tuu_{xxt} + tfuu_x + tu_{xx}u_{xxt}$$
$$-tfu_xu_{xx} - u_{x^2} - tu_xu_{xt} + uu_{xx} + tuu_{xxt},$$

$$C^2 = -u^2f - tufu_{xxt} + tuf^2u_x + 2uu_{xt} + 2tu_{xxt}u_{xt} - 2ftu_{xt}u_x$$
$$-u_xu_t - tu_{xt}u_t + 4u_{xt}u + 2tu_{ttx}u - 2u_{xxt}u_x + 2fu_x^2 - 2u_{tt}u_x,$$

where $f/f_u \neq Ku = g(u)$.

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References

1. Ibragimov NH (2007) A new conservation theorem. J Math Anal Appl 333: 311-328.
2. Ibragimov NH, Khamitova RS, Valenti A (2011) Self-adjointness of generalized Camassa-Holm equation. J Applied Mathematics and Computation 218: 2579-2583.
3. Clarkson PA, Mansfield EL, Priestley TJ (1997) Symmetries of a Class of Nonlinear Third Order Partial Differential Equations. Math Comput Modelling 25: 195-212.
4. Nadjafikhah M, Shirvani-Sh V (2011) Symmetry classification and conservation laws for higher order Camassa-Holm equation.
5. Ibragimov NH (2007) Quasi-self-adjoint differential equations. Arch ALGA 4: 55-60.
6. Olver PJ (1986) Applications of Lie Group for Differential Equations. Springer-Verlag, New York.
7. Olver PJ (1995) Equivalence, invariant and symmetry. Cambridge University Press, Cambridge.

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