Stability and Practically Stability of Impulsive Integro-Differential Systems by Cone-Valued Lyapunov Functions

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Abstract

Stability and practically stability comparison criteria of impulsive integro-differential systems with fixed moments of impulse effects are established by cone-Lyapunov functions through comparing with impulsive ordinary differential equations.

Keywords: Impulsive Integro-Differential System; Cone-Lyapunov Function; Stability

Introduction

Impulsive integro-differential systems which are an important embranchment of nonlinear impulsive differential systems [1], arise from extensive applications in nature-science such as mathematic models of circuit simulation in physics and neuronal networks in biology. Consequently there are some results about stability of such systems by vector Lyapunov functions coupled with Razumikhin techniques [2,3]. However, it's difficult to choose a right vector Lyapunov function because of the restrict conditions. At the same time, the method of cone-valued Lyapunov functions is well known to be advantageous in applications [4]. Hence, the stability results for impulsive integro-differential systems could be improved via the method of cone-valued Lyapunov functions.

Adeeye [5] considered the comparison principle by cone-valued Lyapunov functions for a class of integro-differential system without impulses. But it was not proved and also cannot be applied to impulsive integrodifferential systems. In this paper we shall firstly prove the comparison principle. Then by employing cone-valued Lyapunov functions a new comparison principle for impulsive integro-differential systems with fixed moments of impulse effects is established, which is compared with impulsive differential systems whose stability is relatively easy to solve. Finally the relevant new comparison criteria of stability and practically stability [6] of impulsive-integro-differential systems are obtained too.

In this paper we shall firstly prove the comparison principle. Then by employing cone-valued Lyapunov functions a new comparison principle for impulsive integro-differential systems with fixed moments of impulse effects is established, which is compared with impulsive differential systems whose stability is relatively easy to solve. Finally the relevant new comparison criteria of stability and practically stability [6] of impulsive-integro-differential systems are obtained too.

The remainder of this paper is organized as follows. In section 2, we describe impulsive integro-differential systems and introduce some notions and concepts. In section 3, we get some comparison results of stability and practically stability of the impulsive integro-differential systems with fixed moments of impulse effects by using the method of cone-valued Lyapunov functions.

Preliminaries

Consider the following impulsive integro-differential systems of the form

\[
\begin{align*}
\dot{x}(t) &= f(t, x, Tx), \\
J_k(x(t^+_k)) &= J_k(x(t^-_k)), \\
x(t^0) &= x_0, 
\end{align*}
\]

where \(N\) is the set of all positive integers, \(0 < t_1 < t_2 < \ldots < t_k < \ldots\) and \(t_k \to \infty (k \to \infty)\). \(f \in C([t_1, t_1+1) \times [S(\rho) \times R^r, R^r])(k \in N), f(t,0,0) = 0\), where \(S(\rho) = \left\{ x \in R^r : (x, \rho) \in R^r \right\}\).

\[
\begin{align*}
T_k &= \left[ k(t, x(t)) dt \right], \\
\left\{ (x, t) \in S(\rho) \in [0, \rho_t) \right\}, \\
J_k(x) &= S(\rho) \to R^r, J_k(0) = 0, \forall k \in N \right\} \\
\text{and exists } \rho_1 < \rho_2 \right\}
\end{align*}
\]

In addition, we always assume that \(f, J_k\) satisfy certain conditions. For example, the solution of system (1) exists on \([t_1, +\infty)\) and is unique. We denote by \(x(t) = x(t, t_1, x_1)\) the solution of system (1) with initial value \((t_0, x_0)\). Since \(f(t,0,0) = 0, J_k(0) = 0, k \in N\), then \(x(t) = 0\) is a solution of (1), which is called the trivial solution. Note that the solutions \(x(t)\) of (1) are right continuous, satisfying \(x(t^+_-k) = x(t^-_k) \neq J_k(x(t^+_k))\).

Let \(t_0 = 0\); the following sets are introduced:

\[
G_k = \left\{ (x, t) \in S(\rho) : t < t_k \right\}, \quad G = \bigcup_{k=1}^{\infty} G_k.
\]

For convenience, we define the following classes of functions:

\[
K = \left\{ a \in C[R_+, R] : \text{strictly increasing} \right\},
\]

\[
K_0 = \left\{ a \in C[R_+, R] : \text{strictly increasing and } a(0) = 0 \right\},
\]

\[
\left\{ h(t, x) \in R^r, \left. h(t, x) \in R^r \right\} \right\}.
\]

In addition, we introduce some definitions as follows:

**Definition 1:** Let \(Z \subseteq R^r\) be a cone, that is, \(Z\) is closed, convex with \(\lambda Z \subseteq Z, \lambda \geq 0\) and \(Z \cap (-Z) = \{0\}\) with interior \(Z^0 \neq \emptyset\). For any \(x, y \in R^r\), we let \(x \leq y\) if \(y - x \in Z\) and for any functions \(u, v : R^r \to R^r, u \leq v\) if \(u(t) \leq v(t)\) on \(R_+\). Also let \(Z^+ = \{x \in R^r : \varphi(x) \geq 0, \forall x \in Z\}\) and \(Z^+ = Z \cap \varphi^{-1}([0, \infty))\).

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\[
\varphi(x) = \sum \varphi_i x_i
\]

**Definition 2:** A function \(F : R^n \to R^n\) is said to be quasi-monotone nondecreasing relative to the cone \(Z \subseteq R^n\) if \(x \leq y\) and \(\varphi(y-x) = 0\) for all \(\varphi \in Z^*_+\) implies \(\varphi(F(x)) - F(x) \geq 0\).

**Definition 3:** We shall say that the function \(V_0 : R \times R^n \to Z\) belongs to the class of cone-Lyapunov functions if:

1. \(V \in V_0\) is continuous on \(G_3 \times R^n\) and the following limits exist
   \[
   \lim_{(t,y) \to (t_0,y)} V(t,y), k \in N
   \]
   ii. \(V(t,x) \in V_0\) along system (1) is defined:
   \[
   D^+ V(t,x(t)) + \limsup_{\lambda \to 0} \left( \int_{t}^{t+\lambda} (t+h,x(t)+bf(t,x(t))) - V(t,x(t)) \right). \]

**Definition 4:** The trivial solution of (1) is said to be

i. stable, if for any \(\varepsilon > 0\), every \(t \in R\), there exists a \(\delta = \delta(t_0, \varepsilon) > 0\) such that \(x_0 \leq \delta\) implies \(x(t) \leq \varepsilon\) for all \(t \geq t_0\);

ii. uniformly stable, if \(\delta\) is independent of \(t_0\);

iii. uniformly asymptotically stable, if (ii) and (iv) hold together.

**Definition 5:** The trivial solution of (1) is said to be

i. practically stable, if for given number pair \((\lambda, A)\) with \(0 < \lambda < \lambda\), we have \(x_0 \leq \lambda\) implies \(x(t) \leq A, t \geq t_0\) for some \(t_0 \in R\);

ii. uniformly practically stable, if (i) holds for any \(t_0 \in R\); and some \(t_0 \in R\), we have \(x_0 \leq \lambda\) implies \(x(t) \leq B, t \geq t_0 + T\);

iii. uniformly practically quasi-stable, if (iii) holds for every \(t_0 \in R\).

iv. strongly practically stable, if (i) and (iii) hold together;

v. strongly uniformly practically stable, if (i) and (iv) hold together.

We also consider the comparison differential system:

\[
\begin{cases}
  u' = g(t,u), & t \neq t_k \\
  u(t_k) = \Psi_k (u(t_{k-1})) & k \in N \\
  u(t_0) = u_0
\end{cases}
\]  \hspace{1cm} (2)

where \(g \in C([t_0 + \varepsilon] \times Z, R^n), k \in N\).

In addition we always assume that \(g, \Psi_k\) satisfy certain conditions such that the solution of system (2) exists on \([t_0 + \varepsilon]\) and is unique. We denote by \(u(t) = u(t_0, u_0)\) the solution of system (2) with initial value \((t_0, u_0)\). Note that the solutions \(u(t)\) of (2) are right continuous, satisfying \(u'(t) = u(t_k) = \Psi_k (u(t_{k-1}))\).

**Definition 6:** The trivial solution of (2) is said to be

i. \(\phi_0\) - stable, if for any \(\varepsilon > 0\), every \(t_0 \in R\), there exists a \(\delta = \delta(t_0, \varepsilon) > 0\) such that \(\phi_0(u) \leq \delta\) implies \(\phi_0(u(t)) \leq \varepsilon\), for all \(t \geq t_0\); where \(\phi_0 \in Z^*_+\);

ii. \(\phi_0\) - uniformly stable, if \(\delta\) in (i) is independent on \(t_0\);

iii. \(\phi_0\) - attractive, if for any \(\varepsilon > 0\), every \(t_0 \in R\), there exists a \(\delta = \delta(t_0, \varepsilon) > 0, T = T(t_0, \varepsilon) > 0\), such that \(\phi_0(u) \leq \delta\) implies for all \(t \geq t_0 + T\), where \(\phi_0 \in Z^*_+\);

iv. \(\phi_0\) - uniformly attractive, if \(\delta, T\) in (iii) is independent on \(t_0\);

v. \(\phi_0\) - asymptotically stable, if (i) (and) (iii) hold together;

vi. \(\phi_0\) - uniformly asymptotically stable, if (ii) and (iv) hold together.

**Definition 7:** The trivial solution of (2) is said to be

i. \(\phi_0\) - practically stable, if for given number pair \((\lambda, A)\) with \(0 < \lambda < \lambda\), we have \(\phi_0(u) < \lambda\) implies \(\phi_0(u(t)) < A, t \geq t_0\) for some \(t_0 \in R\), \(\phi_0 \in Z^*_+\);

ii. \(\phi_0\) - uniformly practically stable, if (i) holds for every \(t_0 \in R\);

iii. \(\phi_0\) - practically quasi-stable, if for given number \((\lambda, B, T) > 0\) and some \(t_0 \in R\), \(\phi_0 \in Z^*_+\) we have \(\phi_0(u_0) < \lambda\) implies \(\phi_0(u(t)) < B\) for all \(t \geq t_0 + T\);

iv. \(\phi_0\) - uniformly practically quasi-stable, if (iii) holds for every \(t_0 \in R\);

v. \(\phi_0\) - strongly practically stable, if (i) and (iii) hold together;

vi. \(\phi_0\) - strongly uniformly practically stable, if (ii) and (iv) hold together.

**Main Results**

**Lemma 1:** Assume that

i. \(g \in C([R \times R^+, R^n]), g(u,t)\) is quasi-monotone nondecreasing in \(u\) for each fixed \(t\) on \(Z\) and \(r(t,t_0,u_0)\) is the maximal solution of the system

\[
\begin{cases}
  u'(t) = g(t,u), & t \neq t_k \\
  u(t_k) = \Psi_k (u(t_{k-1})) & k \in N \\
  u(t_0) = u_0
\end{cases}
\]  \hspace{1cm} (2)

ii. \(V \in C([R \times R^+, Z], V(t,x))\) is locally Lipschitz in \(x\) relative to the cone \(Z\) and \(D^+ V(t,x(t)) \leq \varepsilon\) for all \(t \geq t_0 + T\);

\[
g(t,V(t,x(t)))(t \geq t_0) \text{ for any solution } x(t) = x(t_0, x_0) \text{ of }
\]  \hspace{1cm} (2)

Then \(V(t_0, x_0) \leq u_0\) implies \(V(t,x(t)) \leq r(t,t_0,u_0), t \geq t_0\).

**Proof:** Let \(x(t) = x(t_0, x_0)\) is any solution of system in (ii), satisfying \(V(t_0, x_0) \leq u_0\).

Set \(m(t) = V(t,x(t))\), for small enough \(h < 0\) from (ii) \(V(t,s)\) is locally Lipschitz in \(s\) relative to the cone \(Z\), therefore \(m(t+h) - m(t) \leq \varepsilon(h) = \int_{t}^{t+h} (t+h,s)(-V(t+h,s)+bf(t,s,\lambda)+V(t+h,s)-V(t,s))\) for \((t,s,\lambda) \in R^+ \times (t_0, x_0) \times \varepsilon(h)\), when \(h \to 0\) we have \(D^+ m(t) \leq \varepsilon(h) \leq \varepsilon(t,x(t)) \leq g(t,m(t))\).

For small enough \(\forall N > 0\), consider system

\[
\begin{cases}
  u'(t) = g(t,u) + \eta \varepsilon, & t \neq t_k \\
  u(t_k) = \Psi_k (u(t_{k-1})) & k \in N \\
  u(t_0) = u_0, t \in R
\end{cases}
\]  \hspace{1cm} (2)

where \(\eta \in Z\).

The solution of it is, then we have.

To prove the conclusion we only need to prove

\[
m(t) \leq u(t, \varepsilon), t \geq t_0.
\]  \hspace{1cm} (3.1)
If it is not true, there exists $t_1 > t_0$, such that $u(t_1, e) - m(t_1) \in \partial Z$, and $u(t_1, e) - m(t) \in Z$, if $t \in [t_0, t_1)$.

From (i) $g(t, u)$ is quai-monotone non-decreasing in $u$ for each fixed $t$ on $Z$, so there exists $\varphi \in \mathcal{P}$, such that

$$\varphi(u(t_1, e) - m(t_1)) = 0 \quad \text{and} \quad \varphi(g(t_1, u(t_1, e)) - g(t_1, m(t_1))) \geq 0.$$ 

Set $\bar{\varphi}(t) = \varphi(u(t, e) - m(t)), t \in [t_0, t_1]$. Obviously $\bar{\varphi}(t) > 0, t \in [t_0, t_1]$ and $\bar{\varphi}(t_1) = 0$.

Therefore, $D \bar{\varphi}(t) < 0$.

But

$$D \bar{\varphi}(t) = \varphi(D u(t, e) - D m(t)) > \varphi(g(t, u(t, e)) + \eta - g(t, m(t))) \geq 0,$$

a contradiction. So (3.1) holds, thus lemma 1 holds.

**Lemma 2:** Assume that

1. $g \in C([t_0, t_1] \times Z, R^+), k \in N, g(t, u)$ is quasi-monotone non-decreasing in $u$ for each fixed $t$ on the cone $Z$ and $r(t) = r(t, t_0, u_0)$ is the maximal solution of system (2) on $Z$;

2. $\Psi : Z \to Z(k \in N)$ is strictly increasing on $Z$;

3. For any solution $x(t) = x(t, t_0, x_0)$ of system (1) and $V \in V_0^*$, $D V(t, x(t)) \leq 0, t \neq t_1, k \in N$;

4. $V(t, t_1, J_1(x(t_1))) \leq \Psi_1(x(t, x))$

Then implies

Proof: For any $t_0 \in R^+$, and $t_0 \in [t_0, t_1 + 1]$, for some $k \geq 1$, we designate $t_k = t_0 + 1, t_2, t_3, \ldots$, for convenience, then for $t \in [t_0, t_1]$ from lemma 1 we have $V(t, x(t)) \leq r(t, t_0, u_0)$ where $r(t, t_0, u_0)$ is the maximal solution of system (2) on $[t_0, t_1]$ such that $r(t_0, t_0, u_0) = u_0$. Therefore, $V(t, t_1, x(t_1)) \leq r(t_1, t_0, u_0) = r_1$.

Thus from (iii) we have

$$V(t_1, x(t_1, u_0)) = V(t_1, J_1(x(t_1))) \leq \Psi_1(x(t_1, x)) \leq \Psi_1(r(t_1, t_0, u_0)) = r(t_1, t_0, u_0) = r_1.$$

Again from lemma 1, for $2 \in [t_1, t_2)$, $V(t, x(t)) \leq r(t, t_1, r_1)$; where $r(t, t_1, r_1)$ is the maximal solution of system (2) on $[t_1, t_2]$ such that $r(t_1, t_1, r_1) = r_1$.

Therefore we have

$$V(t, x(t_1, u_0)) \leq r_2(t, t_1, r_1) = r_2,$$

where $r_2(t, t_1, r_1)$ is the maximal solution of system (2) on $[t_1, t_2]$ such that $r_2(t_1, t_1, r_1) = r_1$.

So if we define

$$u^*(t) = \begin{cases} u_0 & t \in [t_0, t_1) \\ r_1(t, t_0, u_0) & t \in [t_1, t_2) \\ r_2(t, t_1, r_1) & t \in [t_2, t_3) \\ \cdots & \end{cases}$$

Then $u^*(t)$ is the solution of system (2), and $V(t, x(t)) \leq u^*(t)$; since $r(t, t_0, u_0)$ is the maximal solution of system (2) on $Z$, we immediately have $V(t, x(t)) \leq u^*(t)$ and $V(t, x(t)) \leq u^*(t)$.

**Theorem 1:** Assume that $\exists a, b \in K$ such that:

1. $b(\|x\|) \leq \alpha(\|x\|), \forall x \in \mathcal{S}(\rho)$;

2. $\mathcal{G}(\|x\|) \leq \mathcal{G}(\|x\|), \forall x \in \mathcal{S}(\rho)$;

3. $V(t, x(t, t_0, x_0)) \in \mathcal{S}(\rho)$

Then the stability properties of the trivial solution of system (2) imply the corresponding stability properties of the trivial solution of system (1).

Proof: For any $\varepsilon < \rho$, every set $V(t) = V(t, x(t))$.

Let the trivial solution of system (2) is $\mathcal{O}$-stable, then for $b(t) > 0$ every $t_0 \in R^+$, there exists $\delta > 0$ such that $\mathcal{O}(\mathcal{G}(u_0) < \delta$ implies $V(t, x(t)) < \rho$.

Set $\delta = \delta_1(e) \forall e \in \mathcal{S}(\rho)$.

Next, we claim that $V(t) < \delta$ implies $V(t, x(t)) < \rho$.

If it is not true, then there exists a solution of system (1) such that $x(t_0, x_0) = 0$ with $x_0 < \delta$, then there exists $t_1 > t_0$ such that $t_1 \leq t_1 < t_1 \in k \in N$ satisfying $V(t_1) \geq \varepsilon$ and $V(t) \leq \varepsilon, t_0 < t < t_1$. From $0 < \varepsilon < \rho$, we have $k(t) = k(t, x(t)) < \rho$. From (iv) we have $D V(t) \leq z g(t, V(t), t) < k(t)$. From $0 < \varepsilon < \rho, k(t) < \rho$. From (iv) we have $D V(t) \leq z g(t, V(t), t) < k(t)$. From $0 < \varepsilon < \rho, k(t) < \rho$. From (iv) we have $D V(t) \leq z g(t, V(t), t) < k(t)$. From $0 < \varepsilon < \rho, k(t) < \rho$. From (iv) we have $D V(t) \leq z g(t, V(t), t) < k(t)$.
holds, thus the trivial solution of system (1) is stable. If the trivial solution of system (2) is $\mathcal{O}_\gamma$-uniformly stable then it is clear that $\delta$ will be independent of $t_0$ and thus we get the uniform stability of the trivial solution of system (1). Assume that the trivial solution of system (2) is $\mathcal{O}_\gamma$-asymptotically stable, consequently we get that the trivial solution of system (1) is stable, then for $\varepsilon = \rho > 0$, there exists $\delta_0 = \delta(t_0, \rho)$, such that $\|x(t)\| < \delta_0$ implies $\|x(t)\| < \rho, t \geq t_0$. From (iv) we have $D\dot{V}(t) \leq z g(t(t_0), t), t \geq t_0$.

For any $0 < \varepsilon < \rho$, every $t_0 \in R$, from the $\mathcal{O}_\gamma$-attractivity of the trivial solution of system (2) we can get that $b(\varepsilon) > 0$, every $t_0 \in R$, there exists $\delta_0 = \delta(t_0, \rho) > 0$ and $T = T(t_0, \rho) > 0$, such that $0 \leq \rho \leq \delta(t_0, u_0)$ implies $\|x(t)\| < \delta(t_0, u_0) + T, t \geq t_0 + T$.

Set $\delta_0 = \min\{\delta(t_0, \rho) \}$.

For $\|x(t)\| < \delta_0$, set $u_0 = V(t_0)$ then from lemma 2 we have $V(t) \leq z \rho(t(t_0), t_0, u_0), t \geq t_0$.

And from (i) $(\mathcal{O}_\gamma, u_0) = (\mathcal{O}_\gamma, V(t_0))$ implies $\|x(t)\| < \delta(t_0, u_0)$ so from the $\mathcal{O}_\gamma$-attractivity of the trivial solution of system (2) we get that $\mathcal{O}_\gamma, r(t(t_0), u_0) < b(\varepsilon), t \geq t_0 + T$ therefore $b(\varepsilon) \leq (\mathcal{O}_\gamma, V(t_0)) \leq (\mathcal{O}_\gamma, r(t(t_0), u_0)) > b(\varepsilon), t \geq t_0 + T$, thus $\|x(t)\| < \rho, t \geq t_0 + T$ so the trivial solution of system (1) is attractive.

Then the trivial solution of system (1) is asymptotically stable. If the trivial solution of system (2) is $\mathcal{O}_\gamma$-uniformly asymptotically stable, then it is clear that $\delta_0 T$ will be independent of $0$, and thus we get the uniform asymptotic stability of the trivial solution of system (1).

**Theorem 2:** Assume that theorem 1 (i)-(v) hold and we have:

(vi) for $0 < A < A \leq \rho$, with $a(A) < b(A)$.

Then the $\mathcal{O}_\gamma$-practical stability properties of the trivial solution of system (2) with respect to $(a(\lambda), b(A))$ imply the corresponding practical stability properties of the trivial solution of system (1) with respect to $(\lambda, A)$.

**Proof:** Set $V(t) = V(t, x(t))$, suppose that the trivial solution of system (2) is $\mathcal{O}_\gamma$ practically stable with respect to $(a(\lambda), b(A))$ then there exists $t_0 \in R$, such that $0 \leq \|x(t)\| < a(\lambda)$ implies $\|x(t)\| < b(A), t \geq t_0$.

For above $t_0 \in R$, next we will prove that $\|x(t)\| < \lambda$ implies $\|x(t)\| < A, t \geq t_0$. If it is not true, then there will exist a solution $(x(t))$ of system (1) such that $x(t_0, t, t_0, x_0) = 0$ with $\|x(t_0, t, t_0, x_0)\| < \delta$, then there exists $t' > t_0$ such that $t_0 > t' > t_0, (k \in N)$ satisfying $\|x(t')\| > A$ and $\|x(t)\| < A, t_0 < t < t_0$. Since $0 < A < \rho$, we have $\|x(t)\| < A < \|x(t')\| < A$ which implies $S \|x(t)\| < A$.

From (iv) we have $D\dot{V}(t) \leq z(g(t), t), t \in [t_0, t]$.

Set $u_0 = V(t_0)$ then from lemma 2 we get that $t \in [t_0, t]$ implies $V(t) \leq z(t(t_0), t, t_0, u_0)$, where $r(t(t_0), t, t_0, u_0) = u_0$. And from $(\mathcal{O}_\gamma, u_0) = (\mathcal{O}_\gamma, V(t_0)) = a(\lambda)$, so from the $\mathcal{O}_\gamma$-practical stability we have $(\mathcal{O}_\gamma, r(t(t_0), t, t_0, u_0)) < b(A), t \geq t_0$.

Then $b(\lambda) < b(\lambda) < b(\lambda) < b(\lambda) < b(\lambda) < b(A)$ a contradiction thus (3.3) holds, so the trivial solution of system (1) is practically stable. Suppose that the trivial solution of system (2) is $\mathcal{O}_\gamma$ uniformly practically stable with respect to $(a(A), b(A))$ then it is clear that $t_0$ will be independent of $t_0$, and thus we get the trivial solution of system (1) is uniformly practically stable with respect to $(\lambda, A)$. Suppose that the trivial solution of system (2) is $\mathcal{O}_\gamma$-strongly practically stable with respect to $(a(A), b(A), (B), T)$, consequently we get that the trivial solution of system (1) is practically stable with respect to $(\lambda, A)$ then exists $t_0 \in R$, such that $\|x(t)\| < \lambda$ implies $\|x(t)\| < A, t \geq t_0$. From (iv) $D\dot{V}(t) \leq z(g(t), t), t \geq t_0$. And since the trivial solution of system (2) is $\mathcal{O}_\gamma$-practically quasi-stable with respect to $(a(A), b(A), T)$, we have that $0 \leq (\mathcal{O}_\gamma, u_0) < a(\lambda)$ implies $\|x(t)\| < A, t \geq t_0 + T$.

For $\|x(t)\| < \lambda$, set $u_0 = V(t_0)$ then from lemma 2 we get $V(t) \leq z(r(t(t_0), t_0, u_0)), t \geq t_0$.

And from (ii) $(\mathcal{O}_\gamma, u_0) = (\mathcal{O}_\gamma, V(t_0)) = a(\lambda)$, so from the $\mathcal{O}_\gamma$-practical quasi-stability of the trivial solution of system (2) we have $(\mathcal{O}_\gamma, r(t(t_0), t, t_0, u_0)) < b(B), t \geq t_0 + T$ and $b(\lambda) \leq (\mathcal{O}_\gamma, V(t_0)) \leq (\mathcal{O}_\gamma, r(t(t_0), t, t_0, u_0)) < b(B), t \geq t_0 + T$ thus $\|x(t)\| < (B), t \geq t_0 + T$, so the trivial solution of system (1) is practically quasi-stable with respect to $(\lambda, A, B, T)$.

Therefore the trivial solution of system (1) is strongly practically stable with respect to $(\lambda, A, B, T)$.

If the trivial solution of system (2) is $\mathcal{O}_\gamma$-strongly uniformly practically stable with respect to $(\lambda, A, B, T)$ then it is clear that the above proof establish for every $t_0$, therefore we get the trivial solution of system (1) is strongly uniformly practically stable with respect to $(\lambda, A, B, T)$.

**References**
