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Research Article

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Solution of Voltra-Fredholm Integro-Differential Equations using Chebyshev Collocation Method

Deepmala[1], Vishnu Narayan Mishra[2]*, HR Marasi[3], H Shabanian[4] and M Nosrati Sahlan[4]

[1]Mathematics Discipline, PDPM Indian Institute of Information Technology, Design and Manufacturing, Jabalpur-482005, India
[2]Applied Mathematics and Humanities Department, S.V. National Institute of Technology, Surat-395007, Gujarat, India
[3]Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Tabriz, Tabriz, Iran
[4]Department of Mathematics, Faculty of Sciences, University of Bonab, Bonab, Iran

Abstract

In this paper, we use chebyshev polynomial basis functions to solve the Fredholm and Volterra integro-differential equations. We directly calculate integrals and other terms are calculated by approximating the functions with the Chebyshev polynomials. This method affects the computational aspect of the numerical calculations. Application of the method on some examples show its accuracy and efficiency.

Keywords: Integro-differential equation; Chebyshev polynomial; Collocation method

Introduction

We consider the integro-differential equations of Fredholm, Volterra and Fredholm-Volterra types in the forms

$$Dy(x) = f(x) + \lambda \int_{-1}^{1} k(x,t)y(t)dt, \quad (1)$$

$$Dy(x) = f(x) + \lambda \int_{-1}^{x} k(x,t)y(t)dt, \quad (2)$$

and

$$Dy(x) = f(x) + \lambda_1 \int_{-1}^{x} k_1(x,t)y(t)dt + \lambda_2 \int_{-1}^{1} k_2(x,t)y(t)dt, \quad (3)$$

where λ, λ_1, and λ_2 are real parameters. The functions *f(x)*, *k(x,t)*, $k_1(x,t)$ and $k_2(x,t)$ are known, *y(x)* is the unknown function to be determined and *D* is a linear differential operator. We suppose, without loss of generality, that the interval of integration is [-1,1]. Many problems in engineering and mechanics can be transformed into integral equations. For example it is usually required to solve Fredholm integral equations(FIE) in the calculations of plasma physics [1]. The numerical solution of these equations is a well-studied problem and a large variety of numerical methods have been developed to rapidly and accurately obtain approximations to *y(x)*. Overviews and references to the literature for many existing methods are available in [2,3]. Collocation methods [2-6], Sinc methods [7], global spectral methods [8], methods for convolution equations [9], Newton-Gregory methods [10], Runge-Kutta methods [11,12], qualocation methods [13] and Galerkin methods [14] are several of the many approaches that have previously been considered. In this paper the aim is to obtain the solution as a truncated Chebyshev series defined by

$$y(x) \approx y_N(x) = \sum_{j=0}^{N} a_j T_j(x), \quad (4)$$

where $T_j(x)$ denotes the Chebyshev polynomials of the frist kind, a_j are unknown Chebyshev coefficients and *N* is any chosen positive integer. The Chebyshev collocation points are defined by

$$x_i = \cos(\frac{i\pi}{N}), i = 0,1,\ldots,N \quad (5)$$

The paper is organized as follows: In Section Approximations we describe numerical approximations for differential operator and functions of integro-differential equation. The numerical results are presented in Section Numerical examples.

Methods and Approximations

Let *D* be a linear differential operator of order *v* with polynomial coefficients defined by

$$D := \sum_{r=0}^{v} p_r(x) \frac{d^r}{dx_r} \quad (6)$$

We shall write for $p_r(x)$

$$p_r(x) = \sum_{j=0}^{\alpha_r} p_{rj} x^j, \quad (7)$$

Where α_r is the degree of $p_r(x)$.

Let *y(x)* be the exact solution of the integro-differential equation

$$Dy(x) = f(x) + \lambda \int_a^b K(x,t)y(t)dt, \quad x \in [a,b], \quad (8)$$

with

$$\sum_{k=1}^{v} \left[c_{jk}^{(1)} y^{(k-1)}(a) + c_{jk}^{(2)} y^{(k-1)}(b) \right] = d_j, \quad j = 1,\ldots,v, \quad (9)$$

Where *f(x)* and *k(x,t)* are given continuous functions and λ, a, b, $c_{jk}^{(1)}$, $c_{jk}^{(2)}$ and d_j some given constants.

Matrix representation for *Dy(x)*

Let $\underline{V} := \{v_0(x), v_1(x), \ldots\}$ be a polynomial basis given by $\underline{V} = V\underline{X}$ where $\underline{X} = (1, x, x^2, \ldots)^T$ and *V* is a non-singular lower triangular matrix with degree $(v_i(x)) \leq i$ for *i*=0,1,2,…. According to [17] the effect of differentiation, shifting and integration on the

***Corresponding author:** Vishnu Narayan Mishra, Applied Mathematics and Humanities Department, S.V. National Institute of Technology, Surat-395007, Gujarat, India, Tel: +91 99133 87604; E-mail: vishnunarayanmishra@gmail.com

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coefficients vector $\tilde{a}_n = (\tilde{a}_0, \tilde{a}_1, ..., \tilde{a}_n, 0, 0, ...)$ of a polynomial $u_n(x) = \tilde{a}_n \underline{X}$ is the same as that of post-multiplication of $\tilde{a}_n$ by the matrices *η*, *rμ* and *l* respectively,

$$\eta = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \mu = \begin{bmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \iota = \begin{bmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & \frac{1}{2} & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

Where

$$\eta = \begin{bmatrix} 1 & 0 & \cdots & \cdots \\ 0 & 0 & 0 & \cdots \\ 0 & 2 & 0 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \mu = \begin{bmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & 1 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}, \iota = \begin{bmatrix} 0 & 1 & \cdots & \cdots \\ 0 & 0 & \frac{1}{2} & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

We recall now the following theorem given by Oritez and Samara [15].

Theorem 4.1.1 For any linear differential operator *D* defined by Eq. (6) and any series

$$y(x) := \underline{AT}, \underline{A} := [a_0, a_1, a_2, ...], \underline{\tilde{A}} = [\tilde{a}_0, \tilde{a}_1, \tilde{a}_2, ...]$$

$$\underline{A} := [a_0, a_1, a_2, ...],$$

we have

$$Dy(x) = \underline{\tilde{A}} Q \underline{X} = \underline{A} Z \underline{T},$$

where

$$Q = \sum_{i=0}^{v} \eta^i p_i(\mu), \text{ and}$$

$$Z = TQT^{-1}, T = \begin{bmatrix} 1 & 0 & 0 & \cdots \\ 0 & 1 & 0 & \cdots \\ -1 & 0 & 2 & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

Function approximation

The solution of Eqs. (1), (2) and (3) can be expressed as a truncated Chebyshev series. Therefore, the approximate solution (4) can be written in the matrix from

$$y(x) = T^T(x)A, \tag{10}$$

where

$$T(x) = [T_0(x), T_1(x), ..., T_N(x)]^T \quad A = [a_0, a_1, ..., a_N]^T.$$

Consequently, using Theorem 2.1 and substituting Eq. (10) in Eq. (1), we get

$$A^T ZT(x) = f(x) + \lambda \int_{-1}^{1} K(x,t)[T^T(t)A]dt \tag{11}$$

Now using the chebyshev collocation points (5) in Eq. (11) we obtain the following new system of algebraic equations

$$A^T ZT(x_i) = f(x_i) + \lambda \int_{-1}^{1} K(x_i,t)[T^T(t)A]dt,$$
$$x_i = \cos(\frac{i\pi}{N}), \qquad i = 0,1,2,...,N, \tag{12}$$

and so, unknown coefficients a_j are found.

Definition 4.2.1 The polynomial $y_n(x) = A_N \underline{T} \, \underline{T} = T(x)$, will be called an approximate solustion of Eqs. (8) and (9), if the vector $A_N = [a_0, a_1, ..., a_N]$ is the solution of the system of liner algebraic equations (12).

Similarly we can develop the method for the problem defined in the domain [0, 1]

$$Dy(x) = f(x) + \lambda \int^{1} k(x,t)y(t)dt$$

In this case we obtain the solution in terms of shifted Chebyshev polynomials $T_j^*(x)$ in the form

$$y(x) = \sum_{j=0}^{N} a_j^* T_j^*(x), 0 \le x \le 1,$$

where $T_j^*(x) = T_j(2x-1)$ Similar to the previous procedure and using the collocation points defined by

$$x_i = \frac{1}{2}(1 + \cos(\frac{i\pi}{N})), i = 0,1,2,...,N, \tag{13}$$

one can get the following system of algebraic equations

$$A^T Z^* T^*(x_i) = f(x_i) + \lambda \int_0^1 K(x_i,t) A T^{*T}(t)dt,$$
$$i = 0,1,2,...,N, \tag{14}$$

where $T^*(x) = [T_0^*(x), T_1^*(x), ..., T_N^*(x)]$ and ($Z^* = T^* Q T^{*-1} T^*(x) = \underline{T}^* = T^* \underline{X}$).

Solving the nonlinear system, unknown coefficients a_j are found. Similarly, we obtain the fundamental equation for Volterra and Fredholm-Volterra integral equation. In this study, instead of approximating integral terms, we directly calculate integrals. Examples show that this method affects the computational aspect of the numerical calculations.

Results and Numerical Examples

The results obtained in previous sections are used to introduce a direct efficient and simple method to solve integro-differential equations of Volterra and Fredholm type.

Example 5.1 We consider the following Fredholm integro-differential equation of the second kind

$$y'(x) = y(x) + 1 - \frac{4}{3}x + \int_0^1 xty(t)dt,$$
$$y(0) = 0$$

The exact solution is *y(x) =cx*. We assume the solution of *y(x)* as a truncated Chebyshev series

$$y(x) = a_0 T_0^*(x) + a_1 T_1^*(x), \quad 0 \leqslant x \leqslant 1. \tag{15}$$

Here, we have

$$f(x) = 1 - \frac{4}{3}x, \quad k(x,t) = xt, \quad \lambda = 1, \quad N = 1$$

$$Dy(x) = y'(x) - y(x), v = 1$$

The fundamental equation of the problem is defined by

$$A^T Z^* T^*(x_i) = 1 - \frac{4}{3}(x_i) + \int_0^1 x_i t A T^{*T}(t)dt, \; x_0 = 1, \; x_1 = 0,$$

where $T^*(x_i) = [T_0^*(x_i), T_1^*(x_i)] = [1, 2x_i - 1], A = [a_0, a_1]^T$,

$$T^*(t) = [T_0^*(t), T_1^*(t)] = [1, 2t - 1],$$

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$$\eta = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \mu = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$T^* = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}, \quad T^{*-1} = \begin{bmatrix} 1 & 0 \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

Therefore, using Theorem 4.1.1 we obtain

$$Q = \sum_{i=0}^{v} \eta^i p_i(\mu) = \sum_{i=0}^{1} \eta^i p_i(\mu) = -1 + \eta = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix},$$

$$Z^* = T^* Q T^{*-1} = \begin{bmatrix} -\frac{3}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

The system yields the solution

$$a_0 = \frac{1}{2},\ a_1 = \frac{1}{2}$$

Substituting these values in (15), we get the exact solution of the problem

$$y(x) = \frac{1}{2} + \frac{1}{2}(2x - 1) = x$$

Example 5.2 *We consider the following Fredholm-Volterra integro-differential equation*

$$y'(x) = -2\sin(x) - x^2\sin(2x) - 2x\cos(2x) + 2\sin(2x) - 2e^x + 5e^{x-1} + 2x + \int_0^x \cos(x+t)y(t)dt + \int_0^1 e^{x-t} y(t)dt,$$

$$y(0) = 0$$

The exact solution is $y(x) = x^2$. Let us suppose that $y(x)$ is approximated by Chebyshev series

$$y(x) = \sum_{j=0}^{3} a_j T_j^*(x), \quad 0 \leqslant x \leqslant 1$$

Using the procedure in section Approximations, we obtain the approximate solution of the problem.

In Table 1, we compare the numerical results of the problem by the proposed method of N=3 with the method discussed in an earlier study [16].

Example 5.3 *We consider the following Fredholm integro-differential equation of the second kind*

X	The method discussed in [16]	Presented method N=3
0	4.930×10^{-4}	3.3590×10^{-15}
0.1	2.240×10^{-3}	3.7764×10^{-15}
0.2	1.571×10^{-3}	4.2889×10^{-15}
0.3	1.514×10^{-3}	4.8711×10^{-15}
0.4	7.015×10^{-3}	5.5788×10^{-15}
0.5	1.6336×10^{-2}	6.2727×10^{-15}
0.6	1.1862×10^{-2}	7.0499×10^{-15}
0.7	4.971×10^{-3}	7.8825×10^{-15}
0.8	4.338×10^{-3}	8.9928×10^{-15}
0.9	1.6068×10^{-2}	9.8809×10^{-15}
1.0	----	1.0880×10^{-14}

Table 1: Comparison of the absolute errors of example (3.2).

x	Method in [13]	presented method N=2	presented method N=4
0	0.0187621362	0.0036921151	0.002472602103
1/15	0.0200637354	0.0036087818	0.002472602236
2/15	0.0212780889	0.0035254484	0.002472602365
3/15	0.0215334191	0.0034421151	0.002472602492
4/15	0.0205212581	0.0033587818	0.002472602617
5/15	0.0192905931	0.0032754484	0.002472602740
6/15	0.0181294338	0.0031921151	0.002472602862
7/15	0.0170353464	0.0031087818	0.002472602984
8/15	0.0160143431	0.0030254484	0.002472603106
9/15	0.0150618428	0.0029421151	0.002472603229
10/15	0.0141627843	0.0028587818	0.002472603353
11/15	0.0133104506	0.0027754484	0.002472603479
12/15	0.0125010566	0.0026924451	0.002472603607
13/15	0.0117309887	0.0026087818	0.002472603788
14/15	0.0109968073	0.0025254484	0.002472603873
1.0	0.0102952225	0.0024421151	0.002472604012

Table 2: Comparison of the absolute errors of example (3.3).

X	Exact	Presented method N=11
-1	0.36787944	0.36787445212
-0.8	0.44932896	0.44932342567
-0.6	0.54881164	0.54881194564
-0.4	0.67032005	0.67032187409
-0.2	0.81873075	0.81897235137
0	1	0.99988319565
0.2	1.22140276	1.22140675944
0.4	1.49182470	1.49189543017
0.6	1.82211880	1.82211675430
0.8	2.22554093	2.22554075420
1.0	2.71828183	2.71828147693

Table 3: Comparison of numerical results for example (3.4).

$$y''(x) - y(x) = -1/20\int_0^1 t^{39} y(t)dt - x^2 - 2x + 2521/688800,$$

$$\begin{cases} y(0) - y'(0) = 0 \\ y(1) - y'(1) = 9 \end{cases}$$

The exact solution is $y(x) = x^2 + 2x + 2$. Talking $N = 2, 4$, the approximate solutions are obtained by this method. Results are compared with those of the methods in literature [17]as shown in Table 2.

Example 5.4 *We consider the following Volterra integro-differential equation of the second kind*

$$y''(x) + xy'(x) - xy(x) = e^x - 2\sin(x) + \int_{-1}^{1} \sin(x)e^{-t} y(t)dt, \quad -1 \le x \le 1$$

$$y(0) = 1, \quad y'(0) = 1$$

The exact solution is $y(x) = e^x$. See Table 3 for the numerical results.

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