Journal of Physical Mathematics
ISSN: 2090-0902

Bature and Musa, J Phys Math 2017, 8:3
DOI: 10.4172/2090-0902.1000245

Research Article | Open Access

Stability Analysis of the 2-Point Diagonally Implicit Super Class of Block Backward Differentiation Formula with Off-Step Points

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Abstract

In this paper, the stability region of the 2-point diagonally implicit super class of block backward differentiation formula with off-step points is to obtain and demonstrate its suitability for solving stiff ODEs. The stability region must cover the entire negative half-plane for the method to be suitable for solving stiff differential equations, analysis also shows that the method is zero stable.

Keywords: Implicit block method; Diagonally implicit method; Super class; Block backward differentiation formula; Off-step points; A-stability zero stability; Stability region; Stiff initial value problems

Introduction

Consider a system of first order stiff initial value problems (IVPs) of the form:

$$y' = \mathbf{f}(x, \mathbf{y}) \quad x \in [a,b] \quad y(x_0) = y_0 \tag{1}$$

System (1) is said to be stiff if all its eigenvalues have negative real part, and the stiffness ratio (the ratio of the magnitudes of the real parts of the largest and smallest eigenvalues) is large. Developing numerical methods for solving eqn. (1) in terms of accuracy, stability, convergence, computational expense, and data-storage requirements remains a major challenge in modern numerical analysis. However, some numerical methods developed for solving eqn. (1) have been introduced in Ababneh et al. [1], Abasi et al. [2,3], Babangida et al., Dhalquist [4,5], Curtiss and Hirschfelder [6], Cash [7], Ibrahim et al. [8-11], Musa et al. [12-19], and Suleiman, et al. [20] among others. According to researchers the stability problem appears to be the most serious limitation of block methods. The aim this work is to investigate the linear stability properties of the 2-point diagonally implicit super class of block backward differentiation formula with off-step points and demonstrate its suitability for solving eqn. (1). We start with some basic definition of stability of a multistep method given introduced in Ibrahim et al. [11].

Definition 1.1: A general k-step linear multistep method is defined as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

where α_j and β_j are constants and $\alpha_k \neq 0$ α_0 and β_0 cannot both be zero at the same time. For any k step method, α_k is normalized to one. The method (2) is said to be explicit if β_k=0 and implicit if $\beta_k \neq 0$.

Definition 1.2: The first and the second characteristic polynomials of the eqn. (2) are defined by:

$$\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j \tag{3}$$

and

$$\sigma(\xi) = \sum_{j=0}^{k} \beta_j \xi^j \text{, respectively.}$$

Definition 1.3: A Linear Multistep Method is said to be zero stable if no root of the first characteristics polynomial has modulus greater than one and that any root with modulus one is simple [21].

Definition 1.4: A Linear Multistep Method is said to be A-stable if its stability region covers the entire negative half-plane.

Stability of the Method

In this section, we introduce the basic definition of a block method described in Fatunla [22] and Chu [23], Babangida and Musa [24] reported by Ibrahim et al. [11].

Definition 2.1: Let Y_m and F_m be vectors defined by

$$Y_m = [y_n, y_{n+1}, y_{n+2}, \ldots, y_{n+r-1}]^t,$$

$$F_m = [f_n, f_{n+1}, f_{n+2}, \ldots, f_{n+r-1}]^t \text{, respectively.}$$

Then a general k-block, r-point method is a matrix finite difference equation of the form

$$Y_m = \sum_{i=1}^{k} A_i Y_{m-1} + h \sum_{i=0}^{k} B_i f_{m-1} \tag{4}$$

where all A_i's and B_i's are properly chosen $r \times r$ matrix coefficients and m=0,1,2,... represents the block number, n=mr the first step number in the m-th block and r is the proposed block size.

Definition 2.2: The Block Method (4) is said to be zero-stable if the roots $R_{(j,j)}$=1(1)k of the first characteristic polynomial $\rho(R) = det[\sum_{i=0}^{k} A_i R^{k-i}] = 0, Ao = -I$, satisfies $|Rj| \leq 1$. If one of the roots is +1, we call this root the principal root of $\rho(R)$.

Here, we will apply the same approach to formulas that has been derived, called diagonally implicit 2-point super class of block backward differentiation formula with off-step points. These formulas are given by

$$y_{n+\frac{1}{2}} = -\frac{7}{20} y_{n-1} + \frac{27}{20} y_n - \frac{9}{20} hf_n + \frac{3}{5} hf_{n+\frac{1}{2}},$$

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Received May 19, 2017; **Accepted** September 04, 2017; **Published** September 08, 2017

Citation: Bature B, Musa H (2017) Stability Analysis of the 2-Point Diagonally Implicit Super Class of Block Backward Differentiation Formula with Off-Step Points. J Phys Math 8: 245. doi: 10.4172/2090-0902.1000245

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$$y_{n+1}=\frac{11}{141}y_{n-1}-\frac{50}{47}y_n+\frac{280}{141}y_{n+\frac{1}{2}}-\frac{12}{47}hf_{n+\frac{1}{2}}+\frac{16}{47}hf_{n+1},$$

$$y_{n+\frac{3}{2}}=-\frac{3}{88}y_{n-1}+\frac{13}{22}y_n-\frac{21}{11}y_{n+\frac{1}{2}}+\frac{207}{88}y_{n+1}-\frac{9}{44}hf_{n+1}+\frac{3}{11}hf_{n+\frac{3}{2}} \quad (5)$$

$$y_{n+2}=\frac{19}{1005}y_{n-1}-\frac{29}{67}y_n+\frac{316}{201}y_{n+\frac{1}{2}}-\frac{189}{67}y_{n+1}+\frac{892}{335}y_{n+\frac{3}{2}}-\frac{12}{67}hf_{n+\frac{3}{2}}+\frac{16}{67}hf_{n+2}.$$

The method (5) can be rewritten in matrix form as follows:

$$\begin{pmatrix}1&0&0&0\\-\frac{280}{141}&1&0&0\\\frac{21}{11}&-\frac{207}{88}&1&0\\-\frac{316}{201}&\frac{189}{67}&-\frac{892}{335}&1\end{pmatrix}\begin{pmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{pmatrix}=\begin{pmatrix}0&-\frac{7}{20}&0&\frac{27}{20}\\0&\frac{11}{141}&0&-\frac{50}{47}\\0&-\frac{3}{88}&0&\frac{13}{22}\\0&\frac{19}{1005}&0&-\frac{29}{67}\end{pmatrix}\begin{pmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{pmatrix}+ \quad (6)$$

$$\begin{pmatrix}0&0&0&-\frac{9}{20}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}\begin{pmatrix}f_{n-\frac{3}{2}}\\f_{n-1}\\f_{n-\frac{1}{2}}\\f_n\end{pmatrix}+h\begin{pmatrix}\frac{3}{5}&0&0&0\\-\frac{12}{47}&\frac{16}{47}&0&0\\0&-\frac{9}{44}&\frac{3}{11}&0\\0&0&-\frac{12}{67}&\frac{16}{67}\end{pmatrix}\begin{pmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{pmatrix}$$

Definition 2.3: Let Y_m and F_m be vectors defined by

$Y_m=[y_{n+1},y_{n+2},\ldots,y_{n+r}]^T$ $F_m=[f_{n+1},f_{n+2},\ldots,f_{n+r}]^T$, *r=2, and n=2m*

Method (5) can be written in matrix form as follows:

$$A_0Y_m=A_1Y_{m-1}+h\left(B_0F_{m-1}+B_1F_m\right). \quad (7)$$

Where

$$A_0=\begin{pmatrix}1&0&0&0\\-\frac{280}{141}&1&0&0\\\frac{21}{11}&-\frac{207}{88}&1&0\\-\frac{316}{201}&\frac{189}{67}&-\frac{892}{335}&1\end{pmatrix}, A_1=\begin{pmatrix}0&-\frac{7}{20}&0&\frac{27}{20}\\0&\frac{11}{141}&0&-\frac{50}{47}\\0&-\frac{3}{88}&0&\frac{13}{22}\\0&\frac{19}{1005}&0&-\frac{29}{67}\end{pmatrix}, B_0=\begin{pmatrix}0&0&0&-\frac{9}{20}\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}, B_1=\begin{pmatrix}\frac{3}{5}&0&0&0\\-\frac{12}{47}&\frac{16}{47}&0&0\\0&-\frac{9}{44}&\frac{3}{11}&0\\0&0&-\frac{12}{67}&\frac{16}{67}\end{pmatrix}$$

$$Y_{m-1}=\begin{pmatrix}y_{n-\frac{3}{2}}\\y_{n-1}\\y_{n-\frac{1}{2}}\\y_n\end{pmatrix}=\begin{pmatrix}y_{2m-\frac{3}{2}}\\y_{2m-1}\\y_{2m-\frac{1}{2}}\\y_{2m}\end{pmatrix}=\begin{pmatrix}y_{2(m-1)+\_r\ constant\ and\ hence\ could\ be+\frac{1}{2}}\\y_{2(m-1)+1}\\y_{2(m-1)+\frac{3}{2}}\\y_{2(m-1)+2}\end{pmatrix}, Y_m=\begin{pmatrix}y_{n+\frac{1}{2}}\\y_{n+1}\\y_{n+\frac{3}{2}}\\y_{n+2}\end{pmatrix}=\begin{pmatrix}y_{2m+\frac{1}{2}}\\y_{2m+1}\\y_{2m+\frac{3}{2}}\\y_{2m+2}\end{pmatrix},$$

$$F_{m-1}=\begin{pmatrix}f_{n-\frac{3}{2}}\\f_{n-1}\\f_{n-\frac{1}{2}}\\f_n\end{pmatrix}=\begin{pmatrix}f_{2m-\frac{3}{2}}\\f_{2m-1}\\f_{2m-\frac{1}{2}}\\f_{2m}\end{pmatrix}=\begin{pmatrix}f_{2(m-1)+\_r\ constant\ and\ hence\ could\ be+\frac{1}{2}}\\f_{2(m-1)+1}\\f_{2(m-1)+\frac{3}{2}}\\f_{2(m-1)+2}\end{pmatrix} F_m=\begin{pmatrix}f_{n+\frac{1}{2}}\\f_{n+1}\\f_{n+\frac{3}{2}}\\f_{n+2}\end{pmatrix}=\begin{pmatrix}f_{2m+\frac{1}{2}}\\f_{2m+1}\\f_{2m+\frac{3}{2}}\\f_{2m+2}\end{pmatrix}$$

Substituting scalar test equation y'=λy ($\lambda<0$, λ complex) into eqn. (7) and using $\lambda h=\bar{h}$ gives

$$A_0Y_m=A_1Y_{m-1}+\bar{h}\left(B_0Y_{m-1}+B_1Y_m\right) \quad (8)$$

The stability polynomial of (5) is given by

$$Det\left[\left(A_0-\bar{h}B_1\right)t-\left(A_1+\bar{h}B_0\right)\right]=0 \quad (9)$$

i.e.,

$$R(t,\bar{h})=t^4-\frac{24874}{18425}t^3+\frac{6449}{18425}t^2+\frac{49463}{346390}t^2\bar{h}+\frac{10734}{865975}\bar{h}^2t^2+\frac{1447}{157450}\bar{h}t^3-\frac{4304}{15745}\bar{h}^2t^3+\frac{8283}{157450}\bar{h}^3t^3$$
$$\frac{4304}{15745}\bar{h}^3t^2-\frac{729}{173195}\bar{h}^4t^3-\frac{251472}{173195}\bar{h}t^4+\frac{129973}{173195}\bar{h}^2t^4-\frac{28704}{173195}\bar{h}^3t^4+\frac{2304}{173195}\bar{h}^4t^4=0 \quad (10)$$

For zero stability, we set $\bar{h}=0$ in eqn. (10) to obtain

$$t^4-\frac{24874}{18425}t^3+\frac{6449}{18425}t^2=0. \quad (11)$$

Solving eqn. (11) for t gives the following roots:

t=0, t=0, t=0.350014 and t=1. (12)

From the definition 1.3, method (5) is zero-stable.

The stability region of method (5) is shown Figure 1.

From the definition 1.4, method (5) is A-stable.

Tested Problems

To validate the efficiency of the method developed, the following stiff IVPs are solved:

(1) $y_1'=-20y_1-19y_2$ $\quad y_1(0)=2,$ $\quad 0\le x\le 20$

$y_2'=-19y_1-20y_2.$ $\quad y_2(0)=0.$

Exact solutions: $y_1(x)=e^{-39x}+e^{-x}$,

$Y_2(x)=e^{-39x}+e^{-x}$,

Eigenvalues: -1 and -39 Source: (Musa et al. [25])

$y_1'=198y_1+199y_2$

$y_1'=198y_1+199y_2$ $\quad y_1(0)=1$ $\quad 0\le x\le 10$

$y_2'=-398y_1-399y_2$ $\quad y_2(0)=-1$

Exact solution: $y_1(x)=e^{-x}$

$y_2(x)=-e^{-x}$ Eigen values: -1 and -200.

Numerical Results

The numerical results for the test problems given in section 3 are tabulated. The problems are solved with 2-point diagonally implicit super class of block backward differentiation formula with off-step points.

The notations used in the tables are listed below:

2ODISBBDF=2-point diagonally implicit super class of block backward differentiation formula with off-step points method (Table 1).

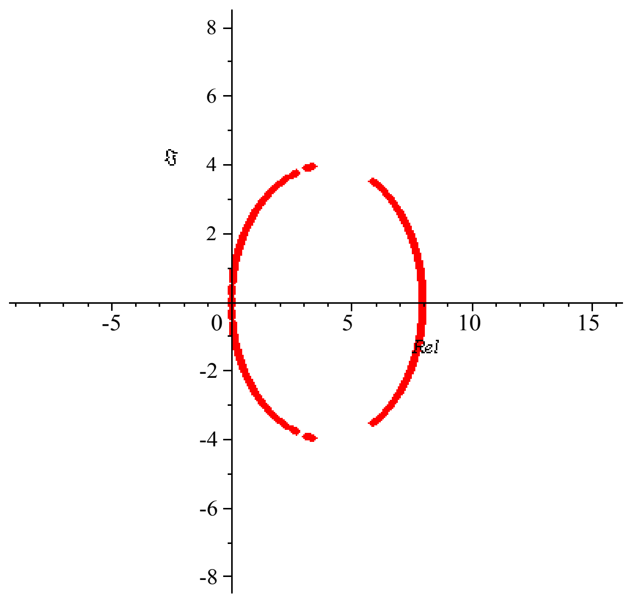

Figure 1: Stability Region of the 2-Point DISBBDF with off-step points.

h	MAXE for Problem 1	MAXE for Problem 2
10^{-2}	3.81561e-002	1.03577e-004
10^{-4}	1.64714e-005	1.12034e-008
10^{-6}	1.70657e-009	1.96752e-010

Table 1: Numerical result for problem 1 and 2.

- h=Step size.
- MAXE=Maximum error.

From the Table 1, the zero stability of 2ODISBBDF method is indicated by the decrease in error as the step length h tends to zero. The accuracy also improves as the step length is reduced.

Similarly, the solution at any fixed point improves as the step length reduced. This can be seen from the above table when h is reduced (from 0.01, 0.0001, and 0.000001). The maximum error indicates that the numerical result becomes closer to the exact solution. Thus, the computed solution tends to the exact solution as the step length tends to zero.

Conclusion

The stability analysis of the 2-point diagonally implicit super class of block backward differentiation formula with off-step points for solving stiff IVPs has been studied. The analysis has shown that the method is zero and A-stable. Based on the numerical results, it can be concluded that the Block Method is suitable for stiff problems because of its A-stability property.

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