

Solving Differential Equations by New Wavelet Type Transform Method Based on the Wavelets and Symmetry Groups

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Abstract

The wavelets are important functions in the harmonic analysis. Up to our knowledge, apply wavelets to solve differential equations are limited to ODEs or PDEs with approximate and numerical solutions. In this paper, the novel methods based on the wavelets with two independent variables according to differential invariants are proposed. In fact, the transform groups are constructed that can be acted on the existing solutions and produced the new solution. These groups are based on the symmetry groups (obtained by the Lie symmetry method and other equivalence methods) and mother wavelets. The new method based on the wavelets are presented, new mother wavelets are produced, the corresponding wavelet type transform groups are provided and applied for solving the differential equations. Our method can be used for ODEs and PDEs at every order and give us the analytic solutions according to the existing solutions (from every method without any exception).

Keywords: Wavelet; Mother wavelet; Wavelet type transform; Invariant solution; Degree reduction

Introduction

The wavelets are very useful tools for analyzing problems. They are important functions in the functional and harmonic analysis. The first wavelet was introduced by Alphered Haar (the Hungarian mathematician) [1]. Nowadays, the wavelets have numerous applications in some fields of science and technology; such as seismology, image processing, signal processing, coding theory, biosciences, financial mathematics, fractals and so on [2]. The application of wavelets for solving differential equations limited to ODEs or PDEs with the numerical solutions in the special conditions. The famous wavelets such as Haar, Daubechie, Coiflet, Symlet, CDF, Mexican hat and Gaussian are extendible to two or more variables with tensor product. The wavelets with two or more variables (that in connection with PDEs are useful) is very important. In this paper, we make some new wavelets with two variables in different method; these wavelets depend on the differential invariants of differential equations. Therefore, we can use their transform groups for solving differential equations. This method called the Wavelet Type Transform Method (WTTM). We will show the performance of WTTM with examples.

We propose the method based on the wavelets with two independent variables according to the symmetry groups. This method based on the group action of wavelet transforms correspond to the suitable mother wavelets on the differential equations. In fact, by wavelet transforms, we act on the solution spaces obtained by other methods (like the Lie symmetry method) and construct the new solutions according to the existing solutions. For this reason, this method is called the wavelet type transform method (WTTM). Recently, the similar method based on the Fourier transforms has been proposed by Craddock [3]. Here, first briefly explain the Lie symmetry method that will be applied for obtaining the differential invariants.

The Lie symmetry groups theory of differential equations was developed by Sophus Lie [2]. Such Lie groups are invertible point transforms of both the dependent and independent variables of the differential equations. The symmetry group method provide the collection of tools for analyzing differential equations and is of great importance to understand and to construct solutions of differential equations. Several applications of Lie groups in the theory of differential

equations were discussed in the literature, the most important ones are: the reduction of order of ordinary differential equations (ODEs), the construction of invariant solutions, mapping the solutions to other solutions and the detection of linear transforms [4,5].

The remainder of this paper is organized as follows. In section 2, we recall some needed results to construct differential invariants, the mother wavelets and the wavelet type transforms. In section 3, the wavelet type transform method (WTTM) is proposed. In sections 4, the proposed method will be demonstrated by examples. Finally, the conclusions and future works are presented.

Preliminaries

In this section, we remember some definitions, conceptions and theorems from related areas.

The Lie symmetry method

In this section, we recall the general procedure for determining symmetries for any system of partial differential equations [2,4,6,7].

To begin, let us consider the general case of a nonlinear system of partial differential equations of order n th in p independent and q dependent variables is given as a system of equations:

$$\Delta_v(x, u^{(n)})=0, v=1, \dots, l, \quad (1)$$

Involving $x=(x^1, \dots, x^p)$, $u=(u^1, \dots, u^q)$ and the derivatives of u with respect to x up to n , where $u^{(n)}$ represents all the derivatives of u of all orders from 0 to n . We consider a one-parameter Lie group of infinitesimal transforms acting on the independent and dependent variables of the system (1) as below:

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$$(\tilde{x}^i, \tilde{u}^j) = (x^i, u^j) + s(\xi^i, \eta^j) + O(s^2), \quad (i=1, \dots, p, \quad j=1, \dots, q)$$

Where s is the parameter of the transform and ξ^i, η^j are the infinitesimals of the transforms for the independent and dependent variables, respectively. The infinitesimal generator v associated with the group of transforms can be written as $v = \sum_{i=1}^p \xi^i \partial_{x^i} + \sum_{j=1}^q \eta^j \partial_{u^j}$, A symmetry of a differential equation is a transform which maps solutions of the equation to other solutions. The invariance of the system (1) under the infinitesimal transforms leads to the invariance conditions (Theorem 2.36 of ref. [4]):

$$Pr^{(n)}v[\Delta_i(x, u^{(n)})] = 0, \quad \Delta_i(x, u^{(n)}) = 0, \quad v = 1, \dots, l,$$

where $Pr^{(n)}$ is called the n^{th} -order prolongation of the infinitesimal generator and given by $Pr^{(n)}v = v + \sum_{\alpha=1}^q \sum_J \phi_J^\alpha(x, u^{(n)}) \partial_{u_J^\alpha} z$, where $J=(j_1, \dots, j_k), 1 \leq j_k \leq p, 1 \leq k \leq n$ and the sum is over all J 's of order $0 < \#J \leq n$. If $\#J=k$, the coefficient ϕ_J^α of $\partial_{u_J^\alpha}$ will only depend on k -th and lower order derivatives of u , and $\phi_\alpha^J(x, u^{(n)}) = D_J \left(\phi_\alpha - \sum_{i=1}^p \xi^i u_i^\alpha \right) + \sum_{i=1}^p \xi^i u_{J,i}^\alpha$, where $u_i^\alpha := \partial u^\alpha / \partial x^i$ and $u_{J,i}^\alpha := \partial u_J^\alpha / \partial x^i$.

One of the most important properties of these infinitesimal symmetries is that they form a Lie algebra under the usual Lie bracket. The first application of symmetry group methods is to construct the new solutions from known existing solutions. The second is when a nonlinear system of differential equations admits infinite symmetries, so it is possible to transform it to a linear system. Infact, for every vector field, we establish the following characteristics system.

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\phi}$$

By solving the system, we obtain differential invariants corresponding to the vector fields. The PDE is expressed in the coordinates (x, t, u) , so for reducing this equation, we should search for its form in the specific coordinates. Those coordinates will be constructed by searching for independent invariants (y, v) corresponding to the infinitesimal generators. Thus by using the chain rule, the expression of equation in the new coordinate, allows us to reduce the order of the differential equation [6].

The wavelet theory

The wavelets are important functions in the mathematics and other scientific fields. In this section, we introduce wavelets as functions belong to $L^2(R^2)$ (The space of squared integrable functions with integral norm).

Definition 1: The function ψ is a wavelet, if it satisfies in the following admissible condition:

$$C_\psi = \int_{R^2} \frac{|F(\psi)(\omega)|^2 d\omega}{|\omega|} < \infty.$$

Where $F(\psi)(\omega)$ is the Fourier transform of wavelet ψ and defined as follows:

$$F(\psi)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{R^2} \exp(-ix \cdot \omega) \psi(x) d\omega z,$$

C_ψ is called the wavelet coefficient of ψ . Here, $\omega = (\omega_1, \omega_2)$ and $x = (x_1, x_2)$ belong to R_2 [7].

Definition 2: The wavelet ψ is called mother wavelet, if it satisfies in the following properties,

$$\int_{R^2} \psi(x) dx = 0 \tag{2}$$

$$\int_{R^2} |\psi(x)|^2 dx < \infty \tag{3}$$

$$\lim_{|\omega| \rightarrow \infty} F(\psi(\omega)) = 0 \tag{4}$$

Note that, the first property equivalents to $C_\psi > 0$ (the admissible condition) for mother wavelet ψ [1].

Indeed, the mother wavelets have the admissible condition, n-zero moments and exponential decay properties. The mother wavelet have two parameters: the translation parameter $b=(b_1, b_2)$ and scaling parameter $a>0$. The mother wavelet corresponding to (a, b) is:

$$\psi_{a,b}(x) = \psi\left(\frac{x-b}{a}\right) = \psi\left(\frac{x_1-b_1}{a}, \frac{x_2-b_2}{a}\right)$$

If the function ψ don't satisfy in the some properties of mother wavelets, or approximately satisfies, ψ is called quasi-wavelet. The quasi-wavelets have numerous applications in the applied mathematics and other scientific fields for solving PDEs [8].

Definition 3: The wavelet transform corresponding to the mother wavelet ψ for the function $f \in L^2(R^2)$ with parameters (a, b) defined as follows:

$$W_\psi(f)(a, b) = \frac{1}{\sqrt{|a|} C_\psi} \int_{R^2} \psi_{a,b}(x) \cdot f(x) dx.$$

Thus, the wavelet transform depends on the wavelet ψ , the function f , and the parameters (a, b) .

Theorem 1: The wavelet transform as integral operator is from $L^2(R^2)$ to $L^2(R^3)$ that satisfies in the following properties:

- Linearity: $W_\psi[\alpha f(x) + \beta g(x)] = \alpha W_\psi[f(x)] + \beta W_\psi[g(x)]$,
- Translation: $W_\psi[f(x-k)] = W_\psi[f(x)](a, b-k), \quad \forall k \in R^2$,
- Scaling: $W_\psi\left[\frac{1}{\sqrt{s}} f\left(\frac{x}{s}\right)\right] = W_\psi[f(x)]\left(\frac{a}{s}, \frac{b}{s}\right)$,
- Wavelet shifting: $W_{\psi(x-k)}[f(x)] = W_\psi[f(x)](a, b+ak)$,
- Linear combination: $W_{\alpha v_1 + \beta v_2}[f(x)] = \alpha W_{v_1}[f(x)] + \beta W_{v_2}[f(x)]$,
- Wavelet scaling: $W_{\frac{\psi(x/s)}{\sqrt{|s|}}}[f(x)] = W_\psi[f(x)](as, b)$.

Proof [1]: Indeed, the wavelet transforms are isometries [7,9], this property and the admissible condition imply that the wavelet transform is invertible. The inversion formula for the wavelet transform $W_\psi(f)$ is:

$$f(x) = f(x_1, x_2) = \frac{1}{C_\psi} \int_{R^2 \times R^2} W_\psi f(a, b) \psi_{a,b}(x) \frac{da db_1 db_2}{a^3}$$

In fact, for obtaining the function $f(x)$ corresponds to the wavelet transform $W_\psi(f)$, the inversion formula (also called the synthesis formula) could be employed.

The Wavelet Type Transform Method

The wavelet type transform method (WTTM) has 3 following steps:

• Apply equivalence algorithms (for example, the Lie symmetry method) on DE, and obtain the general solutions as follows:

$$\tilde{u}(x, t; \varepsilon) := \rho(\exp(\varepsilon v_k)) u_0,$$

• Propose the suitable mother wavelet (as a test function) based on the symmetry groups and differential invariants. According to mother wavelet ψ , similar to the wavelet transforms definition, we have:

$$U(x, t) = \int \psi(x, t; \varepsilon) \cdot \rho(\exp(\varepsilon v_k)) u_0(x, t; \varepsilon) d\varepsilon,$$

This is a wavelet type transform that its kernel is an invariant solution of differential equation.

• For every generator vector field v_k from the symmetry groups, by considering the suitable values (0,1) for the parameters ε_i (this parameter for symmetry group G_i equals 1 and for every symmetry group G_j such that $j \neq i$ equals 0), and the correspondent version of mother wavelet ψ based on the differential invariants of the symmetry groups G_i (that's $\psi(x, t; \varepsilon_i)$), construct $U(x, t)$.

Indeed, $U(x, t)$ is a wavelet transform and on the other hand (for every invariant solution $\tilde{u}(x, t; \varepsilon)$), it is a wavelet invariant solution.

This method provides a link between harmonic and wavelet analysis and Lie symmetries. In fact, the new solutions can be constructed by integral kernel operators, where the kernels are invariant solutions (obtained by the symmetries of the equation) and test functions are suitable mother wavelets (constructed based on the differential invariants). Therefore, this method converges to the solution of differential equations.

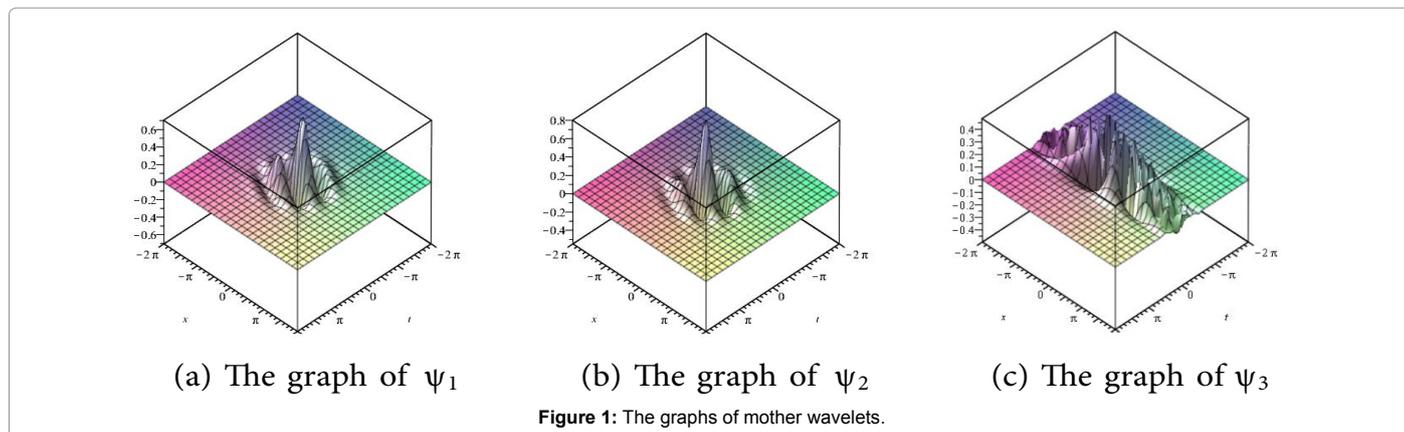
Examples

Here, we demonstrate WTTM by examples. We implement WTTM on the heat, KdV and GFKPP equations and obtain solutions. finally, WTTM results will be proposed. First, we should apply the Lie symmetry method on these equations, for more details and calculations [4]. The following table proposes the mother wavelets according to the differential invariants of symmetry groups: (Table 1).

By the little computation, it can be seen that offered functions have

Symmetry group	Differential invariants	Mother wavelets
Translation	xct, u	$(4/5)\exp((x^2+t^2)/2)\sin(\pi(x-ct)/2)$
Scaling	$(x/t), (x/\sqrt{t}), (u/t^*)$	$(4/5)\exp(-(x^2+t^2)/2)\cos(\pi(xct)/2)$ $\exp(-(x^2+15t^2)/20)\cos(x/t)\sin(x/t)$
Galilean Boost	$t, u\exp(x^2/4t)$	$\exp(-(x^2+at^2)/b)\cos(x/t)\sin(x/t)$

Table 1: Symmetry groups, Differential invariants and Mother wavelets.



properties (2)-(3) of the mother wavelets. Figure 1 shows the graphs of mother wavelets. The some properties are clear from this figure (some computations and plots done by the maple 2016).

We know that the general solution (from the Lie symmetry method) of the heat equation $u_t = u_{xx}$ as follows:

$$\tilde{u}(x, t; \varepsilon) = \frac{\exp\left(\varepsilon_3 - \frac{\varepsilon_5 x + \varepsilon_6 x^2 - \varepsilon_5^2 t}{1 + 4\varepsilon_6 t}\right) u\left(\frac{\exp(-\varepsilon_4)(x - 2\varepsilon_3 t)}{1 + 4\varepsilon_6 t}, \frac{\exp(-2\varepsilon_4)}{1 + 4\varepsilon_6 t} - \varepsilon_2\right)}{\sqrt{1 + 4\varepsilon_6 t}},$$

Thus for the translation symmetry group in the line of x , all ε_i (except $i=1$) are zero. So,

$$\tilde{u}(x, t; \varepsilon) = u(x - \varepsilon_1, t).$$

But for the vector field $v_1 = \partial_x + 2t\partial_t$, with the differential invariant $(x-2t)$ (the translation parameter $c=2$), we have:

$$\tilde{u}(x, t; \varepsilon) = u(x - 2\varepsilon_1, t - \varepsilon_2).$$

for the corresponding mother wavelet $\psi_1(x, t) = \frac{4}{5}\exp\left(-\frac{x^2+t^2}{2}\right)\sin\left(\frac{\pi(x-2t)}{2}\right)$, we get,

$$\psi_1(x, t; \varepsilon) = \frac{4}{5}\exp\left(-\frac{(x-2\varepsilon_1)^2 + (t-\varepsilon_2)^2}{2}\right)\sin\left(\frac{\pi(x-2\varepsilon_1-2t+2\varepsilon_2)}{2}\right)$$

Therefore:

$$\int \int_{\mathbb{R}^+ \times \mathbb{R}^+} \frac{4}{5}\exp\left(-\frac{(x-2\varepsilon_1)^2 + (t-\varepsilon_2)^2}{2}\right)\sin\left(\frac{\pi A}{2}\right)\tilde{u}(x, t; \varepsilon) d\varepsilon_1 d\varepsilon_2. \quad (5)$$

Where $A=(x-2t+2(\varepsilon_2-\varepsilon_1))$. Indeed U is the wavelet transform $W_{\psi_1}(x, t)$ on the related symmetry group v_1 that its invariant solution as below:

$$\tilde{u}(x, t; \varepsilon) = k\exp(-2A) + l,$$

By replacing this solution in the integral (5) and integration respect to $\varepsilon_1, \varepsilon_2$, we get to the wavelet invariant solution of heat equation based on $\psi_1(x, t)$. The calculation for mother wavelet ψ_2 is similar.

Now, let us consider mother wavelet ψ_4 related to the Galilean boost vector field $v_4=2t \partial_x x u \partial_u$ with differential invariant solution

$u(x,t) = \frac{k}{\sqrt{t}} \exp\left(-\frac{x^2}{4t}\right)$. The correspondent wavelet type transform as follows:

$$\int_{\mathbb{R}^+} \exp\left(-\frac{x^2 + 15t^2}{20}\right) \cos\left(\frac{x + 2\epsilon t}{t}\right) \sin\left(\frac{x + 2\epsilon t}{t}\right) \cdot \frac{k}{\sqrt{t}} \exp\left(-\frac{(x + 2\epsilon t)^2}{80t}\right) d\epsilon,$$

Where $\epsilon = \epsilon_4$.

Now, as another example, we consider the KdV equation $u_t + u_{xxx} + uu_x = 0$. First, the general solution from the Lie symmetry methods as follows:

$$\tilde{u}(x,t;\epsilon) = \exp(-2\epsilon_4) u(\exp(-\epsilon_4)x - \epsilon_1, \exp(-\epsilon_4)t - \epsilon_2).$$

For the vector field $v_1 = \partial_x + 2\partial_t$ with the differential invariant $(x-2t)$ and for mother wavelet ψ_1 , the wavelet type transform as follows:

$$\iint \frac{4}{5} \exp\left(-\frac{(x-2\epsilon_1)^2 + (t-\epsilon_2)^2}{2}\right) \sin\left(\frac{\delta(x-2t+2(\epsilon_2-\epsilon_1))}{2}\right) \tilde{u}(x,t;\epsilon) d\epsilon_1 d\epsilon_2,$$

The calculations are similar for ψ_2 . By considering the invariant solution as follows:

$$u(x,t) = 6 \operatorname{sech}^2 \left[\frac{\sqrt{2}}{2} (x-2t) + \delta \right].$$

The wavelet types transform corresponding to ψ_2 as below:

$$\iint \frac{24}{5} \exp\left(-\frac{(x-2\epsilon_1)^2 + (t-\epsilon_2)^2}{2}\right) \cos\left(\frac{\pi A}{2}\right) \operatorname{sech}^2 \left[\frac{\sqrt{2}}{2} A + \delta \right] d\epsilon_1 d\epsilon_2,$$

where all integrals are calculated on $\mathbb{R}^+ \times \mathbb{R}^+$.

In the following, the generalized FKPP equations are analyzed by WTTM. Under the scaling symmetry group with parameter $\lambda = \exp(\epsilon)$ and for invariants $(x/t, x/t^2)$, by the mother wavelet ψ_3 the general solution as follows:

$$\tilde{u}(x,t;\epsilon) = \exp(2\epsilon) u(\exp(-\epsilon)x, \exp(-2\epsilon)t),$$

Thus,

$$\psi_3(x,t;\epsilon) = \exp\left(-\frac{\exp(2\epsilon)x^2 + 15\exp(4\epsilon)t^2}{20}\right) \cos\left(\frac{\pi x}{\exp(\epsilon)t}\right) \sin\left(\frac{\pi x}{\exp(\epsilon)t}\right),$$

and we get,

$$\int_{\mathbb{R}^+} \exp\left(2\epsilon - \frac{\exp(2\epsilon)x^2 + 15\exp(4\epsilon)t^2}{20}\right) \cos(B) \sin(B) u(\exp(-\epsilon)x, \exp(-2\epsilon)t) d\epsilon,$$

Where $B = \frac{\pi x}{\exp(\epsilon)t}$. Now, we replace $u(x,t)$ by the solution of

GFKPP equation, for example, in the primary version of GFKPP equation (that is $u_t = u_{xx} + u - u^2$), the analytic solution is:

$$u(x,t) = \frac{1}{1 + F(x)\exp(-t)}.$$

So, we have:

$$\int_{\mathbb{R}^+} \frac{\exp\left(2\epsilon - \frac{\exp(2\epsilon)x^2 + 15\exp(4\epsilon)t^2}{20}\right) \cos(B) \sin(B)}{1 + F(\exp(\epsilon)x)\exp(-\exp(2\epsilon)t)} d\epsilon,$$

and for the generalized version with $f(x)$, we get,

$$\int_{\mathbb{R}^+} \exp\left(2\epsilon - \frac{\exp(2\epsilon)t}{f(\exp(\epsilon)x)} - \frac{\exp(2\epsilon)x^2 + 15\exp(4\epsilon)t^2}{20}\right) \cos(B) \sin(B) G(\exp(\epsilon)x) d\epsilon,$$

Hence, if the functions $f(x), F(x), G(x)$ are determined, we can easily obtain the wavelet invariant solutions.

Conclusions and Future Works

In this paper, we proposed the novel method based on the wavelets and symmetry groups. We constructed new mother wavelets with two variables according to the differential invariants (in the different style), then calculated the corresponding wavelet type transforms, applied them on the invariant solutions of PDEs and hereafter, the solutions were obtained. Unlike other applications of wavelets and wavelet transforms in the numerical analysis and other fields of the differential equations, our method results in the analytic and exact solutions. Indeed, we have relied upon constructing solutions from other solutions, on the other words, WTTM uses the existing solutions (from every method without any exception), then acts on them and produces the new solutions. In this method, the crucial step is to provide the proper mother wavelet according to the differential invariants, therefore this method can be used to solve the PDEs that equivalence methods (like the Lie symmetry method) can be applied on those. In fact, our contributions are new wavelets and the new wavelet method that with its wavelet type transform acts on the solution manifolds and produces new solutions from existing ones. By this method, we can produce a considerable range of integral operators. In the future works, by implementing WTTM on other PDEs, we will propose the suitable mother wavelets for every differential invariant and symmetry group and generalized the method for solving PDEs at every order and every number of independent variables.

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